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Loop Quantum Gravity and its Geometrical Applications

Par

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Contents

List of Figures

Chapter 1

Introduction

During the last years, the search for a quantum version of gravity was a challenge that physicists had seriously taken, they have never stopped to understand the geometry of spacetime at the infinitesimally scales. It seems the only way to solve this problem can be done by combining general relativity (GR) [1] with the quantum effect of quantum mechanics (QM) [2]. In the 60s, J. Wheeler and B. DeWitt presented a first attempt to proceed spacetime dynamics with quantization program from a canonical point of view [3], they used the induced spatial 3-metric as a configuration variable. Unfortunately, this program encountered several major difficulties due to the problem of introducing an invariant measure on the metric space. In the late 80s, an intense interest to a new quantum geometry called Loop Quantum Gravity (LQG) has been devoted [4, 5, 6, 7, 8, 9, 10]. Loop Quantum Gravity is a non-perturbative[1], background-independent[2] and a quantum field theory of geometry itself. It is based on the quantum implementation of the Holst Hamiltonian[3] by using Dirac quantization program [11, 12], with the Ashtekar-Barbero variables [13, 14].

In order to construct the starting kinematical Hilbert space for Loop Quantum Gravity, one has to use the well known representation of the holonomy-flux algebra

[1]LQG theory quantizes the full metric without taking any perturbative fluctuations.

[2]LQG theory conserves the diffeomorphism symmetry of GR.

[3]Holst formulation is an equivalence way to describe general relativity. It will be discussed in section 2.7.

[15]. It is represented by the space of all cylindrical wave functional through holonomies defined by the $su(2)$ connection along a system of smooth oriented paths and flux variables as the smeared electric field along the dual surface for each path. After we introduce the invariant Haar measure of the space of holonomies, the kinematical Hilbert space will be well-defined. A useful basis state of the quantum geometry known as the Penrose's spin networks is frequently used [16]. Spin network arises as a generalization of Wilson loops [17] necessary to deal with mutually intersecting loops "nodes" which is represented by a space of intertwiners at each node. One can construct well defined operators such as the area and volume acting on links and nodes of smooth paths system respectively. From the spectrum of the geometrical operators, the fuzziness and discreteness property of space is predicted [18, 19, 20, 21, 22, 23, 24, 25].

In this thesis, we will construct a new geometrical information from the spin network states, based on the polyhedra interpretation of intertwiners [26, 27], which is the value of the 3d-Ricci scalar curvature and the edge length as a function of volume and boundary areas operators [28]. The main idea of our work comes from the determination of the volume and the boundary area of a fixed region in a Riemannian manifold as a function of the scalar curvature inside that region as well as its parameterization. One can then invert the resulting functions to get the explicit formula of the scalar curvature in terms of volume and boundary area of a fixed region. Similar idea was explored by using a geodesic polyhedron shape[4] [29]. Thus, we can use the new proposed scalar curvature operator related to a fixed polyhedron measure and try to determine its spectrum in order to know what kind of space in which the intertwiner state is represented. Thus, one can describe the intertwiner state by a curved chunk of a curved polyhedron. In our approach, an example of a such monochromatic 4-valent node state was studied in details

[4]Geodesic polyhedron is the convex region enclosed by the ntersection of geodesic surfaces. A geodesic surface is a surface with vanishing extrinsic curvature and the intersection of two such surfaces is necessarily a geodesic curve. Geodesic tetrahedron is a special case of a geodesic polyhedron with four faces.

and its associated Kapovich-Millson phase space [26, 27] (i.e. the space of all equilateral Euclidean tetrahedron shapes) was constructed. Moreover, we will show the absence of a regular Euclidean tetrahedron from the volume orbit of relevant shapes in that phase space, instead of this it is possible to find a regular tetrahedron correspondence in the context of a non-zero constant curvature tetrahedron. It is worth to mention that the phase space of curved tetrahedron shapes idea has been initiated in ref. [30]. In our present paper [29], full expressions of volume and boundary face area of a regular tetrahedron in a constant curvature space (in terms of the scalar curvature and the edge length) are explicitly derived than inverted to get the exact form of the 3d- Ricci scalar curvature and the edge length. At the quantum level, we obtain two well defined operators acting on the monochromatic 4-valent nodes state. Their spectra show that all quantum atoms of space can be represented by chunks of regular hyperbolic tetrahedron of a negative curvature. It also produces the Euclidean regular tetrahedron in the semi-classical limit.

