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MEMOIRE

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Par

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Intitulé

Dirac Equation in the Formalism of Fractal Geometry

Soutenu le :

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Introduction

We know that the laws of physics are described by two important theories, the first one is the theory of relativity which includes the classical mechanics which is based on Galilean relativity, special and general relativities. All these theories describe the macro physical world. The second theory is quantum mechanics which describes the microphysical world. If we assume that the macro physical world includes the microphysical world as a limit, these two theories must be linked somehow. The two worlds must be described by a same theory or by two dependent theories.

The problem with the actual theories that they are formulated on completely different grounds.

For instance general relativity is a theory based on fundamental physical principles which are the principle of general covariance, whereas the quantum theory is an axiomatic theory.

So the different constructions of the classical and the quantum theory leads to a fission in physics yielding two opposite worlds according to the scale: the smallest and largest. That is why the modern physics seems incomplete, several problems are still posed.

At the small scale the standard model of Weinberg-Salam-Glashow leads to the observed structure of elementary particles and coupling constants, but this model is not able to predict a theoretical basis to the number of elementary particles or their masses. In summary some problems were solved but the problem of the quantization of masses and charges is still unresolved.

The idea behind this work is the possibility is that quantum and classical domains may have a similar nature. The aim is to find a theory which depends of the scale. If the scale is less than a fundamental length which has to be specified, we recover quantum theory and if the scale is greater that this length we find classical mechanics.

This theory baptized scale relativity has been formulated by Nottale in 1992. This theory is based on fractal geometry with the assumption that the Einstein’s principle of relativity applies not only to laws of motion but also to laws of scale.

In scale relativity we can treat quantum mechanics without using quantum principles in other words we do need to use the correspondence principle.

In this work we attempt to derive the Dirac equation in the formalism of fractal geometry without any need to quantum mechanics postulates.

This dissertation is organized as follow: in chapter 1 is devoted to a review of fractal geometry and scale relativity.

Then in chapter 2 we will consider the behavior of quantum mechanical paths in the light of the fractal geometry.

In chapter 3 we will derive the Schrödinger’s equation from Newton's fundamental equation of dynamics without using the axioms of quantum mechanics. The method used is the stochastic mechanics according to Nelson.

The chapter 4 we will apply the principle of scale relativity to the quantum mechanics by defining the covariant derivative operator and we will treat some applications.

In chapter 5 we write the Schrödinger’s equation by using the hypothesis of Nottale and the complex Klein-Gordon equation is derived.

We end up with chapter 6 where we have derived the Dirac equation from the Newton's equation in the spirit of Nottale hypothesis. The dissertation ends up by a perspective for future work and some appendices.

Chapter 01

Fractal Geometry and the Theory of Scale Relativity

Introduction

We know that the theory of Kaluza-Klein which attempted to unify the gravitation and electromagnetism on a geometrical approach based on curvature and or torsion of spacetime was unsuccessful. After that the advance of quantum gauge theories led to the hope that unification may rather be reached by the quantized fields associated to particles, but until now this approach was vain. The main problem is how to quantize gravity. Up to now there is no acceptable quantum version of gravity.

The geometrical attempts to unification failed because of the following remark:

The observed properties of the quantum world cannot be reproduced by Riemannian geometry. Indeed we know that quantum mechanics and field theories are based on flat spacetime, whereas general relativity is formulated in curved spacetime. In the first theories, spacetime is passive and may be considered as a scene on which physical phenomena occur, however in general relativity spacetime is dynamical or active, in other words the scene on which physical phenomena occur may contribute to the physical phenomena.

Until now there is no satisfactory geometrical approach of the quantum properties of microphysics. For this reason, Nottale suggested in 1992 a possible way towards the construction of a spatial-temporal theory of the microphysical world, based on the concept of fractal space-time. His theory is based on the extension of the principle of relativity to include in addition to the ordinary relativity which is based on motion, another type of relativity: the relativity of scale.

In this theory, Nottale assumes that spacetime is non-differentiable. One can see easily why it is possible for the space-time to non differentiable at small scale. Indeed the fact that in the micro world the notions of velocity and acceleration are totally absent since quantum theory is in essence non differential in contrast to classical mechanics. Nottale extended Einstein's principle of relativity by assuming that the principle of relativity applies not only to motion transformations, but also to scale transformations. In this way he included the resolution of measurements as a state of the system in addition to the usual coordinates (x, y, z, t) .

From Fractal Objects to Fractal Space

The word "fractal" comes from the Latin word "fractus", which means "fragmented" or "fractured". It was Benoît Mandelbrot a French mathematician who used this term for the first time in 1975.

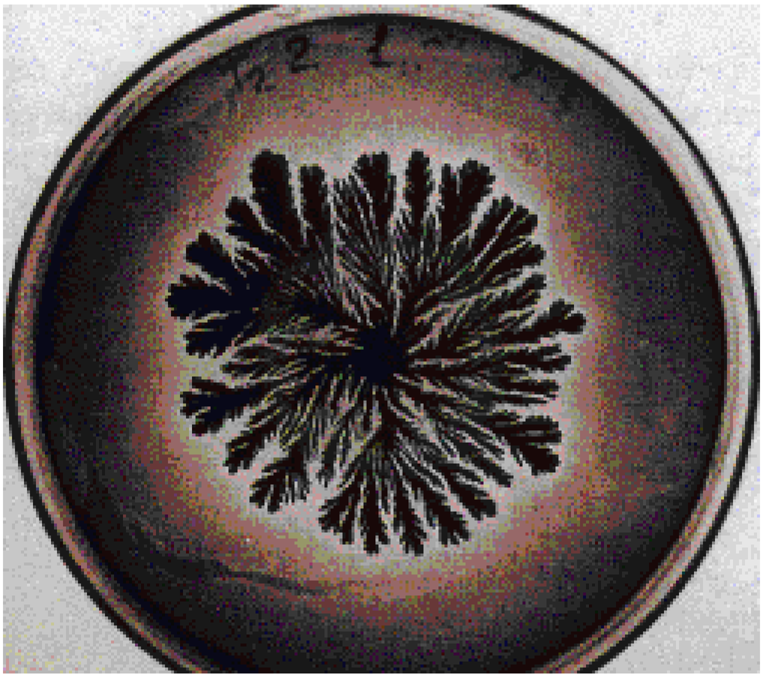
Fractals are objects, curves, functions, or sets, whose form is extremely irregular or fragmented at all scales [1]. The study of fractal objects is generally referred to as fractal geometry.

We can see in nature a lot of objects which have fractal structure such as mountains, coastlines, rivers, plants, clouds.

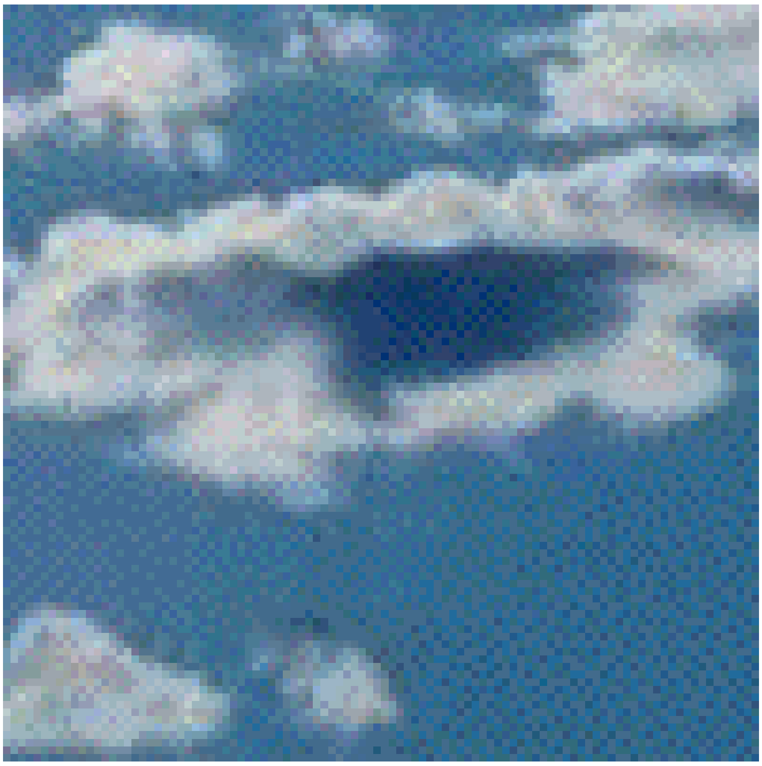
In humans branches of arteries and blood vessels have a fractal structure, as well as a number of other things including: kidney structure, skeletal structure, heart and brain waves and the nervous system.

Examples of natural fractals

Lightning



Bacteria

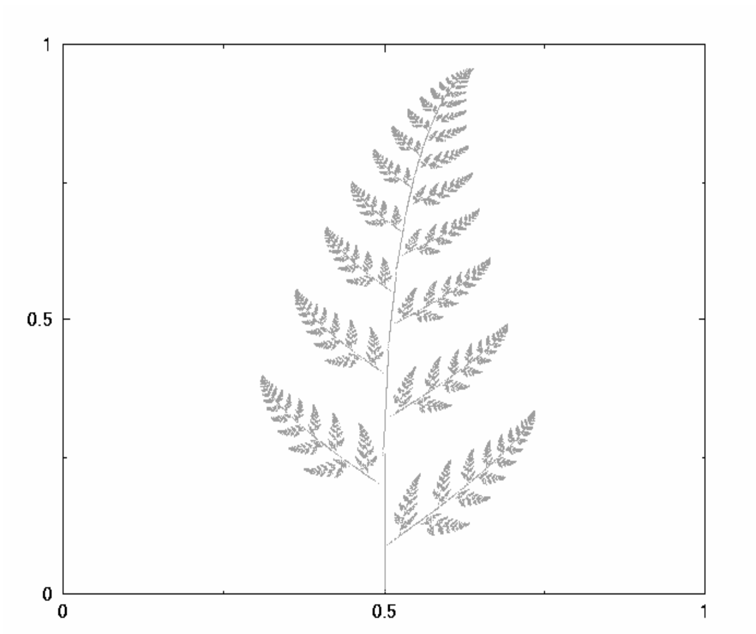


Clouds

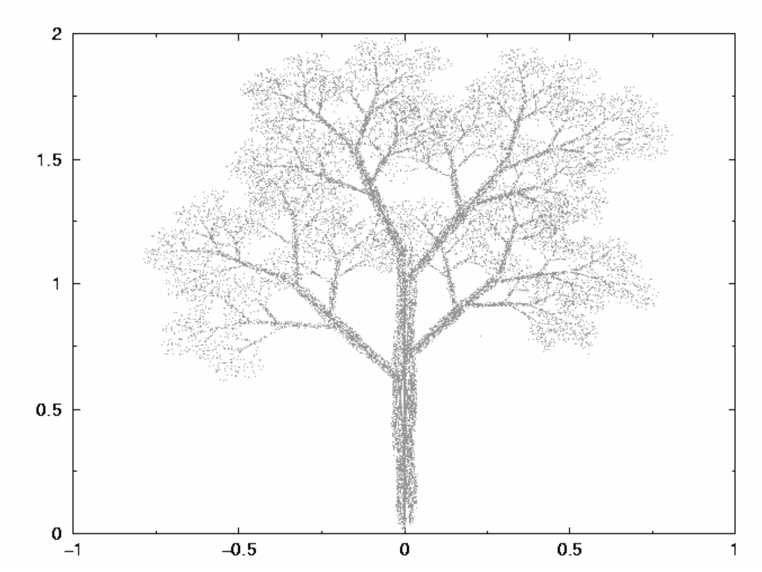
Trees

Using a computer by using some algorithms we can obtain some natural things such as plants and trees. See the following figures.

Man made fractals



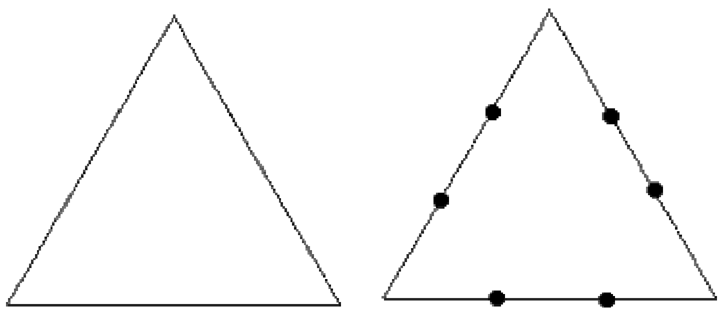
A Fractal Plant



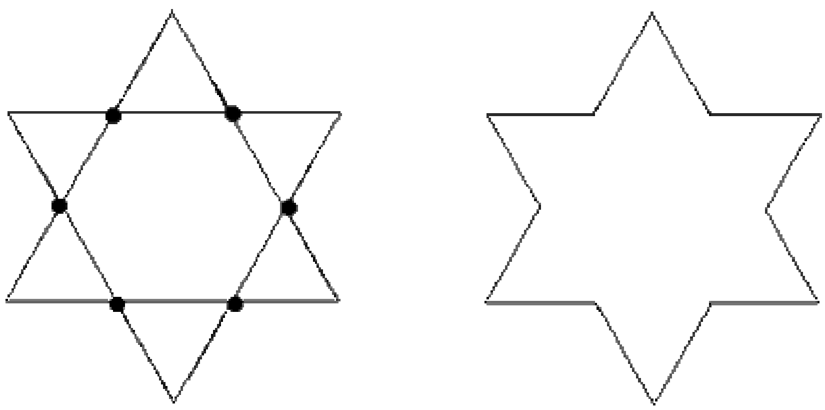
A fractal tree

To have an idea on fractals, let us make one known as Koch snowflake.

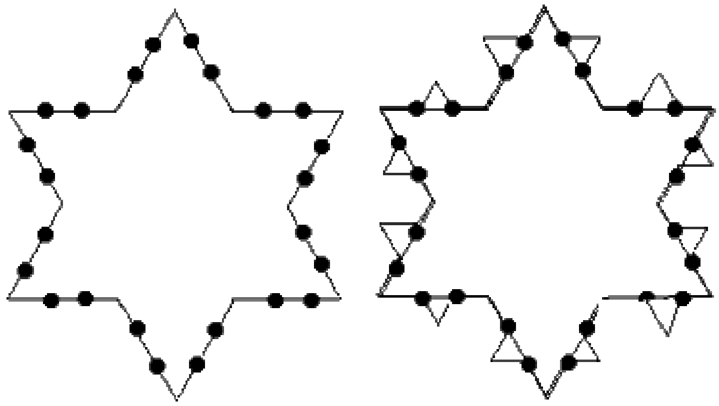
Consider a triangle



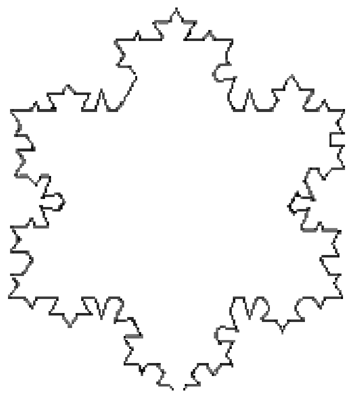
Now let us add a small triangle to each edge, we obtain the following figure



We repeat the previous procedure which means adding a small triangle to each edge which gives

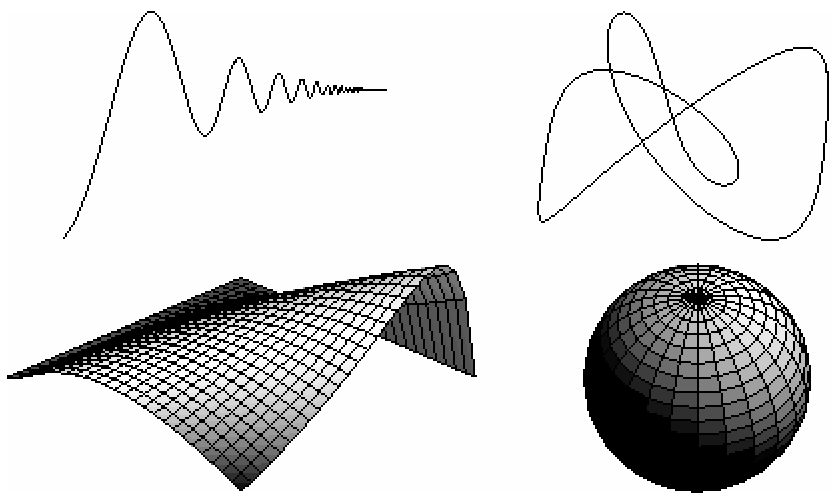


More iterations gives



The fractal obtained is called Koch snowflake.

Classical geometry based on Euclidean geometry deals with objects of integer dimensions: points are zero dimensional objects, lines and curves are one dimensional, however plane figures such as squares and circles are two dimensional, and cubes and spheres are three dimensional solids.



The problem is as Mandelbrot quoted in his book *"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a* *straight line.*" is that natural phenomena are better described by fractal geometry than the Euclidean geometry.

Fractal is characterized by non-integer dimension, which is a dimension between two whole numbers. So while a straight line has a dimension of one, a fractal curve will have a dimension between one and two. The more the flat fractal fills a plane, the closer it approaches dimension two [2].

The dimension used in Euclidean geometry is called a topological dimension which is the "normal" idea we have on dimension; a point has topological dimension 0, a line has topological dimension 1, a surface has topological dimension 2, a volume has topological dimension 3.

In fractal geometry there is another type of dimension called Hausdorff-Besicovitch dimension or fractal dimension.

So fractals are usually defined as sets of topological dimension D_T and fractal dimension D, such that $D > D_T$

Roughly speaking, fractal dimension can be calculated by taking the limit of the quotient of the log change in object size and the log change in measurement scale or resolution, as the measurement scale approaches zero.

Let us calculate the Hausdorff (fractal) dimension D for a famous example of an everywhere continuous but nowhere differentiable curve called the Koch curve.

Its construction is shown in the following figure. The Koch curve is the final product of an infinite sequence of steps like those in the figure, where in each step in the construction, the length of the curve increases by a factor of $\frac{4}{3}$. So the final curve being the result of an infinite number of steps is infinitely long although it occupies a finite area.

Suppose that we consider the Koch curve resolving distances greater than some scale Δx and measure its length to be ℓ then, if we improve our resolution so that

$\Delta x' = \left(\frac{1}{3}\right)\Delta x$, the next level of resolution in the curve will become visible and we will measure a new length $l' = \left(\frac{4}{3}\right)l$. Since the conventional definition of length, when applied to curve like the Koch curve, gives a quantity which depends on the resolution with which the curve is examined (even for very small Δx), this definition is not very useful. That is why Hausdorff has proposed a modified definition of length to be used in these cases, which is called the Hausdorff length L given by

$$L = \ell(\Delta x)^{D-1}$$

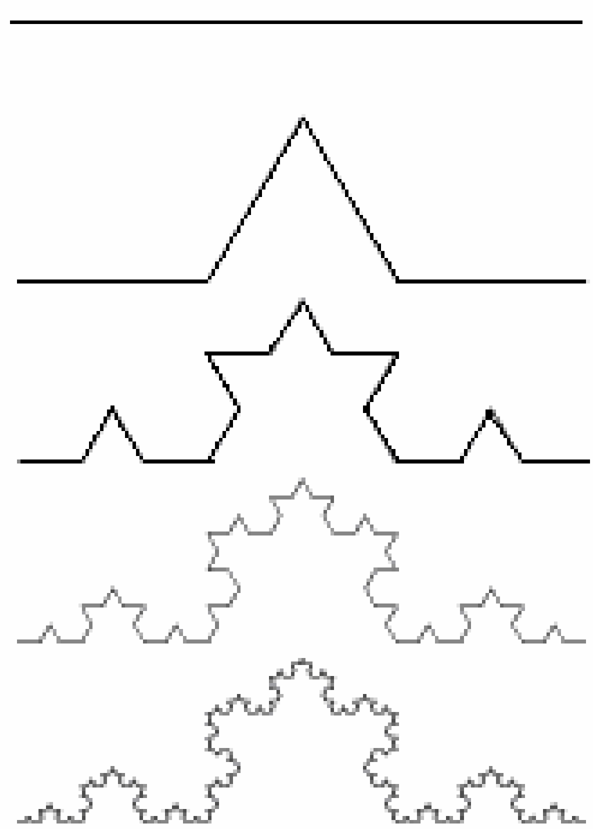
where ℓ is the usual length measured when the resolution is Δx, D is a number chosen so that L will be independent of Δx, at least in the limit $\Delta x \to 0$.

For the Koch curve, we can determine D by requiring that

$$L' = \ell'(\Delta x')^{D-1} = \left(\frac{4}{3}\ell\right)\left(\frac{1}{3}\Delta x\right)^{D-1} = L = \ell(\Delta x)^{D-1}$$

This implies that

$$D = \frac{\ln 4}{\ln 3} \approx 1.26$$



The Construction of the Koch curve

Relativity of scales

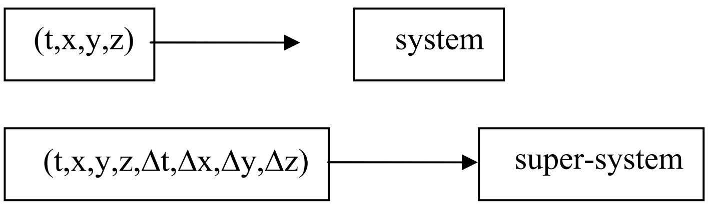
In the well-know theories the coordinate systems is subjected to transformations corresponding to changing the origin and axes, but we ignore the units. In spite of their introduction for measuring lengths and times, which is made necessary by the relativity of every scales in nature, we know that the measuring of length (time) is physical when it relative to another length (time), what we actually do is to measure the ratio of the lengths of two bodies (times of two clocks), in the same way as the absolute velocity of a body has no physical meaning, but only the relative velocity of one body with respect to another as demonstrated by Galileo, so we can say that the length of a body or the periods of a clock has no physical meaning, but only the ration of the lengths of two bodies and the ration of the periods of two clocks.

The resolutions of measurement are related to the units, and their interpretation is changed according to the scale, while classically we can interpreted it as a precision of measurements (measuring with two different resolutions yields the same result with different precisions) for example we can measure the length of a table by a ruler and a palmer. In microphysics where classical mechanics is no more applicable and it has to be replaced by quantum mechanics, changing the resolution of measurement dramatically affects the results.

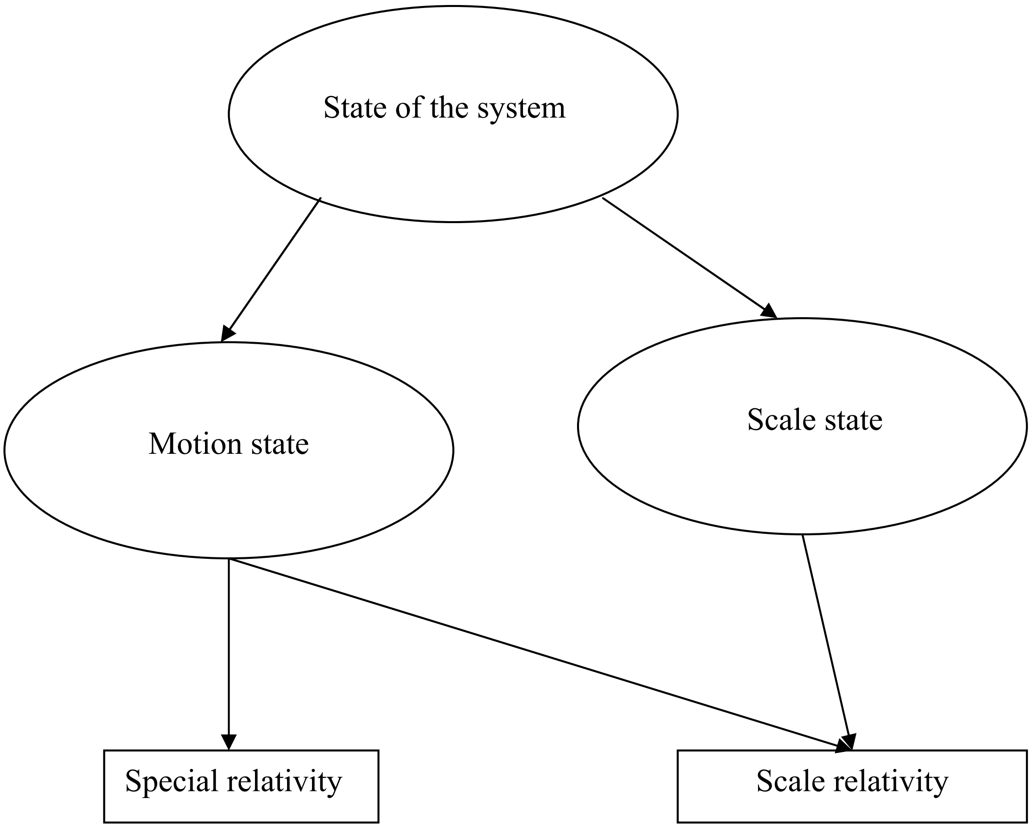
Indeed, if your ruler measure centimetres, what sense does an angstrom make?

The results of measurements explicitly depend on the resolution of the apparatus, as indicated by Heisenberg's relation, for this reason we suggest the introduction of resolution into the description of coordinate systems (as a state of scale), which is the basis of theory of scale relativity [3].

To realize this idea of scale relativity, firstly we propose to extend the notion of reference system by defining «super systems» of coordinates which contain not only the usual coordinates (t,x,y,z) but also spatial temporal resolutions $(\Delta t,\Delta x,\Delta y,\Delta z)$.



The second suggestion is the extension of the principle of relativity, according to which the laws of nature should apply to any coordinate super system, in other words, not only general (motion) covariance is needed but also scale covariance.



Special and scale relativities

Let us treat the fundamental behavior of the quantum world. We recall that the wave-particle duality is postulated to apply to any physical system, and that the Heisenberg relations are consequences of the basic formalism for quantum mechanics. The existence of minimal value for the product $\Delta x, \Delta \rho$ is a universal law of nature, but is considered as a property of the quantum objects themselves (it becomes a property of the measurement process because measurement apparatus are in part quantum). But it is remarkable that it may be established without any hint to any particular effective measurement (recall that it arises from the requirement that the momentum and position wave functions are reciprocal Fourier transforms) so we shall assume that the dependence of physics on resolution pre-exists any measurement and that actual measurements do nothing but reveal to us this universal property of nature then a natural achievement of the principle of scale relativity is to attribute universal property of scale dependence to space-time itself

- we finally arrive at the conclusion which is now reached by basing ourselves on the principle of scale relativity rather than on the extension of the principle of motion relativity to non-differentiable motion, namely the quantum space-time is scale-divergent, according to Heisenberg's relations by our definition fractal

- So we conclude that the resolutions are considered as a relative state of scale of the coordinate system, in the same way as velocity describes its state of motion, however according to the Einstein's principle of relativity we derive the principle of scale relativity «the haws of physics must apply to coordinate systems whatever their state of scale», and the principle of scale covariance. « The equations of physics keep the same form (are covariant) under any transformation of scale (contractions and dilatation) »
- from the principle of relativity of motion and the scale relativity ,we obtain the full principle of relativity which will need is the validity of the laws of physics in any coordinate system, whatever its state of motion and of scale
- in more detail we shall see that in this form the principles of scale relativity and scale covariance imply a modification of the structure of space-time at very-small scale in nature which is the fractal structure, then in this space-time structure we find a limiting scale, which is invariant under dilatation, as same as the velocity of light is constant in any coordinate system

So there is an impassable scale in nature plays for scale laws a role similar to that played by the velocity of light for motion laws.

Chapter 02

Fractal Dimension of a Quantum Path

Fractal Dimension of a Quantum Path

We know that the standard interpretation of quantum mechanics has completely abandoned the concept of trajectory by replacing it by the probability amplitude. However Feynman proved that the probability amplitude between two points is equal to integral over all possible paths of $\exp(iS_{cl})$, where S_{cl} is the classical action for each path. This approach is named path integral. The aim of this chapter is to show how Feynman used that approach to prove that the trajectories of the quantum particle are continuous and non-differentiable which means it is fractal. We will see also that accurate calculation of Abbot and Wise leads to fractal dimension D = 2 of quantum path which is a direct consequence of Heisenberg's incertitude's principle.

Path integral approach

Feynman used the path integral approach to understand the behavior of the quantum particle, and he arrived to the conclusion that the path of the quantum particle are highly irregular (as we see in the sketch), and that no mean square velocity exists at any point of the path which means that the paths are continuous and non – differentiable [4].
In other words we shall show in this chapter that the quantum path is fractal.

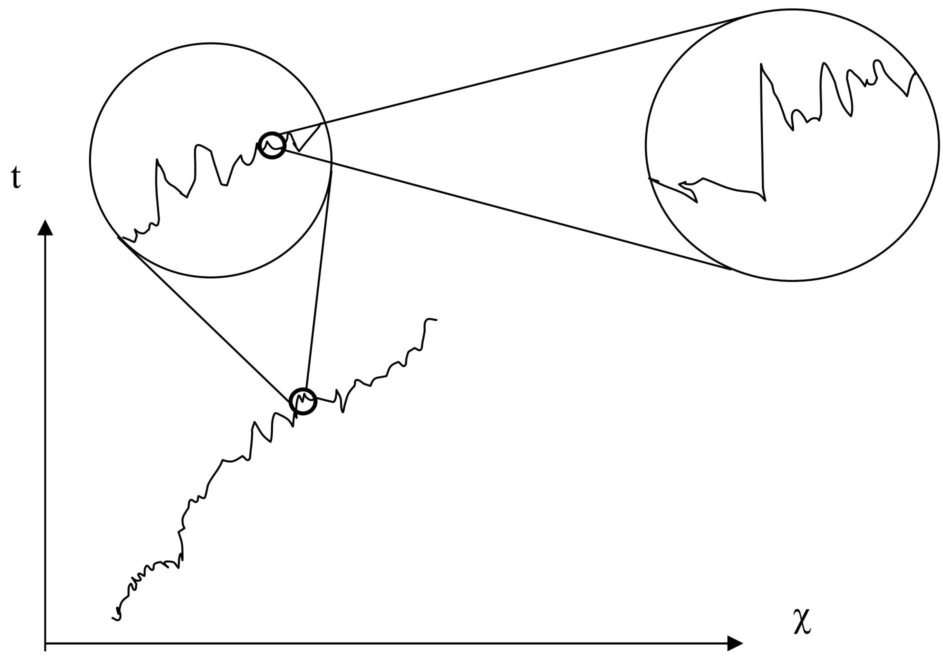


FIG (2-1) sketches the typical path of quantum-mechanical in space-time

We take the special case of one dimensional particle moving in a potential V [x (t)].
The action over the path of the particle is given by

$$S = \int_{t_i}^{t_f} \ell(x,\dot{x},t)dt \qquad (2\text{-}1)$$

where ℓ(x, x, t) is a Lagrangian, defined by

$$\ell = \frac{m\dot{x}^2(t)}{2} - V(x(t)) \qquad (2\text{-}2)$$

So

$$S = \int_{t_i}^{t_f} \left[\frac{m\dot{x}^2}{2} - V(x(t) \right] \qquad (2\text{-}3)$$

Feynman and Hibbs demonstrated the next relation (see Appendix B)

$$\left\langle \frac{\partial F}{\partial x_k} \right\rangle = -\frac{i}{\hbar} \left\langle F \frac{\partial S}{\partial x_k} \right\rangle \qquad (2\text{-}4)$$

where F(x(t)) is a function of x(t).
We divide the time into small intervals of length ε, hence the action S can be written as

$$S = \sum_{i+1}^{N-1} \left[m\frac{(x_{i+1}-x_i)^2}{2\varepsilon} - V(x_i)\varepsilon \right] \qquad (2\text{-}5)$$

when we derive the action with respect to the coordinates we find

$$\frac{\partial S}{\delta x_k} = m\left(\frac{x_{k+1}-x_k}{\varepsilon} - \frac{x_k - x_{k-1}}{\varepsilon} \right) + V'(x_k)\varepsilon \qquad (2\text{-}6)$$

$$\left\langle \frac{\partial F}{\partial x_k} \right\rangle_S = \frac{i\varepsilon}{\hbar} \left\langle F\left[m\frac{x_{k+1}-2x_k+x_{k-1}}{\varepsilon^2} + V'(x_k) \right] \right\rangle \qquad (2\text{-}7)$$

For the special case $F(x) = x_k$ we find

$$\langle 1 \rangle = \frac{i\varepsilon}{\hbar} \left\langle mx_k \left(\frac{x_{k+1}-x_k}{\varepsilon} - \frac{x_k - x_{k-1}}{\varepsilon} \right) + \varepsilon\, x_k V'(x_k) \right\rangle \qquad (2\text{-}8)$$

If we assume that the potential V is a smooth function, then in limit as $\varepsilon \to 0$ we find that $\varepsilon\, x_k V'(x_k)$ is negligible in comparison with the remaining terms, so the result becomes

$$\left\langle m\frac{x_{k+1}-x_k}{\varepsilon} x_k \right\rangle - \left\langle x_k m \frac{x_k - x_{k-1}}{\varepsilon} \right\rangle = \frac{\hbar}{i} \langle 1 \rangle \qquad (2\text{-}9)$$

So we have $\left\langle x_k m \frac{x_k - x_{k-1}}{\varepsilon} \right\rangle$ and $\left\langle x_{k+1} m \frac{x_{k+1} - x_k}{\varepsilon} \right\rangle$ which are two terms differing from each other only is order ε, since they represent the same quantity calculated, at two times differing by the interval ε .

We can substitute the first term into the second one, and we find

$$\left\langle \frac{(x_{k+1} - x_k)^2}{\varepsilon} \right\rangle = -\frac{\hbar}{im\varepsilon} \langle 1 \rangle \tag{2-10}$$

This equation means that the average of the square of the velocity is of the order $\frac{1}{\varepsilon}$, and thus becomes infinite as ε approaches zero. This result implies that the paths of quantum mechanical particle are irregular on a very fine scale, as indicated by fig (2-1). In other words, the paths are non differentiable.

For a short time interval Δt the average velocity is $\frac{[x(t+\Delta t) - x(t)]}{\Delta t}$. The mean square value of this velocity is $-\frac{\hbar}{im\Delta t}$ which is finite but its value becomes larger as the time interval becomes shorter

So

$$\left\langle \left(\frac{\Delta x}{\Delta t} \right)^2 \right\rangle \approx \frac{\hbar}{im\Delta t} \tag{2-11}$$

$$\left\langle (\Delta x)^2 \right\rangle \approx \frac{\hbar \Delta t}{m}$$

or

$$\langle |\Delta x| \rangle^2 \approx \frac{\hbar \Delta t}{m} \tag{2-12}$$

We define the mean length by < L > = N <|Δx| > and T=NΔt

So

$$\langle L \rangle = \frac{T}{\Delta t} \langle |\Delta x| \rangle ,$$

from equation (2-12) we find

$$\langle L \rangle \approx \frac{\hbar T}{m} \frac{(\langle \Delta x \rangle)}{\langle |\Delta x| \rangle^2} = \frac{\hbar T}{m \langle |\Delta x| \rangle}$$

Using the definition of the Hausdorff length

$$\langle L \rangle_{hass} = \langle L \rangle (\Delta x)^{D-1},$$

we find

$$\langle L \rangle_{hauss} = \frac{\hbar T}{m} (\Delta x)^{D-1}$$

for that $\langle L \rangle_{hauss}$ to be independent to the resolution Δx, it must have D=2.

This result means physically that although the particle path is one-dimensional curve, however with time this path will cover an area.

Heisenberg Uncertainty principle

For a quantum particle the position is known with precision Δx. We calculate a mean length of a trajectory which a particle travels during a time T

$$\langle L \rangle = N \langle \ell \rangle \tag{2-13}$$

$\langle \ell \rangle$ is the distance which a particle travels in a period of time Δ t

$$\langle \ell \rangle = \langle v \rangle \Delta t \tag{2-14}$$

so

$$\langle L \rangle = N \Delta t$$

According to the Heisenberg uncertainty principle we have

$$\Delta x \, \Delta p \approx \hbar$$

$$\Delta p \approx \frac{\hbar}{\Delta x}$$

$$\Delta v \approx \frac{\hbar}{m \Delta x}$$

With the assumption that $\langle v \rangle \approx \Delta v$,

we find
$$\langle L \rangle = N \Delta t \frac{\hbar}{m \Delta x} = \frac{\hbar T}{m \Delta x}$$

This expression is an average of length measured with a resolution Δx.
Using the definition of Hausdorff length

$$\langle L \rangle_{hauss} = \langle L \rangle (\Delta x)^{D-1}$$

we obtain

$$\langle L \rangle_{hauss} = \frac{\hbar T}{m \Delta x} (\Delta x)^{D-1}$$

$$\langle L \rangle_{hauss} = \frac{\hbar T}{m} (\Delta x)^{D-2}$$

The dimension of trajectory must be equal 2 to make $\langle L \rangle_{hauss}$ independent of the resolution.

Hence the Hausdorff dimension of the path of a quantum particle is equal to 2.

Now let us study the classical particle path, in this case $\langle L \rangle$ is independent from the resolution (Δx)

so

$$\langle L \rangle_{hauss} = \langle L \rangle (\Delta x)^{D-1}$$

We see that D must be equal 1 to make $\langle L \rangle_{hauss}$ independent of the resolution.

The dimension of quantum trajectory of free particle is $D = 2$, however a classical trajectory has a dimension $D = 1$ [3,5].

Abbot and Wise Work

Abbot and Wise showed that the observed path of a particle in quantum mechanics is a fractal curve with Hausdorff dimension equal to 2 [5,6].

We consider the wave function expression of a free particle which is localized in region of length Δx at time $t = 0$

$$\psi_{\Delta x}(\vec{x}, t=0) = \frac{(\Delta x)^{3/2}}{\hbar^3} \int_{R^3} \frac{d^3 p}{(2\pi)^{3/2}} f\left(\frac{|\vec{p}|\Delta x}{\hbar}\right) e^{\frac{i\vec{p}\,\vec{x}}{\hbar}} \tag{2-15}$$

This packet of wave is obtained by superposition of plan wave $e^{\frac{i\vec{p}\,\vec{x}}{\hbar} - \omega t}$ with coefficient

$$\frac{(\Delta x)^{3/2}}{(2\pi)^{3/2}\hbar^3} f\left(\frac{|\vec{p}|\Delta x}{\hbar}\right)$$

where

$$\omega = \frac{\hbar k^2}{2m}$$

The wave function at the time Δt is given by,

$$\psi_{\Delta x}(\vec{x}, \Delta t) = \frac{(\Delta x)^{3/2}}{\hbar^3} \int_{R^3} \frac{d^3 p}{(2\pi)^{3/2}} f\left(\frac{|\vec{p}|\Delta x}{\hbar}\right) e^{\frac{i\vec{p}\,\vec{x}}{\hbar} - \frac{i|p|^2 \Delta t}{2m\hbar}} \tag{2-16}$$

The normalization condition requires that $f\left(\frac{|\vec{p}|\Delta x}{\hbar}\right)$ has to satisfy the following condition

$$\int dx^2 \left| f\left(\left|\vec{K}\right|\right)\right|^2 = 1 \tag{2-17}$$

where

$$K = \frac{|\vec{p}|\Delta x}{\hbar}$$

We can choose

$$f\left(\left|\vec{K}\right|\right) = \left(\frac{2}{\pi}\right)^{3/2} e^{-\left|\vec{K}\right|^2} \tag{2-18}$$

Indeed we have

$$\int d^3k \left[\left(\frac{2}{\pi}\right)^{3/2} e^{-\left|\vec{k}\right|^2}\right]^2 = \left(\frac{2}{\pi}\right)^{3/2} \left(\sqrt{\frac{\pi}{2}}\right)^3 = 1 \tag{2-19}$$

We define the mean of the distance which a particle travels in time Δt by

$$\langle \Delta\ell \rangle = \int d^3x |\vec{x}| |\psi|^2 \tag{2-20}$$

Let as

$$y = \frac{\vec{x}}{\Delta x}$$

then

$$|\psi|^2 = (\Delta x)^{-3} \left| \int_{R^3} \frac{d^3k}{(2\pi)^{3/2}} f\left(\left|\vec{k}\right|\right) e^{\frac{-i\hbar\Delta t}{2m(\Delta x)^2}k^2 + i\vec{k}\,\vec{y}} \right|^2 \tag{2-21}$$

We set

$$F\left(y, \frac{\hbar\Delta t}{2m(\Delta x)^2}\right) = \int_{R^3} \frac{d^3k}{(2\pi)^{3/2}} f\left(\left|\vec{k}\right|\right) e^{-\frac{i\hbar\Delta t}{2m(\Delta x)^2}k^2 + i\vec{k}\,\vec{y}} \tag{2-22}$$

by substituting in (2-20) we find

$$\langle \Delta\ell \rangle = \Delta x \left[\int_{R^3} d^3y |\vec{y}| \left|F(\vec{y}, b)\right|^2 \right] \tag{2-23}$$

with

$$b = \frac{\hbar\Delta t}{2m(\Delta x)^2} \tag{2-24}$$

We calculate $F(\vec{y}, b)$ in the equation (2-22) by using the equation (2-18)

$$F(\vec{y},b)=\frac{1}{(2\pi)^{3/2}}\left(\frac{2}{\pi}\right)^{3/4}\int_{R^3} d^3k\, e^{i\vec{k}\,\vec{y}-k^2(1+ib)} \qquad (2\text{-}25)$$

$$\left|F(\vec{y},b)\right|^2=(2\pi)^{-\frac{3}{2}}\left(1+b^2\right)^{-\frac{3}{2}}e^{-\frac{|\vec{y}|^2}{2(1+b^2)}} \qquad (2\text{-}26)$$

We substitute in (2-23) to obtain

$$\left\langle \Delta\ell\right\rangle=\Delta x\int d^3y\left|\vec{y}\right|(2\pi)^{-\frac{3}{2}}\left(1+b^2\right)^{-\frac{3}{2}}e^{-\frac{|\vec{y}|^2}{2(1+b^2)}} \qquad (2\text{-}27)$$

By using the integrals

$$\int d^3y=4\pi\int dy\left|\vec{y}\right|^2$$

$$\int_0^\infty e^{-\lambda x^m}x^k dx=\frac{1}{m}\lambda^{-(k+1)/m}\Gamma\left(\frac{k+1}{m}\right)$$

We find

$$\left\langle \Delta L\right\rangle\propto\Delta x\sqrt{1+\frac{\hbar^2(\Delta t)^2}{4m^2(\Delta x)^4}} \qquad (2\text{-}28)$$

or

$$\left\langle \Delta L\right\rangle\propto\frac{\hbar\Delta t}{2m\Delta x}\sqrt{1+\left(\frac{2m(\Delta x)^2}{\hbar\Delta t}\right)} \qquad (2\text{-}29)$$

Let us now study the equation (2-29) when $\Delta x \prec\prec \sqrt{\frac{\hbar\Delta t}{2m}}$, it reduces to

$$\left\langle \Delta L\right\rangle\propto\frac{\hbar\Delta t}{m\Delta x}$$

Since we have

$$\left\langle L\right\rangle_{hauss}=\left\langle L\right\rangle(\Delta x)^{D-1}$$

with

$$\left\langle L\right\rangle=N\left\langle \ell\right\rangle$$

we obtain

$$\left\langle L\right\rangle_{hauss}\propto\frac{\hbar T}{m}(\Delta x)^{D-2}$$

which gives the condition $D=2$ to make $\left\langle L\right\rangle_{hauss}$ independent to Δx.

Transition from classical dimension D=1 to quantum dimension D=2

We consider now the case where the particle has an average momentum P_{av}. The wave function of that particle is

$$\psi_{\Delta x}\left(\vec{x},\Delta t\right)=\frac{\left(\Delta x\right)^{3/2}}{\hbar^3}\int\frac{d^3p}{\left(2\pi\right)^{3/2}}f\left(\frac{\left|\vec{p}\right|\Delta x}{\hbar}\right)e^{\frac{i(\vec{p}+\vec{p}_{moy})}{\hbar}\vec{x}-\frac{i(\vec{p}+\vec{p}_{moy})^2}{2m\hbar}}$$

We set

$$\vec{y}=\frac{\vec{x}}{\Delta x}$$

$$\vec{k}=\frac{\vec{p}\Delta x}{\hbar}$$

After some simplifications we find

$$\psi_{\Delta x}\left(\vec{x},\Delta t\right)=\left(\Delta x\right)^{-\frac{3}{2}}e^{\frac{i}{\hbar}\vec{p}_{moy}\Delta x\left(\vec{y}-\frac{\vec{p}_{moy}\Delta x}{2m\Delta x}\right)}\int_{R^3}\frac{d^3k}{\left(2\pi\right)^{3/2}}f\left(\left|\vec{k}\right|\right)e^{i\left(\vec{y}-\frac{\vec{p}_{m}}{m}\cdots\right.\cdots)^2}k^2}$$

Using the change of variable

$$\vec{y}-\frac{\vec{p}_{moy}\Delta t}{m\Delta x}\rightarrow\vec{y}\qquad\text{(2-30)}$$

We find

$$\psi_{\Delta x}\left(\vec{x},\Delta t\right)=\left(\Delta x\right)^{-\frac{3}{2}}e^{\frac{i}{\hbar}\vec{p}_{moy}\Delta x\left(\vec{y}-\frac{\vec{p}_{moy}\Delta x}{2m\Delta x}\right)}\int_{R^3}\frac{d^3k}{\left(2\pi\right)^{3/2}}f\left(\left|\vec{k}\right|\right)e^{i\vec{y}\vec{k}-\frac{i\hbar\Delta t}{2m(\Delta x)^2}k^2}\qquad\text{(2-31)}$$

Using the expression of $F\left(\vec{y},b\right)$ with $b=\frac{\hbar\Delta t}{2m(\Delta x)^2}$ yields to the expression

$$\psi_{\Delta x}\left(\vec{x},\Delta t\right)=\left(\Delta x\right)^{-\frac{3}{2}}e^{\frac{i}{\hbar}\vec{p}_{moy}\Delta\vec{x}\left(\vec{y}+\frac{\vec{p}_{moy}\Delta t}{m\Delta x}\right)}F\left(\vec{y},b\right)$$

The average distance $\left\langle\Delta L\right\rangle$ which is traveled by the particle during the time Δt is given by

$$\left\langle\Delta L\right\rangle\propto\frac{\hbar\Delta t}{m\Delta x}$$

where

$$b = \frac{\hbar \Delta t}{m(\Delta x)^2} = \text{constant}$$

By using (2-20) and (2-30) we can write

$$\langle \Delta L \rangle = (\Delta x)^4 \int_{R^3} d^3 y \left| \vec{y} + \frac{\vec{p}_{moy} \Delta t}{m \Delta x} \right| |\psi|^2$$

with

$$|\psi|^2 = (\Delta x)^{-3} |F(\vec{y}, b)|^2$$

which gives

$$\langle \Delta L \rangle = \frac{|\vec{p}_{moy}| \Delta t}{m} \int_{R^3} d^3 y \left| \frac{\vec{p}_{moy}}{|\vec{p}_{moy}|} + \frac{m \Delta x}{|\vec{p}_{moy}| \Delta t} \vec{y} \right| |F(\vec{y}, b)|^2$$

or

$$\langle \Delta L \rangle = \frac{|\vec{p}_{moy}| \Delta t}{m} \int_{R^3} d^3 y \left| \frac{\vec{p}_{moy}}{|\vec{p}_{moy}|} + \frac{\hbar}{|\vec{p}_{moy}| \Delta x b} \vec{y} \right| |F(\vec{y}, b)|^2$$

We remind that the Hausdorff length is given by

$$\langle L \rangle_{hauss} = N \langle \Delta L \rangle (\Delta x)^{D-1}$$

Then

$$\langle L \rangle_{hauss} = \frac{|\vec{p}_{moy}| T}{m} \int_{R^3} d^3 y \left| \frac{\vec{p}_{moy}}{|\vec{p}_{moy}|} + \frac{\hbar}{|\vec{p}_{moy}| \Delta x b} \vec{y} \right| |F(\vec{y}, b)|^2 (\Delta x)^{D-1}$$

We have two cases for that the Hausdorff length be independent of Δx

$$D = 1 \quad \text{for} \frac{\hbar}{|\vec{p}_{moy}| \Delta x} \prec\prec 1 \Rightarrow \Delta x \succ\succ \frac{\hbar}{|\vec{p}_{moy}|}$$

is the classical case when the resolution is larger than the quantity $\frac{\hbar}{|\vec{p}_{moy}|}$ which is the particle's wavelength given by the Broglie relation.

$$D = 2 \ \text{for} \ \frac{\hbar}{|\vec{p}_{moy}| \Delta x} \succ\succ 1 \Rightarrow \Delta x \prec\prec \frac{\hbar}{|\vec{p}_{moy}|}$$

In the quantum case when the resolution is smaller than the particle's wavelength.

Conclusion

In this chapter we have calculated the path's dimension of a classical and quantum particle using three methods. The first one consists in the use of the path integral formalism. Following the work of Feynman and Hibbs we have shown that the particle path in quantum mechanics can be described as a continuous and non-differentiable curve. This non-differentiability is one of the properties of fractals. The second method is the use of the Heisenberg uncertainty principle and finally the third method which involves a more accurate calculation is due to Abbot and Wise.

We have shown that the fractal dimension of the quantum path is equal to 2 which means that the "trajectory" of the particle tends to occupy a surface. We have also shown that there is a transition in the Hausdorff dimension from 2 to 1. This transition takes place at the Compton wavelength scale.

Chapter 3

Derivation of Schrödinger Equation from Newtonian Mechanics

EDWARD NELSON Work

Derivation of the Schrödinger Equation from Newtonian Mechanics

In this chapter, we want to show how we can derive the Schrödinger equation without using the quantum axioms. This derivation is based on statistical mechanics and the theory of the Brownian motion [8,9].

Stochastic mechanics

In the stochastic mechanics any particle of mass m is subject to a Brownian motion with diffusion coefficient $\frac{\hbar}{2m}$ and no friction. and they define the Brownian motion with the following properties:

1. The motion is highly irregular and unpredictable which means that we can not draw the tangents of the trajectories.
2. The motion is independent of the particle's nature.
3. The motion is continuous.

In 1905 Einstein adopted a probabilistic description of the Brownian trajectories, and he found that the density of the probability to find a Brownian particle in x at time t satisfies the equation of diffusion

$$\frac{\partial}{\partial t}P(x,t) = \upsilon \Delta P(x,t)$$

with

$$\upsilon = \frac{k_B T}{m\gamma} = \frac{\hbar}{2m}$$

In 1906 Smoluchowski derived the equation which describes the Brownian particle in field of forces $F(x)$

$$\frac{\partial}{\partial t}p(x,t) = -\frac{1}{m\gamma}\frac{\partial}{\partial x}\left(F(x)p(x,t)\right) + \upsilon\frac{\partial^2}{\partial t^2}p(x,t)$$

Any process in time evolution which can be analyzed by the formalism of probability is called stochastic process. We define the absolute probability $W(x,t)$ which satisfies some of properties; however the stochastic process is governed by the conditional probability

$$P(x,t_1 / y,t_2) = \frac{W(x,t_1)}{W(y,t_2)}$$

We call the stochastic process a Markoff process if the conditional probability has the following property

$$\forall t_1 \prec t_2 \prec t_3 \quad P(x_1,t_1;x_2,t_2;.....;x_{n-1},t_{n-1} / x_n,t_n) = P(x_{n-1},t_{n-1} / x_n,t_n)$$

We can say that in the Markoff process the future is independent from the history of the system.