This thesis is organized as follows:

- In chapter 2: we will explore the Hamiltonian formulation of GR starting from the mathematical tools: Hyper-surfaces manifold 2.1 and spacetime foliation 2.2 so we can proceed further to the ADM fomalism 2.3. Then we will also study the first order formulation of GR: Palatini 2.5, Holst 2.7 actions and their Hamiltonian analysis 2.6, 2.8 respectively.

- In chapter 3: we begin by giving an overview of the Dirac quantization program 3.1 then we will construct the notion of the holonomy-flux variables to represents gravity at the quantum scales 3.2. The remaining steps 3.4, 3.6 and 3.7 are to solve the Einstein equation to finally complete the discussion of LQG.

- In chapter 4: The geometrical operator in LQG will be discussed in details: the area for a given surface 4.1 and the volume for a given region 4.2. Then we will explore the geometrical interpretation of quantum geometry state 4.3.

- In chapter 5: we will give a motivation for a new scalar curvature measure 5.1, then a strategy of defining new curvature operator in LQG is presented 5.2. Finally, a 3d- Ricci scalar curvature and edge length operators are constructed for a regular tetrahedron state 5.4.
- In chapter 6: we draw our conclusions.

Chapter 2

The Hamiltonian Formulations of General Relativity

This chapter is based on papers: [31, 32] and textbooks: [1, 33, 34, 35, 36].

The Hamiltonian formulation of General Relativity requires the splitting of spacetime into three dimensional space and one dimensional time that is known by *foliation of spacetime* where the diffeomorphism symmetry of GR must be taken into consideration. It means that we will not fix the splitting of spacetime; rather, we will use arbitrary foliation parameters to preserve the full symmetry. Before introducing the different Hamiltonian formulations of GR, we shall discuss in more details the mathematical tools that we need to perform the the foliation of the spacetime.

2.1 Hyper-Surfaces

Definition 1. *In d-dimensional Pseudo-Riemannian manifold $\mathcal{M}$, a hyper-surface is a $(d-1)$-dimensional submanifold that can be either timelike, spacelike, or null. A particular hyper-surface can be selected by giving an embedding map:*

$$\begin{aligned} e : \Sigma &\hookrightarrow \mathcal{M} \\ \sigma &\longmapsto e(\sigma) \end{aligned} \tag{2.1}$$

where Σ is a $(d-1)$-dimensional space and $S = e(\Sigma) \subset \mathcal{M}$ is the hyper-surface (submanifold), it can also be defined by the parametric functions:

$$\begin{aligned} \mathbb{R}^{d-1} &\longrightarrow \mathbb{R}^{d} \\ \boldsymbol{\sigma} \equiv (\sigma^a) &\longmapsto \boldsymbol{x}(\boldsymbol{\sigma}) \end{aligned} \tag{2.2}$$

where $\{\boldsymbol{\sigma} = (\sigma^a)\}$ with $a = 1, \ldots, d-1$ are coordinates intrinsic to the space Σ, and $\{\boldsymbol{x} \equiv (x^\mu)\}$ with $\mu = 0, 1, \ldots, d-1$ are manifold coordinates system. In more precise mathematical way, we have the following:

$$\begin{array}{ccc} \Sigma & \overset{e}{\hookrightarrow} & \mathcal{M} \\ \boldsymbol{\sigma}\downarrow & & \downarrow \boldsymbol{x} \\ \mathbb{R}^{n-1} & \underset{\boldsymbol{x}\circ e \circ \boldsymbol{\sigma}^{-1}}{\longrightarrow} & \mathbb{R}^{n} \end{array} \tag{2.3}$$

Since every map is surjective when its codomain is restricted to its image, it is useful to define the diffeomorphism $\overline{e}$ as the restricted codomain of the embedding map e to its image:

$$\begin{aligned} \overline{e} : \Sigma &\longrightarrow S = e(\Sigma) \\ \sigma &\longmapsto \overline{e}(\sigma) := e(\sigma) \end{aligned} \tag{2.4}$$

Another way to define a hyper-surface is by imposing a constraint on the coordinates:

$$\begin{aligned} f : \mathbb{R}^{d} &\longrightarrow \mathbb{R} \\ \boldsymbol{x} \equiv (x^\mu) &\longmapsto f(\boldsymbol{x}) \end{aligned} \tag{2.5}$$

the constraint on this coordinates is:

$$f(\boldsymbol{x}) = 0 \tag{2.6}$$

2.1.1 Hyper-surface orthogonal vector fields

We start with a one-parameter family of hyper-surfaces $\{S_C = e_C(\Sigma) \subset \mathcal{M}, C \in \mathbb{R}\}$ defining by a family of embedding maps e_C. It is given by the family of constraints:

$$f(\boldsymbol{x}) = C \quad , \quad C \in \mathbb{R} \tag{2.7}$$

where different members of the family $\{S_C\}$ correspond to different values of the constant C. Consider two neighboring points p and q with coordinates $\boldsymbol{x}_p$ and $\boldsymbol{x}_p + d\boldsymbol{x}_p$ respectively, lying in the same hyper-surface S_C. Then:

$$f(\boldsymbol{x}_p) = C \tag{2.8a}$$

$$f(\boldsymbol{x}_p + d\boldsymbol{x}_p) = C \tag{2.8b}$$

We have then to first order:

$$\begin{aligned}
&f(\boldsymbol{x}_p + d\boldsymbol{x}_p) = C \\
\Rightarrow &f(\boldsymbol{x}_p) + \left.\frac{\partial f}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_p} \cdot d\boldsymbol{x}_p = C \\
\Rightarrow &C + \left.\frac{\partial f}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_p} \cdot d\boldsymbol{x}_p = C \\
\Rightarrow &\left.\frac{\partial f}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_p} \cdot d\boldsymbol{x}_p = 0
\end{aligned} \tag{2.9}$$

Since the displacement $d\boldsymbol{x}_p$ is tangent to the hyper-surface S_C for any $p \in \mathcal{M}$, it implies that the vector field $g^\sharp(df) \in \Gamma T\mathcal{M}$[1][2] is normal to the hyper-surface S_C.

Definition 2. *A unit normal vector field $\boldsymbol{n} \in \Gamma T\mathcal{M}$ can be introduced in the case where the hyper-surfaces $\{S_C\}$ is not null. its norm is defined by:*

$$\forall C \in \mathbb{R},\ \forall p \in S_C \subset \mathcal{M} : \epsilon := g(\boldsymbol{n}_p, \boldsymbol{n}_p) = \begin{cases} -1, & \text{if } S_C \text{ is space-like} \\ +1, & \text{if } S_C \text{ is time-like} \end{cases} \tag{2.10}$$

and we require that $\boldsymbol{n}$ point in the direction of increasing f (future-pointing):

$$df(\boldsymbol{n}) > 0 \tag{2.11}$$

It can then easily be checked that $\underline{\boldsymbol{n}} = g^\flat(\boldsymbol{n})$[3] is given by:

$$\underline{\boldsymbol{n}} = \frac{\epsilon df}{|df|^{1/2}} \tag{2.12}$$

where $|df| = |df(g^\sharp(df))|$ is the norm of the normal vector field $g^\sharp(df)$.

[1] In general, the space of smooth section on the tensor bundle $T^m_n\mathcal{M}$ is defined by

$$\Gamma T^m_n\mathcal{M} := \{\sigma : \mathcal{M} \to T^m_n\mathcal{M} | \sigma \text{ is smooth}\}$$

[2] The sharp map: $g^\sharp : \Gamma T^*\mathcal{M} \to \Gamma T\mathcal{M},\ \underline{\boldsymbol{X}} \mapsto g^\sharp(\underline{\boldsymbol{X}}) := (g^\flat)^{-1}(\underline{\boldsymbol{X}})$

[3] The flat map: $g^\flat : \Gamma T\mathcal{M} \to \Gamma T^*\mathcal{M},\ \boldsymbol{X} \mapsto g^\flat(\boldsymbol{X}) := g(\boldsymbol{X}, \cdot)$
both are linear and isomorphism maps.

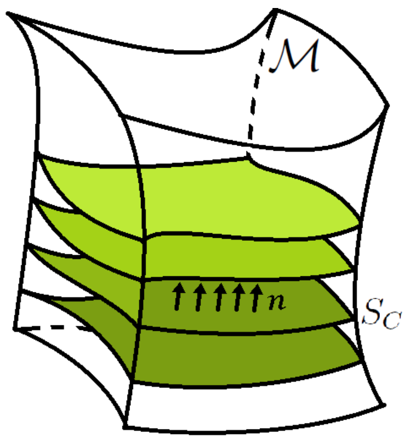


Figure 2.1: Family of hyper-surfaces $\{S_C\}$ embedded on the manifold $\mathcal{M}$ with the orthonormal vector field $\boldsymbol{n}$.