The Fokker-Planck Equation

$$\frac{\partial}{\partial t} p(x_0 / x,t) = -\frac{\partial}{\partial t}\left(a(x)p(x_0 / x,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(b(x)p(x_0 / x,t\right)$$

$a(x)$ is the derive function and $b(x)$ is the function of diffusion

The Wiener and the Ornstein-Uhlenbeck process are considered as a particular case of the Fokker-Planck equation with a certain definition of a(x) and b(x).

For instance we obtain the equation of the Wiener process for

$$a = 0, b = 2\upsilon$$

$$\upsilon = \frac{1}{\beta m \gamma} = \frac{\hbar}{2m}$$

$x(t)$ is a stochastic process when it is not differentiable (case of the Wiener process, in Einstein's theory of Brownian motion), we define the two kind of derivative

The forward derivative $Dx(t)$

$$Dx(t) = \lim_{\Delta t \to 0^+} E_t \frac{x(t+\Delta t) - x(t)}{\Delta t} \tag{3-1}$$

The backward derivative $D_* x(t)$

$$D_* x(t) = \lim_{\Delta t \to 0^+} E_t \frac{x(t) - x(t-\Delta t)}{\Delta t} \tag{3-2}$$

E_t indicates the conditional expectation which given the state of the system at time t.

When $x(t)$ is differentiable, then

$$D_* x(t) = Dx(t) = \frac{dx}{dt}$$

The Ornstein-Uhlenbeck process is obtained for

$$a(v) = -\gamma v,\ b(v) = \frac{2\gamma}{\beta m} = 2\gamma^2 \upsilon$$

in the Fokker-Planck equation of Brownian motion with the presence of a potential V. So the particle acquires an acceleration produced by V given by:

$$K = -\frac{1}{m} gradV$$

Now if we use a Maxwell-Boltzmann distribution of velocity,

$$W(v) = \sqrt{\frac{m\beta}{2\pi}} e^{-\beta \frac{1}{2} mv^2}$$

When we compute the fluctuation of velocity, with an initial condition (v_0, t_0) we find

$$\langle v(t) \rangle = \int_{\Re} dvvW(v) = 0$$

$$\langle v^2 \rangle_{v_0 t_0}(t) = \frac{1}{\beta m}\left(1 - e^{-2(t-t_0)}\right) + v_0^2 e^{-2\gamma(t-t_0)}$$

From the next Ornstein-Uhlenbeck equation of velocity

$$\frac{\partial}{\partial t} P(v,t) = \gamma \frac{\partial}{\partial v}\left(vP(x,t)\right) + \gamma^2 \upsilon \frac{\partial^2}{\partial v^2} P(v,t)$$

The same results are obtained by the Langevin theory

So the system is in equilibrium. We can write the Langevin equations, where $m\beta$ is the friction coefficient

$$dx(t) = v(t)dt \tag{3-3}$$

$$dv(t) = -\beta\, v(t)dt + K\left(x(t)\right)dt + dB(t) \tag{3-4}$$

B is a (white noise) Wiener process representing the residual random impacts, $dB(t)$ are Gaussian with mean 0, and

$$EdB(t)^2 = 6\left(\frac{BkT}{m}\right)dt \tag{3-5}$$

k is the Boltzmann constant, and T is the absolute temperature. dB(t) are independent of x(s), v(s) with $s \le t$, it dependent only of x(s) and v(s) with s > t.

There is an asymmetry in time so we may write

$$dv(t) = -\beta\, v(t)dt + K\left(x(t)\right)dt + dB_*(t) \tag{3-6}$$

In this case dB*(t) is independent of x(s) , v(s) with $s \ge t$

so we calculate the average in (3-1) and in (3-2)

$$Dx(t) = \lim_{\Delta t \to 0^+} E_t \frac{x(t+\Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0^+} E_t \frac{x(t) - x(t-\Delta t)}{\Delta t} = D_* x(t)$$

$$= \frac{dx}{dt} = v(t)$$

Since $x(t)$ is differentiable.

By (3-5) $dB(t)^2$ is of the order $dt^{1/2}$ that means B (t) and v (t) are not differentiable

So $$Dv(t) = -\beta v(t) + K(x(t)) \quad \text{and} \quad D_* v(t) = \beta v(t) + K(x(t))$$

Hence

$$Dv(t) + D_* v(t) = 2K(x(t))$$

and

$$\frac{1}{2} DD_* x(t) + \frac{1}{2} D_* Dx(t) = K(x(t))$$

We define the second derivative of a stochastic process by

$$a(t) = \frac{1}{2} DD_* x(t) + \frac{1}{2} D_* Dx(t) = K(x(t)) = -\frac{1}{m} grad\ V$$

If $x(t)$ is a position, $a(t)$ is defined as an acceleration and $F = ma$ which is the Newton's law.

Kinematics of Markoff Processes

We describe the macroscopic Brownian motion of a free particle moving in a fluid by the Wiener process $w(t)$, which satisfies;

$$Edw_i(t)dw_j(j) = 2\upsilon \delta_{ij} dt \tag{3-7}$$

Where $\upsilon = \dfrac{kT}{m\beta}$ is the diffusion coefficient.

In the fluid where the particle is moving, if there are external forces or currents, the position $x(t)$ of the Brownian particle be decomposed as following

$$dx(t) = b(x(t),t)dt + dw(t) \tag{3-8}$$

b is a vector valued function on spacetime.

Because the Wiener process is a Markoff process $dw(t)$ are independent of $x(s)$ with $s \le t$

So from (3-1) and (3-8) we find

$$Dx(t) = \lim_{\Delta t \to 0+} \frac{x(t + \Delta t) - x(t)}{\Delta t} = b(x(t),t) \tag{3-9}$$

b is the mean forward velocity.

Where we have considered the time t, and $t \le s$, we have an asymmetry in time. So we can write

$$dx(t) = b_*(x(t),t)dt + dw_* \,\mathrm{dx(t)} \tag{3-10}$$

$dw_*(t)$ are independent of $x(s)$ with $s \ge t$

So by (3-1) and (3- 10) we find

$$D_* x(t) = \lim_{\Delta t \to 0+} \frac{x(t) - x(t - \Delta t)}{\Delta t} = b_*\left(x(t), t\right) \qquad (3\text{-}11)$$

b_* is the mean backward velocity.

We assume a motion of an electron with $P\left(x(t), t\right)$ is the probability of position x(t) satisfies the forward Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -div(bP) + \upsilon \Delta P \qquad (3\text{-}12)$$

and the backward Fokker -Planck equation

$$\frac{\partial P}{\partial t} = -div(b_* P) - \upsilon \Delta P \qquad (3\text{-}13)$$

When we add (3-12) to (3-13) we find

$$\frac{\partial P}{\partial t} = -\frac{1}{2}(div(bP) + div(b_* P))$$

$$\frac{\partial P}{\partial t} = -div(\frac{1}{2}(b + b_*)P$$

$$\frac{\partial P}{\partial t} = -div(vP) \qquad (3\text{-}14)$$

which is the equation of continuity, where we have define v by

$$v = \frac{1}{2}\left(b + b_*\right) \qquad (3\text{-}15)$$

We call v the current velocity.

We can expand the function f in Taylor series as

$$\frac{df\left(x(t), t\right)}{dt} = \frac{\partial f}{\partial t}\left(x(t), t\right) + \frac{dx(t)}{dt} \nabla f\left(x(t), t\right) + \frac{1}{2} \sum_{ij} \frac{dx_i(t) dx_j(t)}{dt^2} \frac{\partial^2 f}{\partial x_i \partial x_j}\left(x(t), t\right) \qquad (3\text{-}16)$$

$$df\left(x(t), t\right) = \frac{\partial f}{\partial t}\left(x(t), t\right)dt + dx(t) \nabla f\left(x(t), t\right) + \frac{1}{2} \sum_{ij} \frac{dx_i(t) dx_j(t)}{dt} \frac{\partial^2 f}{\partial x_i \partial x_j}\left(x(t), t\right) \qquad (3\text{-}17)$$

We take (3-16) and we calculate Df and $D_* f$

$$Df\left(x(t),t\right)=\frac{\partial f}{\partial t}\left(x(t),t\right)\frac{dt}{dt}+\left(b\left(x(t),t\right)\frac{dt}{dt}+\frac{dw(t)}{dt}\right)\nabla f\left(x(t),t\right)$$

$$+\frac{1}{2}\sum_{ij}\frac{b_i\left(x(t),t\right)b_j\left(x(t),t\right)}{dt}\frac{\partial^2 f}{\partial x_i \partial x_j}\left(x(t),t\right)$$

$$+\frac{1}{2}\sum_{ij}\frac{dw_i(t)dw_j(t)}{dt}\frac{\partial^2 f}{\partial x_i \partial x_j}\left(x(t),t\right)$$

$$+\frac{1}{2}\sum_{ij}b_i\left(x(t),t\right)dw_j(t)\frac{\partial^2 f}{\partial x_i \partial x_j}\left(x(t),t\right)$$

$$+\frac{1}{2}\sum_{ij}dw_i(t)b_j\left(x(t),t\right)dw_j(t)\frac{\partial^2 f}{\partial x_i \partial x_j}\left(x(t),t\right)$$

Using (3-7) we obtain

$$Df\left(x(t),t\right)=\left(\frac{\partial}{\partial t}+b\nabla+\nu\Delta\right)f\left(x(t),t\right) \qquad (3\text{-}18)$$

In the same way we obtain

$$D_* f\left(x(t),t\right)=\left(\frac{\partial}{\partial t}+b_*\nabla-\nu\Delta\right)f\left(x(t),t\right) \qquad (3\text{-}19)$$

$\left(\frac{\partial}{\partial t}+b\nabla+\nu\Delta\right)$ and $\left(\frac{\partial}{\partial t}+b_*\nabla-\nu\Delta\right)$ are adjoints to each other with respect to ρd^3xdt ,

that is

$$\rho^{-1}\left(\frac{\partial}{\partial t}+b\nabla+\nu\Delta\right)^{+}\rho=-\frac{\partial}{\partial t}-b_*\nabla+\nu\Delta \qquad (3\text{-}20)$$

Where the superscript + denotes the Lagrange adjoint (with respect to ρd^3xdt)

$$\frac{\partial}{\partial t}\left(\rho^{-1}\right)=-\rho^{-2}\frac{\partial}{\partial t}\rho$$

$$\nabla\left(\rho^{-1}\right)=-\rho^{-2}\nabla\rho$$

$$\Delta\left(\rho^{-1}\right)=\nabla\left(\nabla\rho^{-1}\right)$$

$$=\nabla\left(-\rho^{-2}\nabla\rho\right)$$

$$=\nabla\left(-\rho^{-2}\right)\nabla\rho-\rho^{-2}\Delta\rho$$

$$=2\rho^{-3}\nabla\rho\nabla\rho-\rho^{-2}\Delta\rho$$

$$=2\rho^{-3}(\nabla\rho)^2-\rho^{-2}\Delta\rho$$

$$\left[\rho^{-1}\left(\frac{\partial}{\partial t}+b\nabla+\nu\Delta\right)^{+}\rho\right]\rho=-\frac{\partial}{\partial t}\rho-b\nabla\rho+2\vartheta\rho^{-1}(\nabla\rho)^2-\upsilon\Delta\rho$$

$$= -\frac{\partial}{\partial t}\rho - b_* \nabla \rho + \upsilon \Delta \rho$$

So
$$-b\nabla\rho + 2\vartheta\,\rho^{-1}(\nabla\rho)^2 - \upsilon\,\Delta\rho = -b_*\nabla\rho + \upsilon\,\Delta\rho$$

$$-b_*\nabla\rho = -b\nabla\rho - 2\,\upsilon\Delta\rho + 2\upsilon\,\rho^{-1}(\nabla\rho)^2$$

While

$$\oint \Delta\rho\, d^3x dt = 0$$

So
$$b_* = -b - 2\upsilon\frac{\Delta\rho}{\rho}$$

$$b_* = b - 2\upsilon\left(\frac{grad\ \rho}{\rho}\right) \quad (3\text{-}21)$$

Or

$$u = \upsilon\left(\frac{grad\ \rho}{\rho}\right) \quad (3\text{-}22)$$

Where we have defined

$$u = \frac{1}{2}\left(b - b_*\right) \quad (3\text{-}23)$$

According to Einstein's theory of Brownian motion, the eq. (3-23) is the velocity acquired by a Brownian particle in equilibrium with respect to an external force.

By subtracting (3-12) from (3-13) we obtain

$$\frac{\partial\rho}{\partial t} - \frac{\partial\rho}{\partial t} = div(b\rho) - \upsilon\,\Delta\rho - div(b_*\rho) - \upsilon\,\Delta\rho$$

$$0 = 2div\left(\frac{1}{2}(b_* - b)\rho\right) - 2\upsilon\,\Delta\rho$$

$$0 = div(u\rho) - \upsilon\,\Delta\rho \quad (3\text{-}24)$$

$$0 = div\left[u\rho - \upsilon\, grad\ \rho\right] \quad (3\text{-}25)$$

From (3-23) we have

$$u = \upsilon\, grad\ \ell n\rho \quad (3\text{-}26)$$

Using the equation of continuity, we may compute $\frac{\partial u}{\partial t}$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[\upsilon\, grad\ \ell n\,\rho\right]$$

$$= \upsilon\ grad\left(\frac{\partial}{\partial t}\ell n\,\rho\right)$$

$$= \upsilon \; grad\left(\frac{1}{\rho}\frac{\partial \rho}{\partial t}\right)$$

From the equation of continuity, we have

$$\frac{\partial u}{\partial t} = \upsilon \; grad\left(-div(v\rho)\frac{1}{\rho}\right)$$

$$0 = \upsilon \; grad\left(-div \; v\rho\frac{1}{\rho} - v\frac{1}{\rho}grad \; \rho\right)$$

$$0 = -\upsilon \; grad(div \; v) - grad\left(\upsilon \; v\frac{grad \; \rho}{\rho}\right)$$

From (3-26) we obtain

$$\frac{\partial u}{\partial t} = -\upsilon \; grad(div \; v) - grad(v \; u) \tag{3-27}$$

When we apply (3-18) to b_* and (3-19) to b, we find

$$D b_*(x(t),t) = \left(\frac{\partial}{\partial t} + b\nabla + \upsilon \Delta\right) b_*(x(t),t)$$

$$D_* \, b(x(t),t) = \left(\frac{\partial}{\partial t} + b_*\nabla - \upsilon \Delta\right) b(x(t),t)$$

From (3-10) and (3-11) we obtain

$$DD_* x(t) = \left(\frac{\partial}{\partial t} + b\nabla + \upsilon \Delta\right) b_*(x(t),t) \tag{3-28}$$

$$D_* D x(t) = \left(\frac{\partial}{\partial t} + b_*\nabla - \upsilon \Delta\right) b(x(t),t) \tag{3-29}$$

We add (3-28) and (3-29) and multiply the addition by $\frac{1}{2}$

$$\frac{1}{2}DDx(t) + \frac{1}{2}DDx(t) = \frac{1}{2}\frac{\partial}{\partial t}(b + b_*) + \frac{1}{2}(b_*\nabla)b + \frac{1}{2}(b\nabla)b_* - \frac{1}{2}\upsilon\Delta(b - b_*) \tag{3-30}$$

which is the mean acceleration a.

So

$$a = \frac{1}{2}\frac{\partial}{\partial t}(b + b_*) + \frac{1}{2}(b_*\nabla)b + \frac{1}{2}(b\nabla)b_* - \frac{1}{2}\upsilon\Delta(b - b_*) \tag{3-31}$$

From (3-15) and (3-24) we have

$$v = \frac{1}{2}(b + b_*)$$

$$u = \frac{1}{2}(b - b_*)$$

so $b = v + u$ and $b_* = v - u$

Thus (3-31) is equivalent to

$$\frac{\partial v}{\partial t} = a - (v\nabla)v + (u\nabla)u + \upsilon \Delta u \qquad (3\text{-}32)$$

The Hypothesis of Universal Brownian Motion

We consider that the particles move in an empty space, and are subject to a macroscopic Brownian motion with diffusion coefficient υ

$$\upsilon = \frac{\hbar}{2m} \qquad (3\text{-}33)$$

We have not any friction to empty space, this means that the velocities will not exist, hence we cannot describe the state of a particle by a point in phase space as in the Einstein-Smoluchowski theory, and the motion will be described by a Markoff process in coordinate space.

The mean acceleration (a) has no dynamical significance in the Einstein- Smoluchowski theory, that theory applies in the limit of large friction, so that an external force F does not accelerate a particle but merely imparts a velocity $\frac{F}{m\beta}$ to it, in other words to study Brownian motion in a medium with zero Smoluchowski theory, but use Newtonian dynamics as in the Ornstein Uhlenbeck theory.

We consider a particle of mass m, in an external force F, the particle performs a Markoff process, we substitute $a = \frac{F}{m}$ and $\upsilon = \frac{\hbar}{2m}$ in equation (3-27) and (3-32) thus u and v satisfy

$$\frac{\partial u}{\partial t} = -\left(\frac{\hbar}{2m}\right) grad(div\ v) - grad(v u) \qquad (3\text{-}34)$$

$$\frac{\partial v}{\partial t} = \left(\frac{1}{m}\right) F - (v\nabla v) + (u\nabla u) + \left(\frac{\hbar}{2m}\right)\Delta u \qquad (3\text{-}35)$$

consequently, if $u(x, t_0)$ and $v(x, t_0)$ are known and we can solve the problem of the coupled nonlinear partial differential equation (3-27) and (3-28) then the Markoff process will be completely known, thus the state of a particle at time t_0 is described y its position $x(t_0)$ at time t_0.

The velocities u and v at time t_0 notice that $u(x,t_0)$ and $v(x,t_0)$ must be given for all values of x and not just for $x(t_0)$.

The Real Time-independent Schrödinger Equation

We consider the case that the force comes from a potential

$$F = -grad\ V \tag{3-36}$$

Suppose first that the current velocity $v = 0$, from the equation of continuity and (3-22) we obtain

$$\frac{\partial \rho}{\partial t} = 0$$

$$u = \upsilon \frac{grad\ \rho}{\rho}$$

It seems that ρ and u are independent of the time t, so (3-34) and (3-35) become

$$\frac{\partial u}{\partial t} = 0 \tag{3-37}$$

$$u\nabla u + \left(\frac{\hbar}{2m}\right)\Delta u = \left(\frac{1}{m}\right) grad\ V \tag{3-38}$$

by (3-26), u is a gradient, so that we can write

$$(u\nabla)u = \frac{1}{2} grad\ u^2 \text{ and } \Delta u = grad(div\ u) \tag{3-39}$$

So (3-24) becomes

$$grad\left(\frac{1}{2}u^2 + \frac{\hbar}{2m} div\ u\right) = \frac{1}{m} grad\ V \tag{3-40}$$

$$\frac{1}{2}u^2 + \frac{\hbar}{2m} div\ u = \frac{1}{m}V - \frac{1}{m}E \tag{3-41}$$

where E is a constant of integration with the dimensions of energy

If we multiply by $m\rho$ and integrate, after use $u = \upsilon\left(\frac{grad\ \rho}{\rho}\right)$ we obtain

$$\int \frac{1}{2} mu^2 \rho\, d^3x - \frac{\hbar}{2}\int (u\ grad\ \rho) d^3x = \int V \rho\, d^3x - E \tag{3-42}$$

From the last equation of u we obtain

$$u\ grad\ \rho = u\,\rho\,\upsilon^{-1}u = \upsilon u^2\rho = \left(\frac{\hbar}{2m}\right)^{-1} u^2\rho = -mu^2\rho \tag{3-43}$$

So (3-42) becomes

$$E = \int \frac{1}{2} mu^2 \rho\, d^3x + \int V\rho\, d^3x \tag{3-44}$$

E is the average value of $\frac{1}{2}mu^2 + V$ and may be interpreted as the mean energy of the particle.

The equation (3-41) is nonlinear, but it is equivalent to a linear equation by a change of dependent variable, by (3-26) $u = \upsilon\ grad\ \ell n\ \rho$, we put

$$R = \frac{1}{2}\ell n\ \rho \tag{3-45}$$

Eq (3-26) becomes $u = \upsilon\, grad\ 2R$ and we have $\upsilon = \frac{\hbar}{2m}$

$$\frac{m}{\hbar}u = grad\ R \tag{3-46}$$

So R is the potential of $\frac{m}{\hbar}u$ let as write

$$\psi = e^R \tag{3-47}$$

Then ψ is real and $\rho = \psi^2$

$$\psi = e^R = e^{\frac{1}{2}\ell n\,\rho} = \left(e^{\ell n\,\rho}\right)^{\frac{1}{2}} = \rho^{\frac{1}{2}} \Rightarrow \rho = \psi^2$$

From (3-22) we have

$$u = \upsilon\,\frac{grad\ \rho}{\rho} = \frac{\hbar}{2m} grad\ \rho \frac{1}{\rho}$$

$$div\ u = \frac{\hbar}{2m}\,div\left(grad\ \rho\,\frac{1}{\rho}\right) = \frac{\hbar}{2m}\frac{(grad)^2\rho}{\rho^2} - \frac{\hbar}{2m}\frac{grad\ \rho\ grad\ \rho}{\rho^2}$$

$$div\ u = \frac{\hbar}{2m}\frac{(grad)^2\rho}{\rho} - \frac{\hbar}{2m}\frac{grad\ \rho}{\rho}\frac{grad\ \rho}{\rho}$$

$$div\ u = \frac{\hbar}{2m}\Delta\,\rho\frac{1}{\rho} - u\,\frac{grad\ \rho}{\rho} \tag{3-48}$$

$$\frac{\hbar}{2m}div\,u=\frac{\hbar^2}{2m^2}\frac{\Delta\rho}{\rho}-u\frac{\hbar}{2m}\frac{grad\,\rho}{\rho}\qquad /\frac{\hbar}{2m}\frac{grad\rho}{\rho}=u$$

$$\frac{\hbar}{2m}div\,u=\frac{\hbar^2}{2m^2}\frac{\Delta\rho}{\rho}-u^2 \tag{3-49}$$

and we have

$$\Delta\left(\rho^{-\frac{1}{2}}\right)=\nabla\left(\nabla\rho^{\frac{1}{2}}\right)=\nabla\left(\frac{1}{2}\rho^{-\frac{1}{2}}\nabla\rho\right)$$

$$=\frac{1}{2}\nabla\rho^{-\frac{1}{2}}\nabla\rho+\frac{1}{2}\rho^{-\frac{1}{2}}\Delta\rho$$

$$=-\frac{1}{4}\rho^{-\frac{3}{2}}\nabla\rho\nabla\rho+\frac{1}{2}\rho^{-\frac{1}{2}}\Delta\rho$$

$$\Delta\left(\rho^{\frac{1}{2}}\right)\rho^{-\frac{1}{2}}=-\frac{1}{4}\rho^{-2}\nabla\rho\nabla\rho+\frac{1}{2}\frac{\Delta\rho}{\rho}$$

$$\frac{1}{2}\frac{\Delta\rho}{\rho}=\Delta\left(\rho^{\frac{1}{2}}\right)\rho^{-\frac{1}{2}}+\frac{1}{4}\rho^{-2}\nabla\rho\nabla\rho$$

$$\frac{\hbar^2}{2^2m^2}\frac{\Delta\rho}{\rho}=\frac{\hbar^2}{2m^2}\Delta(\sqrt{\rho})\rho^{-\frac{1}{2}}+\frac{1}{4}\frac{\hbar^2}{2^2m^2}\frac{\Delta\rho}{\rho}\frac{\Delta\rho}{\rho}$$

$$=\frac{\hbar^2}{2m^2}\Delta(\sqrt{\rho})\rho^{-\frac{1}{2}}+\frac{1}{2}\frac{\hbar}{2\ m}\frac{\Delta\rho}{\rho}\frac{\hbar}{2m}\frac{\Delta\rho}{\rho}$$

$$\frac{\hbar^2}{2^2m^2}\frac{\Delta\rho}{\rho}=\frac{\hbar^2}{2m^2}\Delta(\sqrt{\rho})\rho^{-\frac{1}{2}}+\frac{1}{2}u^2 \tag{3-50}$$

We substitute in the eq (3-49) we obtain

$$\frac{\hbar}{2m}div\,u=\frac{\hbar^2}{2m^2}\Delta(\sqrt{\rho})\rho^{-\frac{1}{2}}-\frac{1}{2}u^2 \tag{3-51}$$

So

$$\frac{1}{2}u^2+\frac{\hbar}{2m}div\,u=\frac{1}{m}V-\frac{1}{m}E\Rightarrow\frac{\hbar^2}{2m^2}\Delta\left(\sqrt{\rho}\right)\rho^{-\frac{1}{2}}=\frac{1}{m}V-\frac{1}{m}E$$

Product by $\sqrt{\rho}$ and m we find

$$\frac{\hbar^2}{2m^2}\Delta\left(\sqrt{\rho}\right) = (V - E)\sqrt{\rho}$$

$$\Rightarrow \left(\frac{\hbar^2}{2m}\Delta - V + E\right)\sqrt{\rho} = 0$$

We put $\psi = \sqrt{\rho}$

So
$$\left(-\frac{\hbar^2}{2m}\Delta + V - E\right)\psi = 0 \tag{3-52}$$

This is equivalent to the time-independent Schrödinger equation.