2.1.2 Vector decomposition

The embedded hyper-surface $S_C \subset \mathcal{M}$ induces a split of the tangent bundle $T\mathcal{M}$ into *spatial tangent bundle* $T_{\|}\mathcal{M}$ spanned by non-linear independent vector fields tangent to S_C, and *normal bundle* $T_{\perp}\mathcal{M}$ spanned by the unique future-pointing vector field $\boldsymbol{n}$ normal to the hyper-surface S_C:

$$T\mathcal{M} = T_{\|}\mathcal{M} \oplus T_{\perp}\mathcal{M} \tag{2.13}$$

Given a vector field $\boldsymbol{X} \in \Gamma T\mathcal{M}$, for any $p \in S_C \subset \mathcal{M}$, we can uniquely decompose $\boldsymbol{X}$ at p into a part tangent to S_C and a normal part proportional to $\boldsymbol{n}_p$ as follows:

$$\begin{aligned} \boldsymbol{X}_p &= \underbrace{\boldsymbol{X}_p - \epsilon g(\boldsymbol{n}_p, \boldsymbol{X}_p)\ \boldsymbol{n}_p}_{\boldsymbol{X}_p^{\|} \in T_{\|}\mathcal{M}} + \underbrace{\epsilon g(\boldsymbol{n}_p, \boldsymbol{X}_p)\ \boldsymbol{n}_p}_{\boldsymbol{X}_p^{\perp} \in T_{\perp}\mathcal{M}} \\ &= \quad \mathcal{P}^{\|} \cdot \boldsymbol{X}_p \quad + \quad \mathcal{P}^{\perp} \cdot \boldsymbol{X}_p \end{aligned} \tag{2.14}$$

Defining the parallel projector map of a vector field $\boldsymbol{X}$ for any $p \in \mathcal{M}$:

$$\begin{aligned} \mathcal{P}^{\|} : T\mathcal{M} &\longrightarrow T_{\|}\mathcal{M} \\ \boldsymbol{X}_p &\longmapsto \mathcal{P}^{\|}(\boldsymbol{X}_p) := \boldsymbol{X}_p - \epsilon g(\boldsymbol{n}_p, \boldsymbol{X}_p) \equiv X_p^{\|} \end{aligned} \tag{2.15}$$

and the normal projector map of a vector field $\boldsymbol{X}$ for any $p \in \mathcal{M}$:

$$\begin{aligned} \mathcal{P}^{\perp} : T\mathcal{M} &\longrightarrow T_{\perp}\mathcal{M} \\ \boldsymbol{X}_p &\longmapsto \mathcal{P}^{\perp}(\boldsymbol{X}_p) := \epsilon g(\boldsymbol{n}_p, \boldsymbol{X}_p) \equiv X_p^{\perp} \end{aligned} \tag{2.16}$$

From the bilinearity of the metric g, it is easy to prove that the projection maps (2.15,2.16) are $(1,1)$-tensor fields: $\mathcal{P}^{\parallel}$, $\mathcal{P}^{\perp} \in \Gamma T^1_1\mathcal{M}$. Thus, the parallel and normal projectors can be written in terms of components as:

$$\mathcal{P}^{\parallel\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \epsilon n^{\mu} n_{\nu} \tag{2.17a}$$

$$\mathcal{P}^{\perp\mu}{}_{\nu} = \epsilon n^{\mu} n_{\nu} \tag{2.17b}$$

One can easily check the projection property:

$$\mathcal{P}^{\parallel\mu}{}_{\rho}\mathcal{P}^{\parallel\rho}{}_{\nu} = \mathcal{P}^{\parallel\mu}{}_{\nu} \tag{2.18a}$$

$$\mathcal{P}^{\perp\mu}{}_{\rho}\mathcal{P}^{\perp\rho}{}_{\nu} = \mathcal{P}^{\perp\mu}{}_{\nu} \tag{2.18b}$$

It is obvious to see the relations:

$$\forall \boldsymbol{X}^{\parallel} \in \Gamma T_{\parallel}\mathcal{M} : \mathcal{P}^{\parallel} \cdot \boldsymbol{X}^{\parallel} = \boldsymbol{X}^{\parallel} \ , \ \mathcal{P}^{\perp} \cdot \boldsymbol{X}^{\parallel} = \boldsymbol{0} \tag{2.19a}$$

$$\forall \boldsymbol{X}^{\perp} \in \Gamma T_{\perp}\mathcal{M} : \mathcal{P}^{\parallel} \cdot \boldsymbol{X}^{\perp} = \boldsymbol{0} \ , \ \mathcal{P}^{\perp} \cdot \boldsymbol{X}^{\perp} = \boldsymbol{X}^{\perp} \tag{2.19b}$$

These projections can be extended to tensor product of tangent spaces in $\mathcal{M}$ by contracting each vector index in the tensor with the parallel or the normal projectors in (2.15,2.16).