Conclusion

We have exposed in this chapter one of the methods to obtain quantum mechanics from the Newton law. Our aim is to avoid using postulates to construct quantum mechanics. This method consists of treating quantum mechanical effects as stochastic phenomena.

Quantum effect is considered as a Markoff process. Stochastic mechanics enables us to construct a bidirectional velocity which will be used in the subsequent chapters.

Chapter 04

Fractal Geometry and Nottale Hypothesis

Fractal Geometry and Nottale hypothesis

The aim of this chapter is to develop the Nottale hypothesis which can be seen as the covariant derivative for the scale relativity. We shall show that using the Nottale hypothesis one can solve some problems like the energy spectrum of a particle in a box without using the Schrödinger equation.

Fractal behavior

We have seen in the previous chapter, in the Feynman interpretation of the quantum trajectories that non-differentiability means that the velocity $v = \frac{dx}{dt}$ is no longer defined. However from the theory, continuity and non-differentiability implied fractality. This leads to conclusion that the physical function must depend explicitly on the resolution. Hence we can replace the classical velocity on a fine scale which describes the fractal property by a function which depends explicitly on resolution $v = v(\varepsilon)$.

We assume that the simplest possible equation that one can write for v is a first order, differential equation, written in term of the dilatation operator

$$\frac{dv}{d\ \ln\varepsilon} = \beta(v) \tag{4-1}$$

We can use the fact that $v < 1$(c=1) to expand it in terms of Taylor expansion, we get

$$\frac{dv}{d\ \ln\varepsilon} = a + bv + 0(v^2) \tag{4-2}$$

(a,b, independent of ε)

If we take $b = -\delta$ *and* $a = v\delta$, we obtain the solution

$$v = V + k\varepsilon^{-\delta} \tag{4-3}$$

From dimensional analysis we can write $k = \zeta\,\lambda^{\,\delta}$ with

$$\zeta = \zeta(t)\,,\ \left\langle \zeta^2 \right\rangle = 1$$

and λ a constant length-scale.

We find

$$v = V + \zeta\left(\frac{\lambda}{\varepsilon}\right)^{\delta} \tag{4-4}$$

$$D = D_T + \delta$$

$$D_T = 1$$

At large scales

$$\varepsilon \succ\succ \lambda \Rightarrow v \approx V \text{ classical case}$$

While at small scales $\varepsilon \prec\prec \lambda \Rightarrow v \approx \zeta \left(\frac{\lambda}{\varepsilon}\right)^{\delta}$

ε is space-resolution $\varepsilon = \delta x$, while $\varepsilon \prec x$ we have $\frac{\delta x}{c\delta t} \approx \left(\frac{\lambda}{\delta x}\right)^{D-1}$

$$\delta x^{D} = \lambda^{D-1} c\delta t$$

Replacing in equation (4-4), we get

$$dx^{i} = v^{i}dt + \lambda^{1-\frac{1}{D}} \varepsilon^{i} (cdt)^{\frac{1}{D}} \tag{4-5}$$

The first term yields classical physics while the second is one of the sources of the quantum behavior. In general any quantity can be put as a sum of a classical counterpart of this quantity and a fluctuation part which can be considered as the quantum part [3,10-14].

Infinity of geodesics

The scale-relativity hypothesis is that the quantum properties of the microphysical world stem from the properties of the geodesics of a fractal space-time. This means that the quantum effects are the manifestation of the fractal structure of the space-time as the gravitation is a manifestation of the space-time curvature. However when the quantum particle moved between two points in the fractal space-time, it follows one geodesics among an infinity geodesics existing between the two points. We cannot define which geodesic is followed by the particle since all geodesics are equiprobable. It means that we keep the indeterminism property of quantum physics and the predictions can only be of a statistical nature. With this hypothesis we can solve some problems of the quantum physics [3].

Two-valuedness of time derivative and velocity vector

Another consequence of the non differentiable nature of space is the breaking of local differential time reflection invariance, so consider the usual definition of the derivative of a function with respect to time

$$\left(\frac{df}{dt}\right) = \lim_{dt \to 0} \frac{f(t+dt) - f(t)}{dt} = \lim_{dt \to 0} \frac{f(t) - f(t-dt)}{dt} \tag{4-6}$$

In the differentiable case we passed from one another by the transformation $dt \to -dt$, but in the non differentiable case we can not compute the above derivative because the limits are not defined. To solve this problem we use the scale-relativistic method. We suggest that we substitute dt by time-resolution $dt = \delta t$, while the limit $\delta t = 0$ have not a physical meaning, and we get two values of the derivatives f'_+, f'_- ,defined as explicit function of t and of dt

$$f_+{}'(t,dt) = \lim_{dt\to 0}\frac{f(t+dt)-f(t)}{dt},\ f'_-(t,dt) = \lim_{dt\to 0}\frac{f(t)-f(t-dt)}{dt} \qquad (4\text{-}7)$$

When applied to the space variable, we have two velocities

The forward velocity $$V_+ = \lim_{dt\to 0}\frac{x(t+dt)-x(t)}{dt} \qquad (4\text{-}8)$$

The backward velocity $$V_- = \lim_{dt\to 0}\frac{x(t)-x(t-dt)}{dt} \qquad (4\text{-}9)$$

Covariant derivative operators

So we have found that when the particle is at a fine scale it has two velocities, and we have no reason to favor one to another, we must deal the both velocity by the same way and consider the both process the forward and the backward. We have showed that the quantum particle have a fractal trajectory, we obtained above that the elementary displacement for both processes, dX as sum of a classical part $dx = vdt$ and a fluctuation about this classical part $d\varepsilon$, which is a Wiener process satisfying the following relation

$$< d\varepsilon > = 0 \qquad (4\text{-}10)$$

$$\left\langle d\varepsilon^i d\varepsilon^j \right\rangle = \delta^{ij}c^2\left(\frac{\lambda}{cdt}\right)^{2-\frac{2}{D}} \qquad (4\text{-}11)$$

For the quantum particle $D = 2$, so

$$\left\langle d\varepsilon^i d\varepsilon^j \right\rangle = \lambda\delta^{ij}cdt \qquad (4\text{-}12)$$

Now we consider the both process the forward (+) and the backward (-)

$$dX^i_\pm = dx^i_\pm + d\varepsilon^i_\pm\ (t) \qquad (4\text{-}13)$$

$$dX^i_\mp = v^i_\mp\ dt + d\varepsilon^i_\mp\ (t) \qquad (4\text{-}14)$$

v^i_+ Forward velocities, v^i_- backward velocities

$$v^i_+ = \frac{d}{dt_+}(x(t)) \qquad (4\text{-}15)$$

$$v^i_- = \frac{d}{dt_-}(x(t)) \qquad (4\text{-}16)$$

From Wiener's theory, the fluctuation ε^i can be written as

$$<\frac{d\varepsilon^i_\pm}{dt}\frac{d\varepsilon^j_\pm}{dt}> = \pm\delta^{ij}c^2\left(\frac{\lambda}{cdt}\right)^{2-\frac{2}{D}} \qquad (4\text{-}17)$$

The fractal dimension of typical quantum mechanical path is $D = 2$,

So $$< d\varepsilon_{\pm}^{i} d\varepsilon_{\pm}^{j} > = \pm \lambda \delta^{ij} c dt \qquad (4\text{-}18)$$

$$< d\varepsilon_{\pm}^{i} d\varepsilon_{\pm}^{j} > = \pm 2 \delta^{ij} D dt \qquad (4\text{-}19)$$

Where

$$D = \frac{\lambda c}{2} = \frac{\hbar}{2m}$$

is the diffusion coefficient.

We consider a derivable function f(x(t), dt), so

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{\nabla} f \frac{dX}{dt} + \frac{\partial^2 f}{2\partial X^i \partial X^j} \frac{dX^i dX^j}{dt} \qquad (4\text{-}20)$$

We write the forward (+) derivative and backward derivative $(-)$ $of\ f(x(t, dt))$

$$\frac{d_{\pm} f}{dt} = \frac{\partial f}{\partial t} + \vec{\nabla} f < \frac{d_{\pm} X}{dt} > + \frac{\partial^2 f}{2\partial X^i \partial X^j} < \frac{d_{\pm} X^i d_{\pm} X^j}{dt} > \qquad (4\text{-}21)$$

We replace $d_{\pm} X$ by its expression in (4-20) and (4-21)

$$\frac{d_{\pm} f}{dt} = \frac{\partial f}{\partial t} + \vec{\nabla} f \left\langle \frac{d_{\pm} x}{dt} \right\rangle + \vec{\nabla} f \left\langle \frac{d\varepsilon_{\pm}}{dt} \right\rangle + \quad + \frac{1}{2} \Delta f \left\langle \frac{d_{\pm} x^i d_{\pm} x^j}{dt} \right\rangle + \frac{1}{2} \Delta f \left\langle \frac{d_{\pm} \varepsilon^i d_{\pm} \varepsilon^j}{dt} \right\rangle$$

$$+ \frac{1}{2} \Delta f \left\langle \frac{d_{\pm} x^i d_{\pm} \varepsilon^i}{dt} \right\rangle + \frac{1}{2} \Delta f \left\langle \frac{d_{\pm} \varepsilon^i d_{\pm} x^j}{dt} \right\rangle \qquad (4\text{-}22)$$

We use the equations

$$\frac{d_{\pm} f}{dt} = \frac{\partial f}{\partial t} + \vec{\nabla} f \left\langle \frac{d_{\pm} x}{dt} \right\rangle + \Delta f \left\langle \frac{d_{\pm} \varepsilon^i d_{\pm} \varepsilon^j}{dt} \right\rangle \qquad (4\text{-}23)$$

with the help of equation (4-18) (4-19) to obtain

$$\frac{d_{\pm} f}{dt} = \frac{\partial f}{\partial t} + \vec{\nabla} f\, v_{\pm}{}^{i} \pm D \Delta f \qquad (4\text{-}24)$$

So $$\frac{d_{\pm}}{dt} = \frac{\partial}{\partial t} + v_{\pm}{}^{i} \vec{\nabla} \pm \frac{1}{2} D \Delta \qquad (4\text{-}25)$$

The forward and backward derivatives $\frac{d_{+}}{dt}$ and $\frac{d_{-}}{dt}$ can be combined in term of a complex derivative operator

$$\frac{d\!\!\!/}{dt} = \frac{1}{2}\left(\frac{d_{+}}{dt} + \frac{d_{-}}{dt} \right) - \frac{1}{2} i \left(\frac{d_{+}}{dt} - \frac{d_{-}}{dt} \right) \qquad (4\text{-}26)$$

When we replace $\frac{d_+}{dt}$ and $\frac{d_-}{dt}$ by the expression (4-25) we get

$$\frac{d\!\!\!/}{dt}=\frac{\partial}{\partial t}+v.\nabla-i\frac{1}{2}D\Delta \tag{4-27}$$

While $$v^i=\frac{d}{dt}X^i=V^i-iU^i=\frac{v_+^i+v_-^i}{2}-i\frac{v_+^i-v_-^i}{2} \tag{4-28}$$

We observe that the operator $\frac{d\!\!\!/}{dt}$ includes the total derivative operator $\frac{d}{dt}=\frac{\partial}{\partial t}+v\nabla$ and other imaginary term which vanished at the classical limit

$$\frac{d\!\!\!/}{dt}=\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}+V\vec{\nabla}\right)-(iU\vec{\nabla}+iD\Delta) \tag{4-29}$$

This operator $\frac{d\!\!\!/}{dt}$ is the covariant derivative operator.

Applications

The energy expression in fractal geometry

Using different approaches L. Nottale and independently J.-C. Pissondes established the total energy expression of a particle, the first by using the Newton complex equation of motion but the second by using the conservation law of energy written in terms of complex derivative operator $\frac{d\!\!\!/}{dt}$ [14,16].

Nottale approach

We have the equation of motion

$$m\frac{d\!\!\!/}{dt}v=-\vec{\nabla}\phi \tag{4-30}$$

We replace the operator $\frac{d\!\!\!/}{dt}$ and v by their complex expressions

So

$$m\left(\frac{\partial}{\partial t}+v\vec{\nabla}\right)v-imD\Delta v=-\vec{\nabla}\phi \tag{4-31}$$

For a free particle

$$\phi=0\Rightarrow m\left(\frac{\partial}{\partial t}+v\vec{\nabla}\right)v=imD\Delta v \tag{4-32}$$

Like classical mechanics we have

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V\vec{\nabla} \quad \text{and} \quad m\frac{d}{dt}V = F$$

We make the correspondence with the complex values

$$\frac{\partial}{\partial t} + v\vec{\nabla} = \frac{\partial}{\partial t} + V\vec{\nabla} \tag{4-33}$$

$$m\frac{d}{dt}v = f = -\vec{\nabla}\Phi \tag{4-34}$$

f is called the fractal force and Φ is the fractal potential .

The previous equations give

$$f = imD\Delta v \text{ and } \Phi = -imD\vec{\nabla}v \tag{4-35}$$

So the total energy is

$$E = \frac{P^2}{2m} + \phi + \Phi \tag{4-36}$$

with ϕ external potential and Φ is the fractal potential, in the free particle case $\phi = 0$

So

$$E = m\frac{v^2}{2} - imD\vec{\nabla}v \tag{4-37}$$

J-C Pissondes approach

The Newton's law of mechanics is

$$\begin{aligned} &m\frac{d\vec{V}}{dt} = -\vec{\nabla}\phi \Rightarrow m\vec{V}.\frac{d\vec{V}}{dt} = -\vec{V}.\vec{\nabla}\phi \Rightarrow \\ &m\frac{d}{dt}\left(\frac{V^2}{2}\right) = -V\vec{\nabla}\phi \end{aligned} \tag{4-38}$$

where ϕ is time independent function.

So

$$\frac{d\phi(x)}{dt} = \frac{\partial\phi(x)}{\partial t} + V\vec{\nabla}\phi(x) \Leftrightarrow V\vec{\nabla}\phi(x) = \frac{d\phi(x)}{dt} \tag{4-39}$$

$$m\frac{d}{dt}\left(\frac{V^2}{2}\right) = \frac{d\phi(x)}{dt} \Leftrightarrow \frac{d}{dt}\left(\frac{mV^2}{2} + \phi(x)\right) = 0 \Leftrightarrow \frac{d}{dt}E = 0 \tag{4-40}$$

which is the conservation law of the energy.

By using the change

$$\frac{d}{dt} \to \frac{d}{dt}$$

$$V \to v$$

$$mv\frac{d}{dt}v = -v\vec{\nabla}\phi(x) \qquad (4\text{-}41)$$

We can show that (see appendix B)

$$\frac{d}{dt}(f.g) = f\frac{d}{dt}g + g\frac{d}{dt}f - 2iD\vec{\nabla}\vec{\nabla}fg \qquad (4\text{-}42)$$

By using the eqs (4-41), (4-42) we obtain

$$\frac{d}{dt}(v^2) = 2v\frac{d}{dt}v - 2iD(\vec{\nabla}v)^2 \Leftrightarrow v\frac{d}{dt}v = \frac{1}{2}\frac{d}{dt}(v^2) + iD\left(\vec{\nabla}v\right)^2 \qquad (4\text{-}43)$$

$$mv\frac{d}{dt}v = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + imD(\vec{\nabla}v)^2 \qquad (4\text{-}44)$$

We apply the operator $\frac{d}{dt}$ to the potential $\phi(x)$

$$\frac{d}{dt}\phi(x) = \frac{\partial\phi(x)}{\partial t} + v\vec{\nabla}\phi(x) - iD\Delta\phi(x) \qquad (4\text{-}45)$$

$$\frac{d}{dt}\phi(x) = v\vec{\nabla}\phi(x) - iD\vec{\nabla}\left(\vec{\nabla}\phi(x)\right) \qquad (4\text{-}46)$$

From (4-42) we have

$$\vec{\nabla}\phi(x) = -m\frac{d}{dt}v \qquad (4\text{-}47)$$

We replace in (4-47) which gives

$$v\vec{\nabla}\phi(x) = \frac{d}{dt}\phi(x) - iD\vec{\nabla}\left(m\frac{d}{dt}v\right) \qquad (4\text{-}48)$$

The other way is

$$\vec{\nabla}\left(\frac{d}{dt}v\right) = \vec{\nabla}\left(\frac{\partial v}{\partial t} + v\vec{\nabla}v - iD\Delta v\right)$$

$$= \frac{\partial}{\partial t}(\vec{\nabla}v) + \vec{\nabla}v\vec{\nabla}v + v\vec{\nabla}(\vec{\nabla}v) - iD\Delta(\vec{\nabla}v)$$

$$= \left[\frac{\partial}{\partial t} + v\vec{\nabla} - iD\Delta\right]\vec{\nabla}v + \vec{\nabla}v.\vec{\nabla}v = \frac{d}{dt}(\vec{\nabla}v) + (\vec{\nabla}v)^2 \qquad (4\text{-}49)$$

From (4-34) and (4-35)

$$v\vec{\nabla}\phi(x) = \frac{d}{dt}\phi(x) - iDm\left[\frac{d}{dt}\left(\vec{\nabla}v\right) + (\vec{\nabla}v)^2\right] \qquad (4\text{-}50)$$

We use (4-44) and (4-48), the relation (4-50) becomes

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right)+imD(\vec{\nabla}v)^2=-\frac{d}{dt}\phi(x)+imD\frac{d}{dt}+imD(\vec{\nabla}v)^2$$

$$\Leftrightarrow \frac{d}{dt}\left[\frac{1}{2}mv^2-imD\vec{\nabla}v+\phi(x)\right]=0$$

So the energy is

$$E=\frac{1}{2}mv^2-imD\vec{\nabla}v+\phi(x) \tag{4-51}$$

We put

$$-imD\vec{\nabla}v=\Phi \quad and \quad f=-\vec{\nabla}\Phi$$

In the classical case $v=V$ and $D\to 0$

We have

$$E=\frac{1}{2}mv^2+\phi(x)$$

Particle in a Box

Our aim is to show how we can solve one dimension problem of a quantum particle in a box, by only using the Nottale hypothesis, and without any need to the Schrödinger equation [17]. This means we use only scale covariant derivative and the fundamental equation of dynamics in theirs complex forms

$$-\vec{\nabla}\phi=m\frac{d}{dt}v \tag{4-52}$$

We use the complex velocity

$$v=V-iU$$

Since ϕ, being a potential and it is a real quantity, we can separate (4-52) into real and imaginary parts

$$m\left(\frac{\partial}{\partial t}V-D\Delta U+(V.\vec{\nabla})V-(U.\vec{\nabla})U\right)=-\vec{\nabla}U \tag{4-53}$$

$$m\left(\frac{\partial}{\partial t}U+D\Delta U+(U.\vec{\nabla})U+(U.\vec{\nabla})V\right)=0 \tag{4-54}$$

We consider one dimensional problem with infinite limit boundary and without force (thus ϕ constant).

V is considered as an average classical velocity; $V=0$

So our equations reduce to

$$U(\vec{\nabla}U) + D\Delta U = 0 \qquad (4\text{-}55)$$

$$\frac{\partial}{\partial t}U = 0 \qquad (4\text{-}56)$$

The equation (4-56) means that U is a function of x and does not depend explicitly on time. From (4-55) we have

$$D\vec{\nabla}(\vec{\nabla}U) + \frac{1}{2}\vec{\nabla}U^2 = 0 \Leftrightarrow \vec{\nabla}\left[D(\vec{\nabla}U) + \frac{1}{2}U^2\right] = 0 \qquad (4\text{-}57)$$

which gives

$$\frac{\partial}{\partial x}\left(D\frac{\partial U}{\partial x} + \frac{1}{2}U^2(x)\right) = 0 \qquad (4\text{-}58)$$

Integrating this differential equation gives

$$D\frac{\partial}{\partial x}U + \frac{1}{2}U^2(x) = K'_1 \qquad (4\text{-}59)$$

where $K_1^{'}$ is integration constant

$$\frac{\partial U}{\partial x} + \frac{U^2}{2D} = \frac{K_1^{'}}{D} \Leftrightarrow \frac{U^{'}}{2K_1^{'} - U^2} = \frac{1}{2D} \Leftrightarrow \left(U^{'} = \frac{d}{dx}U\right)$$

$$-\int_{-\infty}^{+\infty}\frac{dU}{U^2 - 2K_1^{'}} = \int_{-\infty}^{+\infty}\frac{dx}{2D} \Leftrightarrow -\int_{-\infty}^{+\infty}\frac{dU}{U^2 + \left(i\sqrt{2K_1^{'}}\right)^2} = \frac{x}{2D} + cste$$

$$-\frac{1}{i\sqrt{2K_1^{'}}}arctg\frac{U}{i\sqrt{2K_1^{'}}} = \frac{x}{2D} + K_2 \qquad (4\text{-}60)$$

Let us introduce the change $K_1^{'} = -K_2$, which gives

$$U(x) = \sqrt{2K_1^{'}}\ \tan\left(-\frac{\sqrt{2K_1}}{2D}x + K_2\right) \qquad (4\text{-}61)$$

where the limit conditions will determine the integration constants K_1 and K_2 u being a difference of velocities can be interpreted as a kind of acceleration. We can thus reasonably suppose that $U \to +\infty$ on the left border (that is $x \to 0$) and $U \to -\infty$ on the right border (That is, conventionally $x \to a$, if our 'box' is of size a)

$$\lim_{x \to a} U(x) = -\infty \Rightarrow K_1 = \frac{2D^2n^2\pi^2}{a^2} \Rightarrow K_1 = -\frac{2D^2n^2\pi^2}{a^2} \qquad (4\text{-}62)$$

$$\lim_{x\to 0} U(x) = -\infty \Rightarrow K_2 = \frac{\pi}{2} \tag{4-63}$$

So

$$U(x) = \frac{2Dn\pi}{a}\tan\left(\frac{n\pi}{a}x + \frac{\pi}{2}\right) \tag{4-64}$$

From equation (4-59) we can write

$$-K_1 = \frac{E}{m}U \Rightarrow E = -mK_1 = \frac{2mn^2\pi^2D^2}{a^2}$$

If $D = \frac{\hbar}{2m}$, the quantum energy expression

$$E = \frac{n^2\pi^2\hbar^2}{2ma^2} \tag{4-65}$$

We can arrive to the same result if we use the energy expression of a free particle

$$E = \frac{mv^2}{2} - imD\vec{\nabla}v, \text{ with } v = V - iU$$

$$V = 0$$

So

$$v = -iU$$

After substituting in the expression of energy, we find

$$E = -\frac{m}{2}\left(\frac{4D^2n^2\pi^2}{a^2}\right)\tan^2\left(-\frac{n\pi}{a}x + \frac{\pi}{2}\right) - mD\vec{\nabla}\left[\frac{2Dn\pi}{a}\tan\left(-\frac{n\pi}{a}x + \frac{\pi}{2}\right)\right]$$

$$E = -\frac{2mD^2n^2\pi^2}{a^2\cos^2\alpha} + \frac{2mD^2n^2\pi^2}{a^2\cos^2\alpha} + \frac{2mD^2n^2\pi^2}{a^2}$$

where

$$\alpha = -\frac{n\pi x}{a} + \frac{\pi}{2}$$

Finally we find

$$E = \frac{2n^2D^2\pi^2}{a^2}m$$

with

$$D = \frac{\hbar}{2m}$$

So

$$E = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

This is the same result when we use the ordinary quantum mechanics.

Conclusion

Using the Nottale hypothesis which consists of replacing the derivative $\frac{d}{dt}$ by $\frac{\not{d}}{dt}$, we recover the Schrödinger equation from Newton equation.

The application to the problem of a free particle in a box gives the known formula of the energy.

Chapter 05

Fractal Geometry and Quantum Mechanics

Fractal Geometry and Quantum Mechanics

Now, we shall postulate that the passage from classical (differentiable) mechanics [18] to the quantum (non differentiable) mechanics can be performed simply by replacing the standard time derivative $\frac{d}{dt}$ by the new complex, operator $\frac{\not{d}}{dt}$. This postulate is called Nottale hypothesis.

Covariant Euler-Lagrange equations

In a general way, the Lagrange function is expected to be a function of the variable x and their time derivatives $\dot{x}$, but in the non-differentiable the number of velocity components $\dot{x}$ is doubled, so that we are led to write [3,7]

$$\ell = \ell(x, \dot{x}_+, \dot{x}_-, t) \tag{5-1}$$

Instead a classical formulation of the Lagrange functions as

$$\ell = \ell(x, v, t)$$

We now keep the classical expression of the Lagrangian but make the substitution the classical velocity $v = \dot{x}$ by its complex form $v = V - iU$ which expressed in term of forward velocity $\dot{x}_+$ and backward velocity $\dot{x}_-$ the as $v = \frac{1}{2}(\dot{x}_+ + \dot{x}_-) - \frac{i}{2}(\dot{x}_+ - \dot{x}_-)$, so the Lagrange function can be written as

$$\ell = \ell\left(x, \frac{1}{2}(\dot{x}_+ + \dot{x}_-) - \frac{i}{2}(\dot{x}_+ - \dot{x}_-), t \right) \tag{5-2}$$

$$\ell = \ell\left(x, \frac{1-i}{2}\dot{x}_+ + \frac{1+i}{2}\dot{x}_-, t \right) \tag{5-3}$$

Therefore we obtain

$$\frac{\partial \ell}{\partial \dot{x}_+} = \frac{\partial v}{\partial \dot{x}_+} \cdot \frac{\partial \ell}{\partial v} = \frac{1-i}{2} \cdot \frac{\partial \ell}{\partial v} \tag{5-4}$$

$$\frac{\partial \ell}{\partial \dot{x}_-} = \frac{\partial v}{\partial \dot{x}_-} \cdot \frac{\partial \ell}{\partial v} = \frac{1+i}{2} \cdot \frac{\partial \ell}{\partial v} \tag{5-5}$$

While the new covariant time derivative operator writes

$$\frac{\not{d}}{dt} = \frac{1-i}{2}\frac{d}{dt_+} + \frac{1+i}{2}\frac{d}{dt_-} \tag{5-6}$$

Let us write the stationary action principle in terms of the new Lagrange function, as written

$$\delta S = \delta \int_{t_1}^{t_2} \ell(x, \dot{x}_+, \dot{x}_-, t)\, dt = 0 \tag{5-7}$$

It becomes

$$\int\left(\frac{\partial\ell}{\partial x}\delta x+\frac{\partial\ell}{\partial\dot{x}_+}\delta\dot{x}_+ +\frac{\partial\ell}{\partial\dot{x}_-}\delta\dot{x}_-\right)dt=0 \tag{5-8}$$

Since

$$\delta\dot{x}_+ =\frac{d(\delta x)}{dt_+}\ \text{and}\ \delta\dot{x}_- =\frac{d(\delta x)}{dt_-} \tag{5-9}$$

Eq (5-1-8) takes the form

$$\int_{t_1}^{t_2}\left(\frac{\partial\ell}{\partial x}\delta x+\frac{\partial\ell}{\partial v}\left[\frac{1-i}{2}\frac{d}{dt_+}+\frac{1+i}{2}\frac{d}{dt_-}\right]\delta x\right)dt=0 \tag{5-10}$$

$$\int_{t_1}^{t_2}\left(\frac{\partial\ell}{\partial x}\delta x+\frac{\partial\ell}{\partial v}\frac{d}{dt}\delta x\right)dt=0 \tag{5-11}$$

To obtain the Lagrange equation from the stationary action principle we must integrate (5-11) by parts, but this integration by parts cannot be performed as usual way because it involves the new covariant derivative.

So we consider the Leibniz rule for the covariant derivative operator $\frac{d}{dt}$ since $\frac{d}{dt}=\frac{\partial}{\partial t}+v\vec{\nabla}-iD\Delta$ is a linear combination of first and second order derivatives, the same is true of its Leibniz rule; this implies an additional term in the expression for the derivative of a product

$$\frac{d}{dt}\left(\frac{\partial\ell}{\partial v}\delta x\right)=\frac{d}{dt}\frac{\partial\ell}{\partial v}\delta x+\frac{\partial\ell}{\partial v}\frac{d}{dt}\delta x-2iD\nabla\frac{\partial\ell}{\partial v}\nabla\delta x \tag{5-12}$$

Since $\delta x(t)$ is not a function of x, the third term on right-hand side of (5-12) vanishes. Therefore the above integral becomes

$$\int_{t_1}^{t_2}\left[\left(\frac{\partial\ell}{\partial x}-\frac{d}{dt}\frac{\partial\ell}{\partial v}\right)\delta x+\frac{d}{dt}\left(\frac{\partial\ell}{\partial v}\delta x\right)\right]dt=0 \tag{5-13}$$

The second point is integration of the covariant derivative we define a new integral as being the inverse operation of covariant derivation

$$\int df=f \tag{5-14}$$

In terms of which one obtains

$$\int_{t_1}^{t_2}d\left(\frac{\partial\ell}{\partial v}\delta x\right)=\left[\frac{\partial\ell}{\partial v}\delta x\right]_{t_2}^{t_2}=0 \tag{5-15}$$

Since $\delta x(t_1) = \delta x(t_2) = 0$ by definition of the variation principle therefore the action integral becomes

$$\int_{t_1}^{t_2} \left(\frac{\partial \ell}{\partial x} - \frac{d}{dt}\frac{\partial \ell}{\partial v} \right) \delta x dt = 0 \qquad (5\text{-}16)$$

And finally we obtain generalized Euler Lagrange equation that read

$$\frac{\partial \ell}{\partial x} - \frac{d}{dt}\frac{\partial \ell}{\partial v} = 0 \qquad (5\text{-}17)$$

Complex probability amplitude and principle of correspondence

Assuming homogeneity of space in the mean leads to defining a complex momentum

$$P = \frac{\partial \ell}{\partial v} \qquad (5\text{-}18)$$

$$p = \vec{\nabla} S \qquad (5\text{-}19)$$

if one now considers the action as a functional of the upper limit of integration (5-7) the variation of the action from a trajectory to another close-by trajectory, when combined with (5-17) yields a generalization of another well-known result, namely that the complex momentum is the gradient of the complex action

$$P = \vec{\nabla} S$$

this equation implies that v is a gradient this demonstrate that the classical velocity v is a gradient (while this was postulated in Nelson's work) (see chapter 3)

We can now introduce a generalization of the classical action S which is complex manifestation consequence to the complex form of velocity in fine scale by the relation [18]

$$v = \frac{1}{m}\vec{\nabla} S \qquad (5\text{-}20)$$

from equation $P = mv$.

We introduce a complex function ψ from the complex action S

$$\psi = \exp\left(\frac{i}{2mD} S \right) \qquad (5\text{-}21)$$

This is related to the complex velocity in the following way

$$v = -2iD\vec{\nabla}(\ell n \psi) \qquad (5\text{-}22)$$

As we shall see in what follows, ψ is solution of the Schrödinger equation and satisfies to Born's statistical interpretation of quantum mechanics, and so can be identified with the wave function or (probability amplitude)

From (5-21) and the relation $P = mv$, we obtain

$$\vec{P}\psi = -2imD\vec{\nabla}(\ell n\,\psi)\psi$$

$$D = \frac{\hbar}{2m}$$

$$\vec{P}\psi = -i\hbar\vec{\nabla}(\ell n\,\psi)\psi$$

$$\vec{P}\psi = -i\hbar\frac{\vec{\nabla}\psi}{\psi}\psi$$

$$\vec{P}\psi = -i\hbar\vec{\nabla}\psi$$

So

$$\vec{P} = -i\hbar\vec{\nabla} \qquad (5\text{-}23)$$

From (5-21) we obtain

$$S = i\hbar\ell n\,\psi$$

$$E = -\frac{\partial S}{\partial t} = i\hbar\frac{\partial}{\partial t}(\ell n\,\psi)$$

$$E = i\hbar\frac{\partial\psi}{\psi\,\partial t}$$

So

$$E = i\hbar\frac{\partial}{\partial t} \qquad (5\text{-}24)$$

Thus the principle of correspondence becomes an equality because, the energy and the impulsion both become complex.

The Schrödinger equation

We consider the Newton equation of dynamics, which is written in terms of complex variable and complex operator as [3, 7, 14]

$$m\frac{d}{dt}v = -\vec{\nabla}\phi \qquad (5\text{-}25)$$

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v\vec{\nabla} - iD\Delta \quad \text{and} \qquad D = \frac{\hbar}{2m}$$

When

$$v = -\frac{i\hbar}{m}\vec{\nabla}\ell n\,\psi \qquad (5\text{-}26)$$

We substitute in equation of motion (5-25), we obtain

$$m\left(\frac{\partial}{\partial t}-\frac{i\hbar}{m}-\vec{\nabla}(\ell n\psi)\vec{\nabla}-\frac{i\hbar}{2m}\Delta\right)\left(\frac{i\hbar}{m}\vec{\nabla}(\ell n\psi)\right)=-\vec{\nabla}\Phi \qquad (5\text{-}27)$$

$$-i\hbar\frac{\partial}{\partial t}(\vec{\nabla}\ell n\psi)-\frac{\hbar^2}{m}\vec{\nabla}\ell n\psi\vec{\nabla}(\vec{\nabla}\ell n\psi)=\vec{\nabla}\Phi \qquad (5\text{-}28)$$

Using the following remark

$$((\nabla\ell nf)^2+\Delta\ell nf=\frac{\nabla f}{f},\Delta\nabla=\nabla\Delta)$$

$$\nabla(\nabla f)^2=2(\nabla f\nabla)(\nabla f),\vec{\nabla}\frac{\partial}{\partial t}=\frac{\partial}{\partial t}\vec{\nabla}$$

We obtain

$$\vec{\nabla}\left[-i\hbar\frac{\partial}{\partial t}(\ell n\psi)\right]-\frac{i\hbar}{m}\vec{\nabla}(\ell n\psi)\Delta\ell n\psi-\frac{\hbar^2}{2m}\vec{\nabla}(\Delta\ell n\psi)=-\vec{\nabla}\phi$$

$$\vec{\nabla}\left[-i\hbar\frac{\partial}{\partial t}(\ell n\psi)-\frac{\hbar^2}{2m}(\vec{\nabla}\ell n\psi)^2-\frac{\hbar^2}{2m}\Delta\ell n\psi\right]=-\vec{\nabla}\phi$$

$$-i\hbar\frac{\partial}{\partial t}(\ell n\psi)-\frac{\hbar^2}{2m}(\vec{\nabla}\ell n\psi)^2-\frac{\hbar^2}{2m}\Delta\ell n\psi=-\phi$$

This yields

$$-i\hbar\frac{\partial\psi}{\psi\partial t}-\frac{\hbar^2}{2m}\left(\frac{\vec{\nabla}\psi}{\psi}\right)^2-\frac{\hbar^2}{2m}\left(\frac{\psi\Delta\psi-(\vec{\nabla}\psi)^2}{\psi^2}\right)=-\phi$$

$$-i\hbar\frac{1}{\psi}\frac{\partial\psi}{\partial t}-\frac{\hbar^2}{2m}\frac{1}{\psi}\Delta\psi=-\phi$$

$$\Rightarrow -i\hbar\frac{\partial}{\partial t}\psi=\left(\frac{\hbar^2}{2m}\Delta+\phi\right)\psi$$

Which is the Schrödinger equation, when has been derived as a geodesics equation in a fractal three space for non-relativistic motion in the framework of Galilean scale relativity.

The Complex Klein-Gordon equation

Now we shall be concerned with relativistic motion in the framework of Galilean scale relativity. we shall derive the complex Klein-Gordon as geodesic equation in a four-dimensional fractal space-time [12].

In the relativistic case, the full space-time continuum is considered to be non differentiable, we consider a elementary displacement dx_μ (μ = 0, 1, 2, 3) of a non differentiable four-coordinate(space and time) along one of the geodesics of the fractal space-time, we can decompose dx_μ in term of v large-scale part $< dx_\mu > = dx_\mu = v_\mu$ ds and a fluctuation $d\varepsilon^\mu$ (Wiener process) such that $< d\varepsilon_\mu > = 0$ by definition, and s is a proper time (relativistic case).

As in the non-relativistic motion case, the non-differentiable nature of space-time yields the breaking of the reflection invariance at the infinitesimal level one is then led to write the elementary displacement along a geodesic of fractal dimension D = 2, respectively for the forward (+) and backward (-) processes, under the form

$$dX_\pm^\mu = dx_\pm^\mu + d\varepsilon_\pm{}^\mu \tag{5-29}$$

$$dX_\pm^\mu = v_\pm{}^\mu ds + d\varepsilon_\pm{}^\mu \tag{5-30}$$

$$d\varepsilon_\pm{}^\mu = a^\mu{}_\pm \sqrt{2D} ds^{\frac{1}{2}} \tag{5-31}$$

$a_\pm{}^\mu$ is a dimensionless fluctuation, and $d\varepsilon_\pm^\mu$ Wiener process satisfy the following relation

$$< d\varepsilon_\pm{}^\mu d\varepsilon_\pm{}^\mu > = \pm 2\eta^\mu ds \tag{5-32}$$

$$< d\varepsilon_\pm{}^\mu > = 0 \tag{5-33}$$

We define the forward and backward derivatives relative to the proper time $\frac{d}{ds_+}$ *and* $\frac{d}{ds_-}$ as

$$\frac{d}{ds_+} x^\mu(s) = v_+{}^\mu, \frac{d}{ds_-} x^\mu(s) = v_-{}^\mu \tag{5-34}$$

We can combined the forward and the backward derivatives to construct a complex derivative operator $\frac{d}{ds}$

$$\frac{d}{ds} = \frac{1}{2}\left(\frac{d}{ds_+} + \frac{d}{ds_-}\right) - \frac{i}{2}\left(\frac{d}{ds_+} - \frac{d}{ds_-}\right) \tag{5-35}$$

When we apply to the position vector, this operator yields a complex four-velocity

$$v^\mu = \frac{d}{ds} X^\mu = V^\mu - iU^\mu = \frac{v_+{}^\mu + v_-{}^\mu}{2} - i\frac{v_+{}^\mu - v_-{}^\mu}{2} \tag{5-36}$$

We have a derivable function $f(x^\mu(s), ds)$

$$\frac{df}{ds} = \frac{\partial f}{\partial s} + \partial_\mu f \frac{dX^\mu}{ds} + \frac{\partial^2 f}{\partial X^\mu \partial X^\nu} \frac{dX^\mu dX^\nu}{ds} \tag{5-37}$$

So

$$\frac{d}{ds_\pm} = \frac{\partial f}{\partial s_\pm} + \partial_\mu f \left\langle \frac{dX^\mu}{ds_\pm} \right\rangle + \partial_\mu \partial_\nu f \left\langle \frac{dX^\mu dX^\nu}{ds_\pm} \right\rangle \tag{5-38}$$

We use (5-30), (5-32), (5-33), we find

$$\frac{df}{ds_{\pm}} = (\frac{\partial}{\partial s} + v_{\pm}^{\mu}\partial_{\mu} \mp D\partial^{\mu}\partial_{\mu})f \tag{5-39}$$

$$D = \frac{\hbar}{2m}$$

We only cansider s-stationary functions, (functions that do not explicitly depend on the proper times), the complex covariant derivative operator reduces to

$$\frac{d\!\!/}{ds} = (v^{\mu} + iD\partial^{\mu})\partial_{\mu} \tag{5-40}$$

Let us now assume that the large-scale part of any mechanical system can be characterized by a complex action S ,the same definition of the action as in standard relativistic mechanics, so we write

$$S = \int_a^b \Lambda(\underline{x}, \underline{u})ds \tag{5-41}$$

$$\Lambda = \frac{mc^2}{ds^2} \qquad dS = \frac{ds}{c} = \frac{\sqrt{dX^{\mu}dX_{\mu}}}{c}$$

$$S = -mc\int_a^b \sqrt{dX^{\mu}dX_{\mu}} \tag{5-42}$$

$$\delta S = -mc\delta\int_a^b \sqrt{dX^{\mu}dX_{\mu}}$$

$$\delta S = -mc\int_a^b \frac{1}{2}\delta(dX^{\mu}dX_{v})(dX^{\mu}dX_{v})^{-\frac{1}{2}}$$

$$\delta S = -mc\int_a^b \delta\, dX^{\mu}\,\frac{dX_{v}}{ds} \tag{5-43}$$

$$\delta S = -mc\int_a^b v^{v}(\delta\, dX^{\mu})$$

Integration by parts yields

$$\delta S = -mc\left[(\delta X^{\mu}v^{v})\right] - \int_a^b \delta X^{\mu}\frac{dv^{v}}{ds}ds \tag{5-44}$$

To get equation of motion, one has to determine $\delta S = 0$ between the same two points, at the limits $(\delta X^{v})_a = (\delta X^{v})_b = 0$.

So

$$\delta S_a^b \Big| = -mc\int \delta X^\mu \frac{dv^\nu}{ds} ds \tag{5-45}$$

We therefore obtain a differential geodesic equation

$$\frac{dv^\nu}{ds} = 0 \tag{5-46}$$

We consider the point (a) as fixed, so that $(\delta x^\nu)^a = 0$

The second point b must be considered as variable $\delta S = -mcv_\nu(\delta X^\nu)_b$, Simply writing $(\delta x^\nu)^b as \delta x^\nu$ gives

So
$$\delta S = -mcv_\nu \delta X^\nu \tag{5-47}$$

The complex four-momentum be written as

$$P_\nu = mcv_\nu = -\partial_\nu S \Rightarrow v_\nu = \frac{\partial_\nu S}{mc} \tag{5-48}$$

Now, the complex S, characterizes completely the dynamical state of the particle, and can introduce a complex wave function

$$\psi = e\frac{iS}{\hbar} \tag{5-49}$$

$$\Rightarrow \frac{iS}{\hbar}\ell n\,\psi \Rightarrow S = -i\hbar\,\ell n\,\psi \tag{5-50}$$

from (5-49) we obtain

$$v_\nu = -\frac{i\hbar}{mc}\partial_\nu \ell n\,\psi \tag{5-51}$$

We now apply the scale-relativistic prescription, replace the derivative by its covariant expression given by eq (5-40), and v_μ the complex four velocity of eq (5-51), in the equation of motion (5-46) we find

$$\frac{d}{dt}v_\nu = (v^u + iD\partial^u)\partial_\mu v_\nu = 0$$

$$v_\nu \partial_\mu v_\nu + iD\partial^\mu \partial_\mu v_\nu = 0$$

We find
$$\frac{-\hbar^2}{m^2c^2}\partial^\mu \ell n\,\psi \partial_\mu \partial_\nu \ell n\,\psi - \frac{\hbar D}{mc}\partial^\mu \partial_\nu \ell n\,\psi = 0 \tag{5-52}$$

We have the relation

$$\partial_\mu \partial^\mu \ell n\ f + \partial_\mu f \partial^\mu \ell n\ f = \frac{\partial_\mu \partial^\mu f}{f}$$

This yield

$$\frac{1}{2}\partial^{\nu}\left(\frac{\partial_{\mu}\partial^{\mu}\psi}{\psi}\right)=\frac{1}{2}\partial^{\nu}\left(\partial^{\mu}\partial_{\mu}\ell n\psi+\partial^{\mu}\ell n\psi\,\partial_{\mu}\ell n\psi\right)$$

$$\frac{1}{2}\partial^{\nu}\left(\frac{\partial_{\mu}\partial^{\mu}\psi}{\psi}\right)=\frac{1}{2}\partial^{\nu}\left(\partial_{\mu}\partial^{\mu}\ell n\psi\right)+\frac{1}{2}\partial^{\nu}\left(\partial^{\mu}\ell n\psi\,\partial_{\mu}\ell n\psi\right)$$

$$=\frac{1}{2}\partial^{\nu}\partial_{\mu}\partial^{\mu}\ell n\psi+\frac{1}{2}\partial^{\nu}\partial^{\mu}\ell n\psi\,\partial_{\mu}\ell n\psi+\frac{1}{2}\partial^{\mu}\ell n\psi\,\partial^{\nu}\partial_{\mu}\ell n\psi$$

$$=(\frac{1}{2}\partial^{\nu}+\partial_{\mu}\ell n\psi)\partial^{\nu}\partial^{\mu}\ell n\psi$$

So
$$\frac{1}{2}\partial^{\nu}\left(\frac{\partial_{\mu}\partial^{\mu}\psi}{\psi}\right)=\left(\partial_{\mu}\ell n\psi+\frac{1}{2}\partial_{\mu}\right)\partial^{\mu}\partial^{\nu}\ell n\psi$$

Dividing (5-53) by the constant factor $D^2=\frac{\hbar^2}{(2mc)^2}$, we obtain the equation of motion of the free particle under the form

$$\partial^{\nu}\left(\frac{\partial^{\mu}\partial_{\mu}\psi}{\psi}\right)=0$$

When we integrate this equation we find

$$\frac{\partial^{\mu}\partial_{\mu}\psi}{\psi}=-K$$

where K is a constant

$$\Rightarrow\partial^{\mu}\partial_{\mu}\psi+K\psi=0$$

The integration constant K is chosen equal to square mass term, $\frac{m^2c^2}{\hbar^2}$

So

$$\partial^{\mu}\partial_{\mu}\psi+\frac{m^2c^2}{\hbar^2}\psi=0$$

This is the Klein-Gordon equation (without electromagnetic field).

Conclusion

Using the Euler-Lagrange equation in the framework of fractal geometry, we have reformulated the Schrödinger and the Klein-Gordon equations. Both of these equations have been obtained by using the Nottale hypothesis without any use of any postulate of quantum mechanics.

Chapter 06

Bi-quaternionic Klein-Gordon Equation and Dirac Equation

Bi -quaternionic Klein - Gordon Equation and Dirac Equation

In this chapter we attempt to write the Dirac equation without using the axioms of quantum mechanics. Indeed, it is known that this equation is obtained from the square root of the Klein-Gordon equation. The latter is obtained from the expression of energy in special relativity and the application of the correspondence principle.

Our aim is to derive the Dirac equation naturally from Klein-Gordon equation when the latter is written in a quaternionic form. We start by a bi –quaternionic covariant derivative operator which leads to the definition of a bi-quaternionic velocity and wave function, which gives us the Klein-Gordon equation in a bi-quaternionic form. The Klein-Gordon equation in bi-quaternionic form allows us to obtain the Dirac equation.

Most of the material used in precedent chapter remains applicable.

Bi- quaternionic covariant derivative operator

Because we are in the relativistic case, and in the scaling domain, we define the space-time coordinates $X^{\mu}(s,\varepsilon_{\mu},\varepsilon_{s})$ as four fractal functions of the proper times s and of the resolutions ε_{μ} and ε_{s} where ε_{μ} for the coordinates and ε_{s} for proper time [7,19].

We assume, that we have a forward shift ds of s which yield to a displacement dX^{μ} of X^{μ} the canonical decomposition is;

$$dX^{\mu} = dx^{\mu} + d\varepsilon^{\mu} \qquad (6\text{-}1)$$

$$\langle\ dX^{\mu}\ \rangle = dx^{\mu} = v_{+}{}^{\mu} ds \qquad (6\text{-}2)$$

$$d\varepsilon^{\mu} = a_{+}{}^{\mu}\sqrt{2D}(ds^{2})^{\frac{1}{2D}},\ \langle a_{+}^{\mu}\rangle = 0,\ \langle (a_{+}^{\mu})^{2}\rangle = 1 \qquad (6\text{-}3)$$

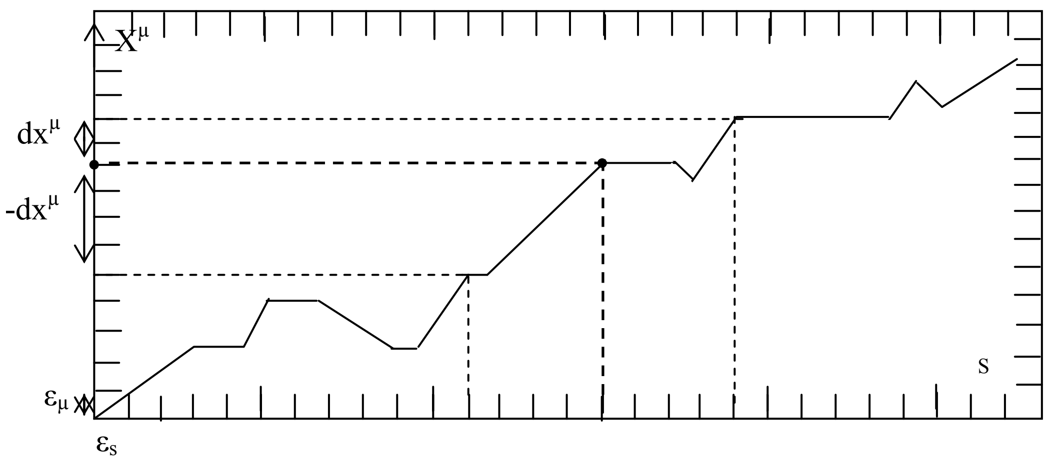


Fig (6-1) Fractal function $X^{\mu}(s,\varepsilon_{\mu},\varepsilon_{s})$

and assume also that we have a backward shift $-ds$ yields a displacement $-dX^{\mu}$ and not necessarily equal to dX^{μ}, so we can write its decomposition as

$$-dX^{\mu} = \delta x^{\mu} + \delta\varepsilon^{\mu} \tag{6-4}$$

$$<-dx^{\mu}> = \delta x^{\mu} = v_{-}{}^{\mu} ds \tag{6-5}$$

$$\delta\varepsilon^{\mu} = a_{-}{}^{\mu}\sqrt{2D}(ds^{2})^{\frac{1}{2D}},\ <a_{-}{}^{\mu}>=0, <(a-^{\mu})>=1 \tag{6-6}$$

Furthermore, we must also consider the breaking of the symmetry $ds \to -ds$, which gives two large-scale forward and backward derivative $\frac{d}{ds_{+}} and \frac{d}{ds_{-}}$. When applied to dx^{μ} (large-scale displacement) yields two large-scale velocities $v_{++}{}^{\mu}$ and $v_{-+}{}^{\mu}$

So we can write

$$dX^{\mu} = v_{++}{}^{\mu}(ds) + d\varepsilon_{++}{}^{\mu} = dx^{\mu} + d\varepsilon^{\mu} \tag{6-7}$$

$$dX^{\mu} = v_{-+}{}^{\mu}(-ds) + d\varepsilon_{-+}{}^{\mu} = dx^{\mu} + d\varepsilon^{\mu} \tag{6-8}$$

$$v^{\mu}{}_{++} = \frac{dx^{\mu}}{ds_{+}} / v^{\mu}{}_{+-} = \frac{dx^{\mu}}{ds_{-}} \tag{6-9}$$

$$d\varepsilon_{+-}{}^{\mu} = a_{+}{}^{\mu}\sqrt{2D}((-ds)^{2})^{\frac{1}{2D}} / d\varepsilon_{++}{}^{\mu} = a_{+}{}^{\mu}\sqrt{2D}((+ds)^{2})^{\frac{1}{2D}} \tag{6-10}$$

$$\frac{d\varepsilon_{++}{}^{\mu}}{ds_{+}} = \omega_{++} = a_{+}{}^{\mu}\sqrt{2D}((+ds)^{2})^{\frac{1}{2D}-\frac{1}{2}}$$

$$\frac{d\varepsilon_{-+}{}^{\mu}}{ds-} = \omega_{-+} = a_{+}{}^{\mu}\sqrt{2D}((-ds)^{2})^{\frac{1}{2D}-\frac{1}{2}} \tag{6-11}$$

Considering now the same forward and backward derivative $\frac{d}{ds_{+}} and \frac{d}{ds_{-}}$ applied to an elementary displacement $-dx^{\mu}$, which yield two velocities $v_{+-}{}^{\mu} and\ v_{--}{}^{\mu}$

So we can write

$$-dX^{\mu} = v_{+-}(ds) + \delta\varepsilon_{+-}{}^{\mu} = \delta x^{\mu} + \delta\varepsilon^{\mu} \tag{6-12}$$

$$-dX^{\mu} = v_{--}(-ds) + \delta\varepsilon_{--}{}^{\mu} = \delta x^{\mu} + \delta\varepsilon^{\mu} \tag{6-13}$$

$$v_{+-} = \frac{\delta x^{\mu}}{ds_{+}} / v_{--} = \frac{\delta x^{\mu}}{ds_{-}} \tag{6-14}$$

$$\delta\varepsilon_{--} = a_{-}{}^{\mu}\sqrt{2D}((-ds)^{2})^{\frac{1}{2D}} \tag{6-15}$$

$$\delta\varepsilon_{+-} = a_{-}{}^{\mu}\sqrt{2D}((ds)^{2})^{\frac{1}{2D}}$$

$$\frac{\delta\varepsilon_{+-}}{ds_+} = \omega_{+-} = a_-{}^{\mu}\sqrt{2D}((ds)^2)^{\frac{1}{2D}-\frac{1}{2}} \tag{6-16}$$

$$\frac{\delta\varepsilon_{--}}{ds_-} = \omega_{--} = a_-{}^{\mu}\sqrt{2D}((-ds)^2)^{\frac{1}{2D}-\frac{1}{2}}$$

By using a Taylor expansion, we can define several total derivatives with respect to s

$$\frac{df}{ds_+} = \frac{\partial f}{\partial s} + \frac{\partial x^{\mu}}{\partial s_+}\frac{\partial f}{\partial x^{\mu}} + \frac{1}{2}\frac{\partial x^{\mu}\partial x^{\nu}}{\partial s_+}\frac{\partial^2 f}{\partial x^{\mu}\partial x^{\nu}} \tag{6-17}$$

$$\frac{df}{ds-} = \frac{\partial f}{\partial s} + \frac{\partial x^{\mu}}{\partial s-}\frac{\partial f}{\partial x^{\mu}} + \frac{1}{2}\frac{\partial x^{\mu}\partial x^{\nu}}{\partial s-}\frac{\partial^2 f}{\partial x^{\mu}\partial x^{\nu}} \tag{6-18}$$

In addition, when considering the two case $(-dx^{\mu})$ and (dx^{μ}) we find four total derivatives.

$$\frac{df}{ds}++ = \frac{\partial f}{\partial s_+} + \frac{\partial x^{\mu}}{ds_+}\frac{\partial f}{\partial x^{\mu}} + \frac{1}{2}\frac{\partial x^{\mu}\ \partial x^{\nu}}{ds_+}\frac{\partial^2 f}{\partial x^{\mu}\partial x^{\nu}} \tag{6-19}$$

$$\frac{df}{ds}-+ = \frac{\partial f}{\partial s_-} + \frac{\partial x^{\mu}}{ds_-}\frac{\partial f}{\partial x^{\mu}} + \frac{1}{2}\frac{\partial x^{\mu}\ \partial x^{\nu}}{ds_-}\frac{\partial^2 f}{\partial x^{\mu}\partial x^{\nu}} \tag{6-20}$$

$$\frac{df}{ds}+- = \frac{\partial f}{\partial s_+} + \frac{-\partial x^{\mu}}{ds_+}\frac{\partial f}{-\partial x^{\mu}} + \frac{1}{2}\frac{-\partial x^{\mu} - \partial x^{\nu}}{ds_+}\frac{\partial^2 f}{-\partial x^{\mu} - \partial x^{\nu}} \tag{6-21}$$

$$\frac{df}{ds}-- = \frac{\partial f}{\partial s_-} + \frac{-\partial x^{\mu}}{ds_-}\frac{\partial f}{-\partial x^{\mu}} + \frac{1}{2}\frac{-\partial x^{\mu} - \partial x^{\nu}}{ds_-}\frac{\partial^2 f}{-\partial x^{\mu} - \partial x^{\nu}} \tag{6-22}$$

We take the first equation and calculate it term by term.

From the equation (6-7) to (6-16) and by using the following equations

$$< \omega_{\pm\pm}{}^{\mu}\omega_{\pm\pm}{}^{\nu} > = \mp 2D\eta^{\mu\nu}ds \tag{6-23}$$

$$<\omega_{\pm\pm}{}^{\mu} > = < \frac{d\varepsilon^{\mu}}{ds_{\pm}} > = 0 \tag{6-24}$$

$$<\frac{d\varepsilon^{\mu}dx^{\nu}}{ds}> = \frac{\langle d\varepsilon^{\mu}\rangle\langle dx^{\nu}\rangle}{ds} = 0 \tag{6-25}$$

$$< \frac{dx^{\mu}d\varepsilon^{\mu}}{ds} > = 0$$

$$<\frac{dx^{\mu}dx^{v}}{ds}> = < \underset{ds\to 0}{v^{\mu}} v^{v}ds > = 0 \qquad (6\text{-}26)$$

$$\frac{\partial X^{\mu}}{\partial s_{\pm}} = \frac{\partial x^{\mu}}{\partial s_{\pm}} + \frac{\partial \varepsilon^{\mu}}{\partial s_{\pm}}$$

$$\omega^{\mu} = a^{\mu}\sqrt{2D}(ds^{2})^{\frac{1}{2D}-\frac{1}{2}}$$

$$= v^{\mu}{}_{\pm +} + \omega^{\mu}{}_{\pm +} \qquad (6\text{-}27)$$

$$\frac{-\partial X^{\mu}}{\partial s_{\pm}} = \frac{\delta x^{\mu}}{\partial s_{\pm}} + \frac{\delta \varepsilon^{\mu}}{\partial s_{\pm}} \qquad (6\text{-}28)$$

$$= v^{\mu}{}_{\pm -} + \omega^{\mu}{}_{\pm -}$$

$$< d\varepsilon > = 0 \Rightarrow < \omega^{\mu} > = 0$$

$$<\frac{dX^{\mu}dX^{v}}{ds}> = <\frac{dx^{\mu}dx^{v}}{ds}> + <\frac{d\varepsilon^{\mu}d\varepsilon^{v}}{ds}> \qquad (6\text{-}29)$$

$$<\frac{-dX^{\mu}\, -dX^{v}}{ds}> = <\frac{\delta x^{\mu}\delta x^{v}}{d\, s}> + <\frac{d\varepsilon^{\mu}d\varepsilon^{v}}{\partial s}> \qquad (6\text{-}30)$$

$$<\frac{dx^{\mu}dx^{v}}{ds}> = < \underset{ds\to 0}{v_{+}{}^{\mu}} v_{+}{}^{v}ds > \to 0 \quad \text{is a same for } (-ds) \qquad (6\text{-}31)$$

$$<\frac{\delta x^{\mu}\delta x^{v}}{ds}> = < \underset{ds\to 0}{v_{-}{}^{\mu}} v_{-}{}^{v}ds > \to 0 \quad \text{is a same for } (-ds) \qquad (6\text{-}32)$$

$$<\frac{d\varepsilon^{\mu}d\varepsilon^{v}}{ds}> = <\frac{a_{+}{}^{\mu}a_{+}{}^{v}2D(ds^{2})^{\frac{1}{2D}}(ds^{2})^{\frac{1}{2D}}}{ds}> \qquad D = 2 \qquad (6\text{-}33)$$

$$= a_{+}{}^{\mu}a_{+}{}^{v}2D\frac{ds}{ds} = a_{+}{}^{\mu}a_{+}{}^{v}2D \qquad (6\text{-}34)$$

$$<\frac{d\varepsilon^{\mu}d\varepsilon^{v}}{ds}> = <\frac{d\varepsilon^{\mu}}{ds}\frac{d\varepsilon^{v}}{ds}ds> = < \omega_{++}{}^{\mu}\omega_{++}{}^{v}> = -2\eta^{\mu v}ds \qquad (6\text{-}35)$$

From the definition

$$<\frac{\delta\varepsilon^{\mu}\delta\varepsilon^{v}}{ds}> = <\frac{\delta\varepsilon^{\mu}}{ds}\frac{\delta\varepsilon^{v}}{ds}ds> = < \omega_{+-}{}^{\mu}\omega_{+}{}^{v}> = -2\eta^{\mu v}ds \qquad (6\text{-}36)$$

Using the same calculation, we find

$$<\frac{d\varepsilon^{\mu} d\varepsilon^{v}}{-ds}> = +2\eta^{\mu v} ds \tag{6-37}$$

$$<\frac{\delta\varepsilon^{\mu} \delta\varepsilon^{v}}{-ds}> = +2\eta^{\mu v} ds \tag{6-38}$$

So when we use the precedent relation, the total derivative with respect to s of a fractal function f becomes

$$\frac{df}{ds_{\pm}} \underset{\mu}{\pm} = \left(\frac{\partial}{\partial s} + v^{\mu} \underset{s\mu}{\pm\pm} \partial_{\mu} \mp D\partial^{\mu}\partial_{\mu} \right) f \tag{6-39}$$

where the $\pm$ sign in the right- hand side is still the inverse of the s-sign.

When we apply these derivatives to the position vector X^{μ}

we obtain
$$\frac{dX^{\mu}}{ds} \underset{s\mu}{\pm\pm} = V_{\underset{s\mu}{\pm\pm}} \tag{6-40}$$

We consider now the four function $-X^{\mu}(s, \varepsilon_{\mu}, \varepsilon_{s})$ because there is no reason for distinguishing $(-X^{\mu})(s, \varepsilon_{\mu}, \varepsilon_{s}))$ and $-(X^{\mu}(s, \varepsilon_{\mu}, \varepsilon_{s}))$ since there is a breaking of the parity $P(\mu = x, y, z)$ and time reversal $T(for\, \mu = t)$ symmetries.

As we are dealing with X^{μ}, we consider that we have a forward shift ds of s which yields a displacement $d(-X^{\mu}) of -X^{\mu}$ and a backward shift $-ds$ yields a displacement $-d(-X^{\mu}) of -X^{\mu}$.

Therefore, we can write the canonical decomposition

$$d(-X^{\mu}) = \tilde{d}x^{\mu} + \tilde{d}\varepsilon^{\mu} \tag{6-41}$$

$$< d(-X^{\mu}) > = \tilde{d}x^{\mu} = \tilde{v}^{\mu} + ds \tag{6-42}$$

$$\tilde{d}\varepsilon^{\mu} = \tilde{a}^{\mu} \sqrt{2D(dt^{2})^{\frac{1}{2D}}} \qquad < (\tilde{a}^{\mu}{}_{+}) >= 0 \qquad < (\tilde{a}^{\mu}{}_{+})^{2} >= 1 \tag{6-43}$$

$$-d(-X^{\mu}) = \tilde{\delta}x^{\mu} + \tilde{\delta}\varepsilon_{\mu} \tag{6-44}$$

$$<-d(-X^{\mu})> = \tilde{\delta}x^{\mu} = \tilde{v}^{\mu}{}_{-} ds \tag{6-45}$$

$$\tilde{\delta}\varepsilon_{\mu} = \tilde{a}^{\mu}{}_{-} \sqrt{2D}(dt^{2})^{\frac{1}{2D}} < \tilde{a}^{\mu}{}_{-} > = 0 < (\tilde{a}^{\mu}{}_{-})^{2} >= 1 \tag{6-46}$$

When we consider the breaking of the $ds \to -ds$ symmetry we find the large scale forward and backward derivatives $\frac{\tilde{d}}{ds_{+}}$ and $\frac{\tilde{d}}{ds_{-}}$.

Furthermore, when we apply the precedent derivatives to $-X^{\mu}$ yielding an elementary displacement $d(-x^{\mu})$, we obtain two large –scale velocities

$$\tilde{v}^{\mu}{}_{\underset{s\mu}{++}} = \frac{\tilde{d}x^{\mu}}{ds_{+}} \; and \; \tilde{v}^{\mu}{}_{\underset{s\mu}{-+}} = \frac{\tilde{d}x^{\mu}}{ds_{-}} \tag{6-47}$$

However, when we consider an elementary displacement $-d(-x^{\mu})$ they yield two other large-scale velocities

$$\tilde{v}^{\mu}{}_{\underset{s\mu}{+-}} = \frac{\tilde{\delta}x^{\mu}}{ds_{+}} \; and \; \tilde{v}^{\mu}{}_{\underset{su}{--}} = \frac{\tilde{\delta}x^{\mu}}{ds_{-}} \tag{6-48}$$

By the same method as above, we obtain new total derivatives with respect to s of a fractal function f, which we can write as

$$\frac{\tilde{d}f}{ds_{\pm}}\underset{\mu}{\pm} = \left(\frac{\partial}{\partial s} + \tilde{v}^{\mu}{}_{\underset{s\mu}{\pm\pm}}\partial_{\mu} \mp D\partial^{\mu}\partial_{\mu}\right)f \tag{6-49}$$

When we apply these derivatives to the position vector X^{μ}, we obtain

$$\frac{\tilde{d}X^{\mu}}{ds}\underset{s\mu}{\pm\pm} = \tilde{v}^{\mu}{}_{\underset{s\mu}{\pm\pm}} \tag{6-50}$$

So from (6-40) and (6-50) we obtain eight large-scale velocities

$$v^{\mu}{}_{\underset{s\mu}{++}}, v^{\mu}{}_{\underset{s\mu}{-+}}, v^{\mu}{}_{\underset{s\mu}{+-}}, \tilde{v}^{\mu}{}_{\underset{s\mu}{--}}, \tilde{v}^{\mu}{}_{\underset{s\mu}{++}}, \tilde{v}^{\mu}{}_{\underset{s\mu}{-+}}, \tilde{v}^{\mu}{}_{\underset{s\mu}{+-}}, \tilde{v}^{\mu}{}_{\underset{s\mu}{--}}$$

If we assume that the breaking of the symmetry $dx^{\mu} \to -dx^{\mu}$ is isotropic (the signs corresponding to the four μ indices are chosen equal), we can use the eight components of velocities to define a bi-quaternionic four-velocity. When we write this velocity we have several choices, but the right choice is the velocity, which leads to complex velocities $v^{\mu} = \left[v^{\mu}{}_{++} + v^{\mu}{}_{-+} - i(v^{\mu}{}_{++} - v^{\mu}{}_{-+})\right]/2$ in the non-relativistic motion and real velocities $v^{\mu} = v^{\mu}{}_{++}$ at the classical limits.

For this reason we write

$$v^{\mu} = \frac{1}{2}(v^{\mu}{}_{++} + \tilde{v}^{\mu}{}_{--}) - \frac{i}{2}(v^{\mu}{}_{++} - \tilde{v}^{\mu}{}_{--}) + \left[\frac{1}{2}(v^{\mu}{}_{+-} + v^{\mu}{}_{-+}) - \frac{i}{2}(v^{\mu}{}_{+-} - \tilde{v}^{\mu}{}_{++})\right]e_1$$

$$+\left[\frac{1}{2}(v^{\mu}{}_{--} + \tilde{v}^{\mu}{}_{+-}) - \frac{i}{2}(\tilde{v}^{\mu}{}_{--} - \tilde{v}^{\mu}{}_{-+})\right]e_2 + \left[\frac{1}{2}(v^{\mu}{}_{-+} + \tilde{v}^{\mu}{}_{++}) - \frac{i}{2}(\tilde{v}^{\mu}{}_{-+} + \tilde{v}^{\mu}{}_{+-})\right]e_3 \tag{6-51}$$

At the limit when $\varepsilon_{\mu} \to dx^{\mu}$ and $\varepsilon_{\mu} \to ds$, every e_1 – term in (6-51) goes to zero, and as $\tilde{v}^{\mu}{}_{--} = v^{\mu}{}_{-+}$ in this limit $v^{\mu} = \left[v^{\mu}{}_{++} + v^{\mu}{}_{-+} - i(v^{\mu}{}_{++} - v^{\mu}{}_{-+})\right]/2$ which is the complex velocity, At the classical limit $v^{\mu}{}_{++} = v^{\mu}{}_{-+}$ so the velocity becomes real: $v^{\mu} = v^{\mu}{}_{++}$.

We obtain the bi-quaternionic velocity v^{μ} when we apply a bi-quaternionic derivative operator $\frac{d}{ds}$ to the position vector X^{μ}

The derivative operator which yields the velocity in eq (6-51) when applied to X^{μ} is writes

$$\frac{d}{ds}=\frac{1}{2}\left(\frac{d}{ds^{++}}+\frac{\tilde{d}}{ds^{--}}\right)-\frac{i}{2}\left(\frac{d}{ds^{++}}-\frac{\tilde{d}}{ds^{--}}\right)+\frac{1}{2}\left[\left(\frac{d}{ds^{+-}}+\frac{d}{ds^{-+}}\right)-\frac{i}{2}\left(\frac{d}{ds^{+-}}-\frac{\tilde{d}}{ds^{++}}\right)\right]e_1$$

$$+\frac{1}{2}\left[\left(\frac{d}{ds^{--}}+\frac{\tilde{d}}{ds^{+-}}\right)-\frac{i}{2}\left(\frac{d}{ds^{--}}-\frac{d}{ds^{-+}}\right)\right]e_2+\left[\frac{1}{2}\left(\frac{d}{ds^{-+}}+\frac{\tilde{d}}{ds^{++}}\right)-\frac{i}{2}\left(\frac{\tilde{d}}{ds^{-+}}+\frac{\tilde{d}}{ds^{+-}}\right)\right]e_3 \quad (6\text{-}52)$$

Substituting by

$$\frac{df}{ds}\underset{s\ \mu}{\pm\pm}=\left(\frac{\partial}{\partial s}+v^{\mu}_{\underset{s\ \mu}{\pm\pm}}\partial_{\mu}\mp D\partial^{\mu}\partial_{\mu}\right)f$$

$$\frac{\tilde{d}f}{ds}\underset{s\ \mu}{\pm\pm}=\left(\frac{\partial}{\partial s}+\tilde{v}^{\mu}_{\underset{s\ \mu}{++}}\partial_{\mu}\mp D\partial^{\mu}\partial_{\mu}\right)f$$

in $\frac{d\!\!\!d}{ds}$ we obtain

$$\frac{d\!\!\!d}{ds}f=\frac{1}{2}\left(\frac{\partial}{\partial s}+v^{\mu}_{++}\partial_{\mu}-D\partial^{\mu}\partial_{\mu}+\frac{\partial}{\partial s}+\tilde{v}^{\mu}_{--}\partial_{\mu}+D\partial^{\mu}\partial_{\mu}\right)$$
$$-\frac{i}{2}\left(\frac{\partial}{\partial s}+v^{\mu}_{++}\partial_{\mu}-D\partial^{\mu}\partial_{\mu}-\frac{\partial}{\partial s}-\tilde{v}^{\mu}_{--}\partial_{\mu}-D\partial^{\mu}\partial_{\mu}\right)$$
$$+\left[\frac{1}{2}(\frac{\partial}{\partial s}+v^{\mu}_{+-}\partial_{\mu}-D\partial^{\mu}\partial_{\mu}+\frac{\partial}{\partial s}+v^{\mu}_{-+}\partial_{\mu}+D\partial^{\mu}\partial_{\mu}\right)$$
$$-\frac{i}{2}\left(\frac{\partial}{\partial s}+V^{\mu}_{+-}\partial_{\mu}-D\partial^{\mu}\partial_{\mu}-\frac{\partial}{\partial s}-\tilde{V}^{\mu}_{++}\partial_{\mu}+D\partial^{\mu}\partial_{\mu}\right)\Big]e_1\Big]$$
$$+\left[\frac{1}{2}(\frac{\partial}{\partial s}+v^{\mu}_{--}\partial_{\mu}+D\partial^{\mu}\partial_{\mu}+\frac{\partial}{\partial s}+\tilde{v}^{\mu}_{+-}\partial_{\mu}-D\partial^{\mu}\partial_{\mu}\right)$$
$$-\frac{i}{2}\left(\frac{\partial}{\partial s}+v^{\mu}_{--}\partial_{\mu}+D\partial^{\mu}\partial_{\mu}-\frac{\partial}{\partial s}-v^{\mu}_{-+}\partial_{\mu}-D\partial^{\mu}\partial_{\mu}\right)\Big]e_2\Big]$$
$$+\left[\frac{1}{2}(\frac{\partial}{\partial s}+V^{\mu}_{-+}\partial_{\mu}+D\partial^{\mu}\partial_{\mu}+\frac{\partial}{\partial s}+\tilde{V}^{\mu}_{++}\partial_{\mu}-D\partial^{\mu}\partial_{\mu}\right)$$
$$-\frac{i}{2}\left(\frac{\partial}{\partial s}+\tilde{V}^{\mu}_{-+}\partial_{\mu}+D\partial^{\mu}\partial_{\mu}+\frac{\partial}{\partial s}+\tilde{V}^{\mu}_{+-}\partial_{\mu}-D\partial^{\mu}\partial_{\mu}\right)\Big]e_3\Big]f$$

$$\frac{df}{ds} = \left(\frac{\partial}{\partial s} + \frac{1}{2}\left(\underset{++}{V^{\mu}} + \underset{--}{V^{\mu}} \right)\partial_{\mu} - \frac{i}{2}\left(\underset{++}{V^{\mu}}\,\partial_{\mu} - \underset{--}{\tilde{V}^{\mu}} \right)\partial_{\mu} + iD\partial^{\mu}\partial_{\mu} \right)$$

$$+ \left[\frac{\partial}{\partial s} + \frac{1}{2}\left(\underset{+-}{V^{\mu}} + \underset{-+}{V^{\mu}} \right)\partial_{\mu} - \frac{i}{2}\left(\underset{+-}{V^{\mu}} - \underset{++}{\tilde{V}^{\mu}} \right)\partial_{\mu} \right] e_1$$

$$+ \left[\frac{\partial}{\partial s} + \frac{1}{2}\left(\underset{--}{V^{\mu}} + \underset{+-}{\tilde{V}^{\mu}} \right)\partial_{\mu} - \frac{i}{2}\left(\underset{--}{V^{\mu}} - \underset{-+}{\tilde{V}^{\mu}} \right)\partial_{\mu} \right] e_2$$

$$+ \left[\frac{\partial}{\partial s} + \frac{1}{2}\left(\underset{-+}{V^{\mu}} + \underset{++}{\tilde{V}^{\mu}} \right)\partial_{\mu} - i\left(\frac{\partial}{\partial s} \right) - \frac{i}{2}\left(\underset{-+}{\tilde{V}^{\mu}} + \underset{+-}{\tilde{V}^{\mu}} \right)\partial_{\mu} e_3 \right] f$$

$$\frac{df}{ds} = \left(\left[\frac{\partial}{\partial s}\left(1 + e_1 + e_2 + (1-i)e_3\right) \right] \right) + iD\partial^{\mu}\partial_{\mu} +$$

$$\left[+ \frac{1}{2}\left(\underset{++}{V^{\mu}} + \underset{--}{\tilde{V}^{\mu}} \right) - \frac{i}{2}\left(\underset{++}{V^{\mu}} - \underset{--}{\tilde{V}^{\mu}} \right) \right]$$

$$+ \left[\frac{1}{2}\left(\underset{+-}{V^{\mu}} + \underset{-+}{V^{\mu}} \right) - \frac{i}{2}\left(\underset{++}{V^{\mu}} - \underset{++}{\tilde{V}^{\mu}} \right) \right] e_1$$

$$+ \left[\frac{1}{2}\left(\underset{--}{V^{\mu}} + \underset{+-}{\tilde{V}^{\mu}} \right) - \frac{i}{2}\left(\underset{--}{V^{\mu}} - \underset{-+}{\tilde{V}^{\mu}} \right) \right] e_2$$

$$+ \left[\frac{1}{2}\left(\underset{--}{V^{\mu}} + \underset{+-}{\tilde{V}^{\mu}} \right) - \frac{i}{2}\left(\underset{-+}{\tilde{V}^{\mu}} + \underset{+-}{\tilde{V}^{\mu}} \right) e_3 \right]\partial_{\mu} \Bigg] f$$

$$\frac{d}{ds} f = \left[\left[1 + e_1 + e_2 + (1-i)e_3 \right] \frac{\partial}{\partial s} + v^{\mu}\partial_{\mu} + iD\partial^{\mu}\partial_{\mu} \right] f$$

Therefore, the bi-quaternionic proper-time derivative operator is given

$$\frac{d}{ds} = \left[1 + e_1 + e_2 + (1+i)e_3 \right] \frac{d}{ds} + v^{\mu}\partial_{\mu} + iD\partial^{\mu}\partial_{\mu} \qquad (6\text{-}53)$$

We consider s-stationary functions (functions which do not explicitly depend on the proper time s) the derivative operator reduces to

$$\frac{d}{ds} = v^{\mu}\partial_{\mu} + iD\partial^{\mu}\partial_{\mu} \qquad (6\text{-}54)$$

Bi-quaternionic stationary action principles

Like the complex case, we give the free motion equation as a geodesic equation

$$\frac{dv_{\mu}}{ds} = 0 \tag{6-55}$$

But now v_{μ} is a bi-quaternionic four velocity, the elementary variation of the action

$$\delta S = -mcv_{\mu}\delta x^{\mu} \tag{6-56}$$

We define the bi-quaternionic four-momentum as [18]

$$P_{\mu} = mcv_{\mu} = -\partial_{\mu}S \tag{6-57}$$

We define the bi-quaternionic wave function as [19, 20]

$$\psi = e^{\frac{iS}{cS_0}} \tag{6-58}$$

$$\partial_{\mu}\psi = \frac{i}{cS_0}\partial_{\mu}S\, e^{\frac{iS}{cS_0}} \tag{6-59}$$

$$\partial_{\mu}\psi = \frac{i}{cS_0}\partial_{\mu}S\psi \tag{6-60}$$

$$\psi^{-1}\partial_{\mu}\psi = \frac{i}{cS_0}\partial_{\mu}S \tag{6-61}$$

So

$$v_{\mu} = \frac{-\partial_{\mu}S}{mc} \tag{6-62}$$

$$v_{\mu} = \frac{iS_0}{m}\psi^{-1}\partial_{\mu}\psi \tag{6-63}$$

We substitute the v_{μ} expression and the covariant derivative operator in the equation of motion, we obtain

$$\frac{d}{ds}v_{\mu} = 0 \tag{6-64}$$

$$\left(v^{\nu}\partial_{\nu} + iD\partial^{\nu}\right)v_{\nu} = 0 \tag{6-65}$$

$$\frac{iS_0}{m}\left(\frac{iS_0}{m}\psi^{-1}\partial^{\nu}\psi\,\partial_{\nu} + iD\partial^{\nu}\partial_{\nu}\right)\left(\psi^{-1}\partial_{\mu}\psi\right) = 0 \tag{6-66}$$

Since

$$S_0 = 2mD$$

We have

$$\psi^{-1}\partial^{\nu}\psi\,\partial_{\nu}\left(\psi^{-1}\partial_{\mu}\psi\right)+\partial^{\nu}\partial_{\nu}\left(\psi^{-1}\partial_{\mu}\psi\right)=0 \tag{6-67}$$

To develop this equation we use the relations of quaternions

$$\psi\psi^{-1}=\psi^{-1}\psi=1 \tag{6-68}$$

$$\psi\,\partial_{\mu}\psi^{-1}=-\left(\partial_{\mu}\psi\right)\psi^{-1} \tag{6-69}$$

$$\psi^{-1}\partial_{\mu}\psi=-\left(\partial_{\mu}\psi^{-1}\right)\psi \tag{6-70}$$

By taking the second one term of equation (6-67) and developing it, we obtain

$$\frac{1}{2}\partial^{\nu}\partial_{\nu}\left(\psi^{-1}\partial_{\mu}\psi\right)=\frac{1}{2}\partial^{\nu}\left(\partial_{\nu}\psi^{-1}\partial_{\mu}\psi+\psi^{-1}\partial_{\nu}\partial_{\mu}\psi\right)$$

$$=\frac{1}{2}\partial^{\nu}\partial_{\nu}\psi^{-1}\partial_{\mu}\psi+\frac{1}{2}\partial_{\nu}\psi^{-1}\partial^{\nu}\partial_{\mu}\psi$$

$$+\frac{1}{2}\partial^{\nu}\psi^{-1}\partial_{\nu}\partial_{\mu}\psi+\frac{1}{2}\psi^{-1}\partial^{\nu}\partial_{\nu}\partial_{\mu}\psi$$

$$\frac{1}{2}\partial^{\nu}\partial_{\nu}\left(\psi^{-1}\partial_{\mu}\psi\right)=\frac{1}{2}\partial^{\nu}\partial_{\nu}\psi^{-1}\partial_{\mu}\psi+\partial^{\nu}\psi^{-1}\partial_{\nu}\partial_{\mu}\psi+\frac{1}{2}\psi^{-1}\partial^{\nu}\partial_{\nu}\partial_{\mu}\psi \tag{6-71}$$

The first term of (6-71) becomes

$$\psi^{-1}\left(\partial^{\nu}\psi\right)\partial_{\nu}\left(\psi^{-1}\partial_{\mu}\psi\right)=\psi^{-1}\left(\partial^{\nu}\psi\right)\left(\partial_{\nu}\psi^{-1}\right)\partial_{\mu}\psi+\psi^{-1}\left(\partial^{\nu}\psi\right)\psi^{-1}\partial_{\nu}\partial_{\mu}\psi$$

$$=\psi^{-1}\left(\partial^{\nu}\right)\left(\partial_{\nu}\psi^{-1}\right)-\left(\partial^{\nu}\psi^{-1}\right)\psi\psi^{-1}\partial_{\nu}\partial_{\mu}\psi$$

$$=\psi^{-1}\left(\partial^{\nu}\right)\left(\partial_{\nu}\psi^{-1}\right)-\left(\partial^{\nu}\psi^{-1}\right)\partial_{\nu}\partial_{\mu}\psi \tag{6-72}$$

The second term of (6-71) vanishes if we use the second term of (6-72)

$$\partial^{\nu}\partial_{\nu}\psi^{-1}=-\partial^{\nu}\left(\psi^{-1}\left(\partial_{\nu}\psi\right)\psi^{-1}\right)$$

$$\partial^{\nu}\partial_{\nu}\psi^{-1}=-\left(\partial^{\nu}\psi^{-1}\right)\left(\partial_{\nu}\psi\right)\psi^{-1}-\psi^{-1}\partial^{\nu}\partial_{\nu}\psi\psi^{-1}-\psi^{-1}\partial_{\nu}\psi\,\partial^{\nu}\psi^{-1}$$

$$\partial^{\nu}\psi^{-1}\partial_{\nu}\psi\,\psi^{-1}=-\partial^{\nu}\psi^{-1}\psi\,\partial_{\nu}\psi^{-1}=+\psi^{-1}\partial^{\nu}\psi\,\partial_{\nu}\psi^{-1}$$

$$=\psi^{-1}\partial_{\nu}\psi\,\partial^{\nu}\psi^{-1}$$

So

$$\partial^{\nu}\partial_{\nu}\psi^{-1}=-2\psi^{-1}\partial^{\nu}\psi\ \partial_{\nu}\psi^{-1}-\psi^{-1}\partial^{\nu}\partial_{\nu}\psi\psi^{-1} \tag{6-73}$$

$$\frac{1}{2}\partial^{\nu}\partial_{\nu}\left(\psi^{-1}\partial_{\mu}\psi\right)=-\psi^{-1}\partial^{\nu}\psi\,\partial_{\nu}\psi^{-1}\partial_{\mu}\psi-\frac{1}{2}\psi^{-1}\partial^{\nu}\partial_{\nu}\partial_{\mu}\psi$$

$$+\partial^{\nu}\psi^{-1}\partial_{\nu}\partial_{\mu}\psi+\frac{1}{2}\psi^{-1}\partial^{\nu}\partial_{\nu}\partial_{\mu}\psi \tag{6-74}$$

$$\psi^{-1}\left(\partial^{v}\psi\right)\partial_{v}\left(\psi^{-1}\partial_{\mu}\psi\right)=\psi^{-1}\left(\partial^{v}\psi\right)\left(\partial_{v}\psi^{-1}\right)\partial_{\mu}\psi-\left(\partial^{v}\psi^{-1}\right)\partial_{v}\partial_{\mu}\psi \qquad (6\text{-}75)$$

When we add (6-74) to (6-75) we find

$$\psi^{-1}\left(\partial^{v}\psi\right)\partial_{v}\left(\psi^{-1}\partial_{\mu}\psi\right)+\frac{1}{2}\partial^{v}\partial_{v}\left(\psi^{-1}\partial_{\mu}\psi\right)=-\psi^{-1}\partial^{v}\psi\,\partial_{v}\psi^{-1}\partial_{\mu}\psi-\frac{1}{2}\psi^{-1}\partial^{v}\partial_{v}\psi\psi^{-1}\partial_{\mu}\psi$$

$$+\partial^{v}\psi^{-1}\partial_{v}\partial_{\mu}\psi+\frac{1}{2}\psi^{-1}\partial^{v}\partial_{v}\partial_{\mu}\psi$$

$$+\psi^{-1}\left(\partial^{v}\psi\right)\left(\partial_{v}\psi^{-1}\right)\partial_{\mu}\psi-\left(\partial^{v}\psi^{-1}\right)\partial_{v}\partial_{\mu}\psi=0$$

So
$$\psi^{-1}\partial^{v}\left(\partial_{v}\psi\right)\psi^{-1}\partial_{\mu}\psi-\psi^{-1}\partial^{v}\partial_{v}\partial_{\mu}\psi=0$$

Using eq (6-73) we can show that

$$\psi^{-1}\partial^{v}\left(\partial_{v}\psi\right)\partial_{\mu}\psi^{-1}\psi+\psi^{-1}\partial^{v}\partial_{v}\partial_{\mu}\psi=0$$

Multiplying this equation by ψ^{-1} to the right and by ψ to the left and using (6-71)

$$\psi\psi^{-1}\partial^{v}\left(\partial_{v}\psi\right)\partial_{\mu}\psi^{-1}\psi\psi^{-1}+\psi\psi^{-1}\partial^{v}\partial_{v}\partial_{\mu}\psi\psi^{-1}=0$$

$$\partial^{v}\left(\partial_{v}\psi\right)\partial_{\mu}\psi^{-1}+\partial^{v}\partial_{v}\partial_{\mu}\psi\psi^{-1}=0$$

where we have used the relation

$$\partial^{v}\partial_{v}\partial_{\mu}=\partial_{\mu}\partial^{v}\partial_{v}$$

So

$$\partial^{v}\partial_{v}\psi\,\partial_{\mu}\psi^{-1}+\partial_{\mu}\partial^{v}\partial_{v}\psi\psi^{-1}=0$$

$$\partial_{\mu}\left[\left(\partial^{v}\partial_{v}\psi\right)\psi^{-1}\right]=0 \qquad (6\text{-}76)$$

If we integrate, we find

$$\left(\partial^{v}\partial_{v}\psi\right)\psi^{-1}+C=0 \qquad (6\text{-}77)$$

when we take the right product by ψ to obtain

$$\partial^{v}\partial_{v}\psi+C\psi=0 \qquad (6\text{-}78)$$

By substituting $C=\frac{m^{2}c^{2}}{\hbar^{2}}$, the equation (6-78) becomes

$$\partial^{v}\partial_{v}\psi+\frac{m^{2}c^{2}}{\hbar^{2}}\psi=0$$

which is the bi-quaternionic Klein-Gordon equation, and we can write

$$\frac{1}{c^{2}}\frac{\partial^{2}\psi}{\partial t^{2}}=\frac{\partial^{2}\psi}{\partial x^{2}}+\frac{\partial^{2}\psi}{\partial y^{2}}+\frac{\partial^{2}\psi}{\partial z^{2}}-\frac{m^{2}c^{2}}{\hbar^{2}}\psi \qquad (6\text{-}79)$$

We now attempt to derive the Dirac equation from the Klein-Gordon equation by using the property of quaternion formalism.

Dirac equation

By using the property of the quaternion and complex imaginary units $e_1^2 = e_2^2 = e_3^2 = i^2 = -1$, where substituting in (6-79) which becomes [19]

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = e_3^2\frac{\partial^2\psi}{\partial x^2}e_2^2 + ie_1^2\frac{\partial^2\psi}{\partial y^2}i + e_3^2\frac{\partial^2\psi}{\partial z^2}e_1^2 \tag{6-80}$$

However, we have used the anticommutative property of the quaternionic units $\left(e_i e_j = -e_j e_i \ \ for\ i \neq j\right)$ and we obtain six vanishing couples of terms.

Then we add this terms to the right-hand side of (6-80), we obtain

$$\begin{aligned}\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\psi}{\partial t}\right) &= e_3\frac{\partial}{\partial x}\left(e_3\frac{\partial\psi}{\partial x}e_2 + e_1\frac{\partial\psi}{\partial y}i + e_3\frac{\partial\psi}{\partial z}e_1 - i\frac{mc}{\hbar}e_3\psi e_3\right)e_2\\ &+ e_1\frac{\partial}{\partial y}\left(e_3\frac{\partial\psi}{\partial x}e_2 + e_1\frac{\partial\psi}{\partial y}i + e_3\frac{\partial\psi}{\partial z}e_1 - i\frac{mc}{\hbar}e_3\psi e_3\right)i\\ &+ e_3\frac{\partial}{\partial z}\left(e_3\frac{\partial\psi}{\partial x}e_2 + e_1\frac{\partial\psi}{\partial y}i + e_3\frac{\partial\psi}{\partial z}e_1 - i\frac{mc}{\hbar}e_3\psi e_3\right)e_1\\ &+ i\frac{mc}{\hbar}\left(e_3\frac{\partial\psi}{\partial x}e_2 + e_1\frac{\partial\psi}{\partial y}i + e_3\frac{\partial\psi}{\partial z}e_1 - i\frac{mc}{\hbar}e_3\psi e_3\right)e_3\end{aligned} \tag{6-81}$$

It seems that eq (6-81) is obtained by applying twice the operator $\frac{\partial}{c\partial t}$ to the bi-quaternionic wave function ψ.

So we obtain

$$\frac{1}{c}\frac{\partial}{\partial t} = e_3\frac{\partial}{\partial x}e_2 + e_1\frac{\partial}{\partial y}i + e_3\frac{\partial}{\partial z}e_1 - i\frac{mc}{\hbar}e_3(\ \ )e_3 \tag{6-82}$$

we show in appendix that the matrices $e_3(\ \ )e_2$, $e_1(\ \ )i$ and $e_3(\ \ )e_1$, are the Conway matrices and can be written in a compact form α^k, with

$$\alpha^k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

where the σ^k being the three Pauli matrices, and $e_3(\ \ )e_3$ is a Dirac β matrix, we substitute the Conway matrices in (6-82) by α^k *and* β, we obtain the non-covariant Dirac equation for a free fermions

$$\frac{1}{c}\frac{\partial\psi}{\partial t} = -\alpha^k\frac{\partial\psi}{\partial x^k} - i\frac{mc}{\hbar}\beta\psi \tag{6-83}$$

Conclusion:

In previous chapters we have noticed that complex numbers enables us to obtain the Schrödinger equation. The aim of this chapter has been to show in explicit manner how the use of the bi-Quaternionic numbers helps us to obtain the Dirac equation.

One of the astonishing remarks which one can mention in this chapter is the link between the laws of physics and the theory of numbers.

Conclusion

In this work we have attempted to obtain the Dirac equation by using the formalism of the scale relativity. In order to reach our goal ,we have followed the following steps.

Chapter 1 is devoted to a review of fractal geometry and scale relativity which lead to define a fractal space- time and introduce new mathematical tools for physics, such as scale-dependent fractal function that allows us to deal with the non differentiability.

Then in chapter 2 we consider the behavior of quantum mechanical paths in the light of the fractal geometry. we have expressed the work of Feynman and Hibbs while were demonstrated that typical quantum mechanical trajectories are characterized by their non differentiability and their fractal structure even the word fractal has not been used . We have also demonstrated that the Heisenberg relation can be translated in terms of the fractal dimension of dimensional four space-time coordinates jumping from D=2 in the quantum and quantum relativistic domain to D=1 in the classical domain . The transition scale has been identified as the de Broglie scale $\lambda_\mu = \frac{\hbar}{p^\mu}$

In chapter 3 we have derived the Schrödinger's equation from Newton's fundamental equation of dynamics without using the tools of quantum mechanics. The method used is the stochastic mechanics according to Nelson.

The chapter 4 we have applied the principle of scale relativity to the quantum mechanics by defining the covariant derivative operator and we have treated some applications.

In chapter 5 we have attempted to write the Schrödinger's equation by using the hypothesis of Nottale also in a same way we have derived the complex Klein-Gorden equation

We end up with chapter 6 where we have derived the Dirac equation from the Newton's equation in the spirit of Nottale hypothesis (the scale covariant derivative)

So we can consider the studies in the previous chapters just as a first step towards a more deep level (all geometrical) where the scale forces are manifestation of fractal geometry and non-differentiability, and we find as a result, lead to a new interpretation of gauge invariance ,and the meaning of the gauge field too.

Prospects and Perspective

Gauge field nature

We consider an electron or any charge particle ,in scale relativity ,which call particle is identify as a geodesic of space –time their trajectories have interns fractals structures ,situate with a resolution $\varepsilon \prec \lambda_c = \hbar/mc$,then we take in account a displacement of the electron ,the principle of scale relativity implicate the appearance of field induce by this displacement to understand, we can give like a model an sight of construction of Einstein gravitation theory from general relativity principle of movement ,in this theory the phenomena of gravitation is identify as a manifestation of space-time curvature which translate by rotation of origin geometrical vector ,indeed on account of space-time non-absolute character , a vector V^{μ} made a displacement dx^{ρ} can not still identical to itself (else, it mean that the space was absolute),so it endure a rotation which write by using the Einstein notation

$$\delta V^{\mu} = \Gamma^{\mu}_{\nu\rho} V^{\nu} dx^{\rho}$$

the Christoffel 's symbols $\Gamma^{\mu}_{\nu\rho}$ appear of course in this transformation, we can calculate it in the following of construction ,in function of derive of metrical potential $g_{\mu\nu}$,which allowed to it be considered as components of gravitation field which generalize Newtonian force

As same as, in the case of electron's fractal structure,we wait that the structure which we find it entailment at certain scale we find it again at other scale after displacement of the electron (in the opposite case the scale's space be absolute, which give a contradiction with the scale relativity principal),so it must appear a dilatation field of resolution induce by the ecclesiastical translation, let us write $e\dfrac{\delta\varepsilon}{\varepsilon} = -A_{\mu}\delta x^{\mu}$

we can write the last equation by covariate derivative

$$eD_{\mu}\ell n(\lambda/\varepsilon) = e\partial_{\mu}\ell n(\lambda/\varepsilon) + A_{\mu}$$

this field of dilatation should be able been define however the scale ,it means however the under-structure was considered, so we take an other scale $\varepsilon' = \rho\varepsilon$ (we considered the Galilean scale relativity) we following the same translation $e\frac{\delta\varepsilon'}{\varepsilon'} = -A'_\mu \delta x^\mu$

The both expression of A_μ are connected by the relation

$$A'_\mu = A_\mu + e\partial_\mu \ell n\rho$$

when $V = \ell n\rho = \ell n(\varepsilon'/\varepsilon)$ characterize relative scale case which explicitly depend to the coordinate, so we are now in the frame of general relativity of scale and the non-linear transformation because the scale velocity which was redefine as a first derivative $\ell n\rho = d\ell nL \ / \ d\delta$

then the equation() including a second derivative of fractal coordinate

$$d^2 \ell nL / \ dx^\mu d\delta$$

So if we now conceder a translation along tow different coordinates, we can write a relation of commutation

$$e(\partial_\mu D_\nu - \partial_\nu D_\mu)\ell n\rho = (\partial_\mu A_\nu - \partial_\nu A_\mu)$$

This relation define a tonsorial field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,which, contrary to A_μ ,is independent to the initial scale

Charge's Nature

In a gauge transformation $A'_\mu = A_\mu - \partial_\mu \chi$,the wave function of a electron of charge (e) become $\psi' = \psi e^{ie\chi}$,the gauge function is the conjugated variable of the charge as the position ,the time and the angle are the conjugated variable respectively of the impulsion ,the energy and the angular moment in the action expressions or in the quantum phase of free particle ,$\theta = i\frac{(px - Et + \sigma\varphi)}{\hbar}$,we know that the impulsion ,the energy and the angular moment are yielding from the space–time nature consequence to its symmetry (translation and rotation) according to Noether's theory,

In the precedent paragraph we reinterpret the gauge transformation as a scale transformation of resolution $\varepsilon \to \varepsilon', \ell n\,\rho = \ell n\left(\frac{\varepsilon}{\varepsilon'}\right)$

In such interpretation the property specific characterize a charged particle is the explicit dependent of the action to the scale it means the wave function with the resolution

So

$$\psi' = \psi e^{i\frac{e^2}{\hbar c}\ell n\,\rho}$$

Now we consider the electron's wave function as a function dependant explicitly with the ratio of the resolution, we can write the differential equation of scale among ψ is solution with the form

$$-i\hbar\frac{\partial\psi}{\partial\left(\frac{e}{c}\ell n\,\rho\right)} = e\psi$$

$\widetilde{D} = -i\hbar\frac{\partial}{\partial\left(\frac{e}{c}\ell n\,\rho\right)}$ is the dilatation operator, the differential equation we can write it as a equation of values proper

$$\widetilde{D}\psi = e\psi$$

So the electrical charge be consider the conservative quantity which yield from the new symmetry of scale, namely, the uniformity of the variable of the resolution $\ell n\,\varepsilon$

Appendix A
The Laws of Scale Transformation

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The Laws of Scale Transformation

In Galilean motion relativity we write the relative velocity as

$$\vec{V}_2 - \vec{V}_1 = \left(\vec{V}_2 - \vec{V}_0\right) - \left(\vec{V}_1 - \vec{V}_0\right) \tag{A-1}$$

We also may write the relative scale as a ratio

$$\rho = \frac{\Delta x_2}{\Delta x_1} = \left(\frac{\Delta x_2}{\Delta x_0}\right)\left(\frac{\Delta x_0}{\Delta x_1}\right) \tag{A-2}$$

It clear that the length of object is define as scale ratio which have physical meaning

In logarithmic representation we have

$$\ell n\rho = \ell n\left(\Delta x_2 / \Delta x_1\right) = \ell n\left(\Delta x_2 / \Delta x_0\right) - \ell n\left(\Delta x_1 / \Delta x_0\right) \tag{A-3}$$

From the equation (A-3) seems that the scale state $V = \ell n\left(\Delta x_2 / \Delta x_1\right)$ is formally equivalent to a velocity.

In accordance to the assumption which treats state of scale and motion as a same, and one we speak to the velocity of the system, we never speak to the absolute velocity, but we speak to the velocity of system relative to an other system, as the velocity the scale of the system can be defined by its ratio to the scale of an other system

We can now write the equation of the scale state

$$V = V_2 - V_1 = \left(V_2 - V_0\right) - \left(V_1 - V_0\right) \tag{A-4}$$

Consider now a φ which transforms under a dilatation $q = \Delta x / \Delta x'$ as

$$\varphi' = \varphi\, q^{\delta} \tag{A-5}$$

We can write the equation above in a linear form

$$\ell n\left(\varphi' / \varphi_0\right) = \ell n\left(\varphi / \varphi_0\right) + \delta\ \ell n\left(\Delta x / \Delta x'\right) \tag{A-6}$$

We assume that the resolution $\Delta x \prec\prec \lambda$ ($\lambda = \frac{\hbar}{p}$ de Broglie length)

We have the Galilean motion transformation

$$x' = x + vt \tag{A-7}$$

$$t' = t$$

When we compare the equations (A-6) and (A-7), we obtain the correspondences

$$x = \ell n\left(\frac{\varphi}{\varphi_0}\right)$$

$$t = \delta$$

$$v = \ell\left(\frac{\Delta x}{\Delta x'}\right)$$

we are particularly interested in the case where $\psi = \ell \ or \ \psi = t$

We showed in chapter 1 that the length of a quantum particle diverges as (A-1-9)

In (A-1-9) $X = \ell n(\ell)$

$t = \delta$

$V = \ell n\left(\frac{\lambda}{\Delta x}\right)$

Then we define the state of scale as

$$V = \ell n\left(\frac{\lambda}{\Delta x}\right) = \frac{d(\ell n(\ \ell))}{d\delta} = \frac{dX}{d\delta} \qquad \text{(A-1-13)}$$

The scale laws (A-1-9) are formerly equivalent to the laws of free motion at constant velocity. We assume that the coordinate system is described by its state of motion and also by its scale. A coordinate system which verifies the equation (A-1-9) is called scale inertial system.

We can suggest now that the laws of nature are identical in all scale inertial systems of coordinates

The anomalous dimension δ is assumed to be invariant as time is invariant in Galilean relativity, where we describe by the equation of the Galilean scale-inertial transformation

$$X' = X + V\delta \qquad \text{(A-14)}$$

$$\delta' = \delta \qquad \text{(A-15)}$$

The law of composition of scale state is, as the velocity, the direct sum

$$W = U + V \qquad \text{(A-16)}$$

which corresponds to the product $\Delta x''/\Delta x = (\Delta x''/\Delta x')(\Delta x'/\Delta x)$ for the resolution.

A-2 Special Scale Relativity

We know that the Lorentez transformation derived from the successive assumptions: linearity, invariance of c (the speed of light), the composition law, existence a neutral element, and the reflection invariance, but we can show that may be derived it from only of the linearity, composition law and reflection invariance

We write the linear transformation of coordinates as

$$x' = a(v)x - b(v)t \tag{A-17}$$

$$t' = \alpha(v)t - \beta(v)x \tag{A-18}$$

We divide on $a(v)$, and we define the velocity $v = \dfrac{b(v)}{a(v)}$ the linear transformation become

$$x' = \gamma(v)(x - vt) \tag{A-19}$$

$$t' = \gamma(v)(A(v)t - B(v)x) \tag{A-20}$$

Where $\gamma(v) = a(v)$, A and B are a new functions of v

Let us take two successive transformations

$$x' = \gamma(u)(x - ut) \qquad \text{(A-21-a)}$$

$$t' = \gamma(u)(A(u)t - B(u)x) \qquad \text{(A-21-b)}$$

$$x'' = \gamma(v)(x' - vt') \qquad \text{(A-22-a)}$$

$$t = \gamma(v)(A(v)t' - B(v)x') \qquad \text{(A-22-b)}$$

We substitute (A-21) in (A-22), we find

$$x'' = \gamma(u)\gamma(v)[1 + B(u)v]\left[x - \frac{u + A(u)v}{1 + B(u)v}t\right] \qquad \text{(A-23-a)}$$

$$t'' = \gamma(u)\gamma(v)[A(u)A(v) + B(v)u]\left[t - \frac{A(v)B(u) + B(v)}{A(u)A(v) + B(v)u}x\right] \qquad \text{(A-23-b)}$$

Then the principle of relativity means that the composed transformation (A-23) keeps the same form (A-21)

$$x'' = \gamma(v)(x - wt) \qquad \text{(A-24-a)}$$

$$t'' = \gamma(v)(A(w)t - B(w)x) \qquad \text{(A-24-b)}$$

We compare (A-24) and (A-23), we find

$$w = \frac{u + A(u)v}{B(u)v} \qquad \text{(A-25-a)}$$

$$\gamma(w)=\gamma(u)\gamma(v)\left[1+B(u)v\right] \tag{A-25-b}$$

$$\gamma(w)A(w)=\gamma(u)\gamma(v)\left[A(u)A(v)+B(v)u\right] \tag{A -25-c}$$

$$\frac{B(w)}{A(w)}=\frac{A(v)B(u)+B(v)}{A(u)A(v)+B(v)u}$$

So we now use the reflection invariance, it means use the transformations $(x\to -x, x'\to -x')$

$$-x'=\gamma(u')(-x-u't)$$

$$t'=\gamma(u')(A(u')t+B(u')x)$$

We compare to (A-21) and take $u'=-u$ consequent to the reflection invariance, we find

$$\gamma(-v)=\gamma(v) \tag{A-26}$$

$$A(-v)=A(v) \tag{A-27}$$

$$B(-v)=B(v) \tag{A-28}$$

Combining the equations (A-25) yields the equation

$$A\left[\frac{u+A(u)v}{1+B(u)v}\right]=\frac{A(u)A(v)+B(v)u}{1+B(u)v} \tag{A-29}$$

Substitute by $v=0$ in the equation, we obtain

$$A(u)\left[1-A(0)\right]=uB(0) \tag{A-30}$$

If $u=0$, we find two solutions $A(0)=0 \quad or \quad A(0)=1$

$$A(0)=0\Rightarrow A(u)=uB(0) \quad with \quad B(0)=0$$

$$\Rightarrow A(u)=0 \tag{A-31}$$

The equation (A-25-d) becomes

$$\frac{B(w)}{A(w)}=\frac{1}{u}\Leftrightarrow A(w)=uB(w) \tag{A-32}$$

$$A(0)=1\Rightarrow B(0)=0 \tag{A-33}$$

Let us now take $v=-u$ in (A-29)

$$A\left[\frac{u-A(u)u}{1-B(u)u}\right]=\frac{A(u)A(u)+B(-u)u}{1-B(u)u} \tag{A-34}$$

We use the property $A(-v)=A(v)$, and the new function F define as

$$F(u)=A(u)-1\Rightarrow A(u)=F(u)+1 \tag{A-35}$$

F verifies the next equation

$$F(0)=0 \tag{A-36}$$

From the equation (A-34) we obtain

$$F\left[\frac{u-A(u)u}{1-B(u)u}\right]+1=\frac{(F(u)+1)^2+B(-u)u}{1-B(u)u} \tag{A-37}$$

$$F\left[\frac{F(u)u}{1-B(u)u}\right]=\frac{F(u)F(u)+2F(u)}{1-B(u)u} \tag{A-38}$$

$$F\left[\frac{F(u)u}{1-B(u)u}\right]=2F(u)\frac{1+F(u)/2}{1-B(u)u} \tag{A-39}$$

We now use the continuity of the F and B at $u = 0$, which means that $\Phi(u) = 1 + F(u)/2$ and $\psi(u) = 1 - B(u)u$ which implies that

$\exists \varepsilon \succ 0 \; then \left|\Phi(u) - \Phi(0)\right| \prec \varepsilon \; and \left|\psi(u) - \psi(0)\right| \; with \; \Phi(0) = 1 and \psi(0) = 1$

Next

$$1 - \varepsilon \prec 1 - B(u)u \prec 1 + \varepsilon$$
$$1 - \varepsilon \prec 1 + F(u)/2 \prec 1 + \varepsilon$$

The function $\Phi(u) \; and \; \psi(u)$ were bounded when $u \in [-\eta_0 \quad \eta_0]$

So we can writ

$$k_1 \prec \Phi \prec k_2 \; and \; k_3 \prec \psi \prec k_4$$
$$k_1 = k_2 = 1 - \varepsilon \quad and \quad k_3 = k_4 = 1 + \varepsilon \quad and \quad \varepsilon \in]0,1[$$

The bounds on $1 + F(u)/2$ and $1 - B(u)u$ change the equation (A-2-23) to the equation

$$2F(u) = F(F(u)u)$$

The continuity of F at u it means

$$\forall \varepsilon \succ 0, \exists \eta \succ 0 \quad then \quad |u| \prec \eta \Rightarrow |F(u)| \prec \varepsilon$$

We put $$F_0 = F(u_0) = 2^{-n} \prec \varepsilon$$

We have $$2F_0 = F(u_0 F_0)$$

$$u_1 = u_0 F_0 = 2^{-n} u_0 \prec u_0 \Rightarrow F(u_1) = 2F_0 = 2^{1-n}$$

After p iteration we obtain

$$F(u_p) = 2^{p-n}$$

And after n iterations

$$F(u_n) = 2^{n-n} = 1 \succ \frac{1}{2}$$

This is in contradiction with the continuity of F since $F(u_n) \succ \varepsilon$

So the composition of velocity in the equation (A-25-a) take the form

$$w = \frac{u + v}{1 + kuv} \qquad \text{(A-40)}$$

The equation (A-25-b) becomes

$$\gamma\left(\frac{u + v}{1 + kuv}\right) = \gamma(u)\gamma(v)(1 + kuv) \qquad \text{(A-41)}$$

We put $u = -v$ the equation (A-41) becomes

$$\gamma(0) = \gamma(v)\gamma(-v)(1 - kv^2) \qquad \text{(A-42)}$$

For $v = 0$

$$\gamma(0) = [\gamma(0)]^2 \Rightarrow \gamma(0) = 1$$

So $$\gamma(v)\gamma(-v) = \frac{1}{1 - kv^2} \qquad \text{(A-43)}$$

When we consider the reflection invariance, which implies $\gamma(v) = \gamma(-v)$, we obtain

$$\gamma(v) = \frac{1}{\sqrt{1 - kv^2}} \qquad \text{(A-44)}$$

We remark that if we put $k = 0$, we find $\gamma(v) = 1$, which describe the Galilean transformation, and if $k = c^2$ (A-44) become $\gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$, which describe the transformation of Lorentz

3-Lorentz Scale Transformation

So we showed how we can derive the Lorentz transformation from only the linearity ,the internal composition and reflection invariant, which lead to the thought that the laws of scale transformation must be also take the Lorentz form ,instead of the Galilean form
We give the Lorentz law of composition of velocities

$$w = \frac{u + v}{1 + \frac{uv}{c^2}} \tag{A-45}$$

Which we can be written it in other way as

$$\frac{w}{c} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{uv}{c^2}} \tag{A-46}$$

We put $U = \frac{u}{c}$, $V = \frac{v}{c}$, $W = \frac{w}{c}$, which are dimensionless quantities

So
$$W = \frac{U + V}{1 + UV} \tag{A-47}$$

We replace W, U and V by logarithms of other numbers taken in base k (A-47) becomes

$$\ell og_k \mu = \frac{\ell og_k \rho + \ell og_k \nu}{1 + \ell og_k \rho \ \ell og_k \nu} \tag{A-48}$$

We divide both members of (A-48) by $\ell n\ k$, we find

$$\ell n \mu = \frac{\ell n \rho + \ell n \nu}{1 + \frac{\ell n\ \rho\ \ell n\ \nu}{\ell n^2 k}} \tag{A-49}$$

Appendix B
Some Explicit Calculations

In the following we shall compute the covariant derivation of a product .We has

$$\frac{d}{dt}(f.g)=\left(\frac{\partial}{\partial t}+\vec{v}\vec{\nabla}-iD\Delta\right)(f.g) \tag{B-1-1}$$

$$\frac{d}{dt}(f.g)=\frac{\partial}{\partial t}(f.g)+\vec{v}\vec{\nabla}(f.g)-iD\Delta(f.g)$$

$$\frac{d}{dt}(f.g)=f\frac{\partial}{\partial t}g+g\frac{\partial}{\partial t}f+\vec{v}f\vec{\nabla}g+\vec{v}g\vec{\nabla}f-iD\vec{\nabla}\left[\vec{\nabla}(f.g)\right]$$

$$\frac{d}{dt}(f.g)=f\frac{\partial}{\partial t}g++\vec{v}f\vec{\nabla}g+g\frac{\partial}{\partial t}f+\vec{v}g\vec{\nabla}f-iD\vec{\nabla}\left[f\vec{\nabla}g+g\vec{\nabla}f\right]$$

$$\frac{d}{dt}(f.g)=f\frac{\partial}{\partial t}g++\vec{v}f\vec{\nabla}g+g\frac{\partial}{\partial t}f+\vec{v}g\vec{\nabla}f-iD\vec{\nabla}f\vec{\nabla}g-iD\vec{\nabla}g\vec{\nabla}f-iDf\Delta g-iDg\Delta f$$

$$\frac{\partial}{\partial t}(f.g)=f\left[\frac{\partial}{\partial t}+\vec{v}\vec{\nabla}-iD\Delta\right]g+g\left[\frac{\partial}{\partial t}+\vec{v}\vec{\nabla}-iD\Delta\right]f-2iD\vec{\nabla}f\vec{\nabla}g$$

So

$$\frac{\partial}{\partial t}(f.g)=f\frac{\partial}{\partial t}g+g\frac{\partial}{\partial t}f-2iD\vec{\nabla}f\vec{\nabla}g \tag{B-1-2}$$

Now we attempt to prove the following relation which it was used in the second chapter .So in the formalism of path integral they give the average of a function $F(x(t))$ by

$$\langle F(x(t))\rangle=\int D(x(t))F(x(t))\,e^{\frac{i}{\hbar}S(x(t))} \tag{B-2-1}$$

$S(x(t))$ is the classical action

If we displace x(t) by $\eta(t)$ fixed

$$D(x+\eta)=D(x(t)) \tag{B-2-2}$$

$$F(x(t)+\eta(t))=F(x(t))+\int\eta(s)\frac{\delta F(s)}{\delta x(s)}ds \tag{B-2-3}$$

Which gives

$$\delta F=\int\eta(s)\frac{\delta F}{\delta x(s)}ds \tag{B-2-4}$$

The average does not affect by the displacement $\eta(t)$, so

$$\langle F(x(t))\rangle=\langle F(x(t)+\eta(t))\rangle \tag{B-2-5}$$

$$\left\langle F(x+\eta)\right\rangle = \int D(x) F(x+\eta) e^{\frac{i}{\hbar}S(x+\eta)} \qquad \text{(B-2-6)}$$

$$\left\langle F(x+\eta)\right\rangle = \int D(x)\left[\left(F(x)+\int \eta(s)\frac{\delta F(s)}{\delta x(s)}ds\right)e^{\frac{i}{\hbar}\left[S+\int \eta \frac{\delta S}{\delta x}ds\right]}\right] \qquad \text{(B-2-7)}$$

If we develop the exponential function we obtain

$$\left\langle F(x+\eta)\right\rangle \approx \int D(x)\left[\left(f(x)+\int \eta(s)\frac{\delta F}{\delta x}ds\right)e^{\frac{i}{\hbar}S}\left(1+\frac{i}{\hbar}\int \eta(s)\frac{\delta S}{\delta x}ds\right)\right]$$

$$\left\langle F(x+\eta)\right\rangle \approx \int D(x)\left[F(x)\; e^{\frac{i}{\hbar}} + \frac{i}{\hbar}F(x)\; e^{\frac{i}{\hbar}S}\int \eta(s)\frac{\delta S}{\delta x}ds + e^{\frac{i}{\hbar}}\int \eta(s)\frac{\delta F}{\delta x}ds\right] \text{(B-2-8)}$$

We denote that

$$\int \eta(s)\frac{\delta S}{\delta x}ds = \delta S \qquad \text{(B-2-9)}$$

So

$$\left\langle F(x+\eta)\right\rangle = \left\langle F(x)\right\rangle + \frac{i}{\hbar}\left\langle F\delta S\right\rangle + \left\langle \delta F\right\rangle \qquad \text{(B-2-10)}$$

From (B-2-5) we conclude that

$$\left\langle \delta F\right\rangle = -\frac{i}{\hbar}\left\langle F\delta S\right\rangle \qquad \text{(B-2-11)}$$

Appendix C
Quaternions

We showed in the chapter six how the Dirac equation was naturally obtained from the Klein-Gordon equation when written in a quaternionic form ,to we get a good understanding we must be know the quaternion and there properties

Definition

A bi-quaternion $\Phi = (\Phi_0, \Phi_1, \Phi_2, \Phi_3)$ is a mathematical being compote to a four complexes numbers components, $\Phi_i \; i = 0, \ldots, 3$

Algebraic properties

We have two quaternions $\Phi = (\Phi_0, \Phi_1, \Phi_2, \Phi_3)$ and $\psi = (\psi_0, \psi_1, \psi_2, \psi_3)$

$$\Phi = 0 \Rightarrow \Phi_i = 0 \text{ , } with \; i = 0,1,2,3$$

So the zero quaternion as $0 = (0, 0, 0, 0)$

$$\Phi = \psi \Rightarrow \Phi_i = \psi_i \text{ , } with \; i = 0,1,2,3$$

We have a complex number α

The multiplication of a quaternion by a complex number α, is write

$$\Phi\alpha = (\alpha\Phi_0, \alpha\Phi_1, \alpha\Phi_2, \alpha\Phi_3)$$

$$\Phi\alpha = \alpha\,\Phi = (\alpha\Phi_0, \alpha\Phi_1, \alpha\Phi_2, \alpha\Phi_3) \quad \text{the commutative property}$$

$$\alpha\,(\Phi + \psi) = \alpha\,\Phi + \alpha\,\psi \quad \text{the distributive property}$$

$$(\alpha + \beta)\Phi = \alpha\,\Phi + \beta\,\Phi \quad \text{the associative property} \qquad \text{(c-1)}$$

Addition of quaternions is commutative and associative

$$\Phi + \psi = \psi + \Phi = (\Phi_0 + \psi_0, \Phi_1 + \psi_1, \Phi_2 + \psi_2, \Phi_3 + \psi_3)$$

$$\Phi + (\psi + \varphi) = (\Phi + \psi) + \varphi$$

$$\Phi - \psi = \Phi + (-1)\psi \qquad \text{(c-2)}$$

The quaternionic product $\Phi.\psi$ of two quaternions is a quaternion

For any complex. Number. α

$$(\alpha\Phi)\psi = \Phi(\alpha\psi) = \alpha(\Phi\,\psi) \qquad \text{(c-3)}$$

We can write the quaternion as

$$\Phi = \Phi_0 + e_1\Phi e_1 + e_2\Phi e_2 + e_3\Phi e_3 \qquad \text{(c-4)}$$

Where the units e_i satisfy the following property

$$e_i e_i = -\delta_{ij} + \sum_k^3 = 1\varepsilon_{ijk} e_k \qquad \text{(c-5)}$$

And ε_{ijk} is antisymmetric three-index with $\varepsilon_{123} = 1$

From these rules, we find that the product of two arbitrary quaternions is

$$\Phi\psi = (\Phi_0\psi_0 - \Phi_1\psi_1 - \Phi_2\psi_2 - \Phi_3\psi_3, \Phi_0\psi_1 + \Phi_1\psi_0 + \Phi_2\psi_3 + \Phi_3\psi_2 + \Phi_0\psi_2 + \Phi_2\psi_0)$$

$$+ \Phi_3\psi_1 - \Phi_1\psi_3, \Phi_0\psi_3 + \Phi_3\psi_0 + \Phi_1\psi_2 - \Phi_2\psi_1) \qquad \text{(c-6)}$$

This product is not in general commutative, but is associative

We define quaternion conjugate as

$$\Phi \to \overline{\Phi} = \left(\Phi_0, -\Phi_1, -\Phi_2, -\Phi_3\right) \tag{c-7}$$

From (c-6), we find that for any Φ, ψ

$$\overline{\Phi}\,\overline{\psi} = \overline{\psi}\overline{\Phi} \tag{c-8}$$

One also defines the scalar product of quaternions

$$\Phi.\psi = \frac{1}{2}\left(\Phi\,\overline{\psi} + \psi\overline{\Phi}\right) = \frac{1}{2}\left(\overline{\Phi}\,\psi + \overline{\psi}\Phi\right) = \Phi_0\psi_0 + \Phi_1\psi_1 + \Phi_2\psi_2 + \Phi_3\psi_3 \tag{c-9}$$

From the scalar product we obtain the norm of a quaternion as

$$\Phi.\overline{\Phi} = \overline{\Phi}\Phi = \Phi_0^2 + \Phi_2^1 + \Phi_2^2 + \Phi_3^2 \tag{c-10}$$

The norm and scalar product are numbers, in general complex

The norm of a product is the product of the norms

$$\overline{\left(\Phi\,\psi\right)}\left(\Phi\,\psi\right) = \left(\Phi\,\psi\right)\overline{\left(\Phi\,\psi\right)} = \left(\Phi\overline{\Phi}\right)\left(\psi\overline{\psi}\right) \tag{c-11}$$

When $\Phi\overline{\Phi} = 0 \Rightarrow \Phi = 0$ or Φ is singular

If $\Phi\overline{\Phi} = 1, \Phi$ is unimodular.

The inverse of Φ since $\Phi \neq 0$ *and* Φ is not singular is defined as

$$\Phi^{-1} = \frac{\overline{\Phi}}{\Phi\overline{\Phi}} \tag{c-12}$$

We have the properties

$$\left(\Phi\,\psi\right)^{-1} = \psi^{-1}\Phi^{-1} \tag{c-13}$$

$$\Phi\Phi^{-1} = \Phi^{-1}\Phi \tag{c-14}$$

The hermitian conjugate of a quaternion Φ is the quaternion

$$\Phi^{+} = \left(-\Phi_0^*, \Phi_1^*, \Phi_2^*, \Phi_3^*\right) \tag{c-15}$$

$$\left(\Phi\,\psi\right)^{+} = \psi^{+}\Phi^{+} \tag{c-16}$$

Quaternion Φ is hermitic if $\Phi^{+} = \Phi$

The complex reflection of a quaternion Φ is the quaternion

$$\Phi^{\times} = \left(-\Phi_1^*, -\Phi_2^*, -\Phi_3^*, -\Phi_4^*\right) \tag{c-17}$$

$$\left(\Phi\,\psi\right)^{\times} = \Phi^{\times}\psi^{\times} \tag{c-18}$$

The effect of applying any two of the operations -, +, * to a quaternion is the same as that of applying the third; for example

$$\left(\Phi^{\times}\right)^{+} = \left(\Phi^{+}\right)^{\times} = \overline{\Phi} \tag{c-19}$$

Note too that, for any complex number a,

$$\left(a\overline{A}\right) = a\overline{A}, \left(aA\right)^{-1} = a^{-1}A^{-1}, \left(aA\right)^{+} = a^{*}A^{+}\left(aA\right)^{\times} = a^{*}A^{\times} \tag{c-20}$$

Quaternions and Conway Matrices

We have four quaternions Φ , ψ , a and b, we consider the transformation again;

$$\Phi \to \psi = a\Phi b \tag{c-21}$$

This transformation is linear for the elements $\left(\Phi_0, \Phi_1, \Phi_2, \Phi_3\right)$, so we can be written in matrix form

$$\psi = M(a,b)\Phi \qquad \text{(c-22)}$$

Where M is a 4×4 matrix and Φ and ψ are 4×1 column matrices the notation indicates that $M(a,b)$ is determined by a and b if we

We now consider tow transformations,

$$\psi = a\Phi b \; and \chi = c\psi d \Rightarrow \chi = ca\Phi bd \qquad \text{(c-23)}$$

Which be written in matrix form

$$\psi = M(a,b)\Phi \; and \; \chi = M(c,d)\psi \Rightarrow \chi = M(c,d)M(a,b)\Phi \qquad \text{(c-24)}$$

$$\chi = ca\Phi bd \Rightarrow \chi = M(ca,bd)$$

So, when we compare the tow parts of (c-24) we find

$$M(c,d)M(a,b) = M(ca,bd) \qquad \text{(c-25)}$$

For $a = c$ and $b = d$, eq (c-25) becomes

$$M(a,b)^2 = M(a^2,b^2) \qquad \text{(c-26)}$$

If we put $a = b = -1$ in eq(c-21) we obtain

$$\psi = (-1)\Phi(-1) = \Phi$$

$$\psi = M(-1,-1)\Phi = \Phi \Rightarrow M(-1,-1) = I \qquad \text{(c-27)}$$

Where I is the unit matrix

If $a^2 = b^2 = 1$, we can write

$$M(a,b)^2 = M(a^2,b^2) = M(-1,-1) = I \qquad \text{(c-28)}$$

From the unit matrix Conway suggested proposes that we can write $M(a,b)$ as

$$M(a,b) = a(\;)b \qquad \text{(c-29)}$$

Now, by using the quaternions (e_1, e_2, e_3, i), which satisfy eq(c-28) we can derive sixteen matrices, but we care to the four matrices which we will use in chapter 6

$$e_3(\;)e_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} e_1(\;)e_i = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$e_3(\;)e_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} e_3(\;)e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It seems that the above matrices satisfy the suggestion of Conway

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Résumé

Le but de ce mémoire est de d'essayer de trouver un lien entre la mécanique classique et la mécanique quantique. Le but de la physique moderne est retrouver la mécanique quantique comme limite de la mécanique classique. L'une des théories contemporaines proposées est la théorie de Nottale dont le but est de reformuler les équations de la mécanique quantique directement de la mécanique classique (le principe fondamental de la dynamique) sans passer par aucun des postulats de la mécanique quantique.

La théorie de Nottale – la relativité d'échelle- consiste dans une généralisation du concept de la relativité pour incorporer en plus de la relativité du mouvement – sur laquelle elle est basée la relativité d'Einstein- une autre relativité celle de l'échelle ou la résolution.

Dans le cadre de la relativité d'échelle l'espace-temps devient fractal d'où la non différentiabilité des coordonnées.

En utilisant la définition de la dérivée fractale covariante qui est une généralisation de la dérivée covariante de la théorie de jauge, on a pu obtenir les équations de Schrödinger, Klein-Gordon et Dirac à partir de la mécanique Newtonienne. Les résultats obtenus montre un lien étroit entre les équations de la physique et la théorie des nombres.

Mots Clefs:

Géométrie Fractale. Relativité d'échelle. La dimension de Haussdorff. Les nombres quaternioniques.

ملخص

هدف هذه المذكرة هو محاولة لفهم العلاقة بين الميكانيكا الكلاسيكية و ميكانيكا الكم ، فالمعلوم انه لا توجد إلى حد الآن نظرية تحاول فهم الميكانيكا الكمية اعتبارا من الميكانيكا الكلاسيكية أو العكس..النظرية المطروحة في هذه المذكرة هي نظرية نوتال و هي محاولة لإعادة صياغة بعض معادلات ميكانيكا الكم انطلاقا من الميكانيكا الكلاسيكية (المبدأ الأساسي للتحريك) بدون استعمال أي مسلمة من مسلمات ميكانيكا الكم.

نستعمل نظرية نوتال و المسماة نسبية السلم و التي تعتمد على تعميم مبدأ النسبية لاينشتاين لتشمل السلم حيث ندمج نسبية الحركة مع نسبية السلم.

في إطار نسبية السلم الزمن-مكان يصبح ذو هندسة كسورية التي تعتمد على لا تفاضل الإحداثيات.

باستعمال مفهوم المشتقة الصامدة بالنسبة لتحويلات السلم و هي تعميم للمشتقة محافظة التغاير المستعملة في نظرية العيارية كي تتحصل على معادلة شرودينغر و معادلة كلاين قوردن ومعادلة ديراك انطلاقا من معادلة نيوتن .

بينت النتائج المتحصل عليها العلاقة الوثيقة بين المعادلات الفيزيائية و نظرية الأعداد.

الكلمات المفتاحية

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Abstract :

The aim of this dissertation is to find a link between quantum and classical mechanics. On of the modern theories which seek to find this link is the scale relativity of L. Nottale. The aim of this theory is to reformulate some of quantum mechanical equations directly from classical mechanics without any use of the postulates of quantum mechanics.

We have used the Nottale theory based of the generalization of the concept of relativity to incorporate another type of relativity: the scale relativity in addition to ordinary relativity of Einstein based on motion relativity. In scale relativity the spacetime will be fractal, which leads to non differentiability of the coordinates.

By using a definition of a covariant fractal derivative proposed by Nottale which similar to the covariant derivative encountered in gauge theories, we can obtain the Schrödinger, Klein-Gordon and Dirac equation from Newtonian Mechanics.

The results obtained in this dissertation show a close relationship between laws of physics and number theory.

Key Words:

Fractal Geometry. Scale relativity. The Haussdorf dimension. Quaternion numbers.