2.1.3 Dual vector decomposition

The embedded hyper-surface $S_C \subset \mathcal{M}$ induces a split of the cotangent bundle $T^*\mathcal{M}$ into *spatial cotangent bundle* $T^*_{\parallel}\mathcal{M}$ spanned by non-linear independent dual vector fields of S_C, and *normal cotangent bundle* $T^*_{\perp}\mathcal{M}$ spanned by the unique future-pointing covector field $\underline{\boldsymbol{n}}$ normal to the hyper-surface S_C:

$$T^*\mathcal{M} = T^*_{\parallel}\mathcal{M} \oplus T^*_{\perp}\mathcal{M} \tag{2.20}$$

Given a covector field $\underline{\boldsymbol{X}} \in \Gamma T^*\mathcal{M}$, for any $p \in S_C \subset \mathcal{M}$, we can uniquely decompose $\underline{\boldsymbol{X}}$ at p into a part cotangent to S_C and a normal covector part proportional to $\underline{\boldsymbol{n}}_p$:

$$\begin{aligned}\underline{\boldsymbol{X}}_p &= \underbrace{\underline{\boldsymbol{X}}_p - \epsilon\underline{\boldsymbol{n}}(g^{\sharp}(\underline{\boldsymbol{X}}_p))\ \underline{\boldsymbol{n}}}_{\underline{\boldsymbol{X}}^{\parallel}_p \in T^*_{\parallel}\mathcal{M}} + \underbrace{\epsilon\underline{\boldsymbol{n}}(g^{\sharp}(\underline{\boldsymbol{X}}_p))\ \underline{\boldsymbol{n}}}_{\underline{\boldsymbol{X}}^{\perp}_p \in T^*_{\perp}\mathcal{M}} \\ &= \qquad \mathcal{P}^{\parallel} \cdot \underline{\boldsymbol{X}}_p \qquad + \qquad \mathcal{P}^{\perp} \cdot \underline{\boldsymbol{X}}_p\end{aligned} \tag{2.21}$$

Defining the parallel projector map of a covector field $\underline{\boldsymbol{X}}$ for any $p \in \mathcal{M}$:

$$\begin{aligned} \mathcal{P}^{\parallel} : T^*\mathcal{M} &\longrightarrow T^*_{\parallel}\mathcal{M} \\ \underline{\boldsymbol{X}}_p &\longmapsto \mathcal{P}^{\parallel}(\underline{\boldsymbol{X}}_p) := \underline{\boldsymbol{X}}_p - \epsilon \underline{\boldsymbol{n}}(g^{\sharp}(\underline{\boldsymbol{X}}_p))\ \underline{\boldsymbol{n}} \equiv \underline{X}^{\parallel}_p \end{aligned} \tag{2.22}$$

and the normal projector map of a covector field $\underline{\boldsymbol{X}}$ for any $p \in \mathcal{M}$:

$$\begin{aligned} \mathcal{P}^{\perp} : T^*\mathcal{M} &\longrightarrow T^*_{\perp}\mathcal{M} \\ \underline{\boldsymbol{X}}_p &\longmapsto \mathcal{P}^{\perp}(\underline{\boldsymbol{X}}_p) := \epsilon \underline{\boldsymbol{n}}(g^{\sharp}(\underline{\boldsymbol{X}}_p))\ \underline{\boldsymbol{n}} \equiv \underline{X}^{\perp}_p \end{aligned} \tag{2.23}$$

From the bilinearity of the metric g, it is easy to prove that the projection maps (2.22,2.23) are $(1,1)$-tensor fields: $\mathcal{P}^{\parallel}, \mathcal{P}^{\perp} \in \Gamma T^1_1\mathcal{M}$. Thus, in terms of components, the parallel and normal projectors can be written as:

$$\mathcal{P}^{\parallel}{}_{\nu}{}^{\mu} = \delta_{\nu}{}^{\mu} - \epsilon n_{\nu} n^{\mu} \tag{2.24a}$$

$$\mathcal{P}^{\perp}{}_{\nu}{}^{\mu} = \epsilon n_{\nu} n^{\mu} \tag{2.24b}$$

One can easily check the projection property:

$$\mathcal{P}^{\parallel}{}_{\nu}{}^{\rho} \mathcal{P}^{\parallel}{}_{\rho}^{\mu} = \mathcal{P}^{\parallel}{}_{\nu}{}^{\mu} \tag{2.25a}$$

$$\mathcal{P}^{\perp}{}_{\nu}{}^{\rho} \mathcal{P}^{\perp}{}_{\rho}{}^{\mu} = \mathcal{P}^{\perp}{}_{\nu}{}^{\mu} \tag{2.25b}$$

It is obvious to see the relations:

$$\forall \underline{\boldsymbol{X}}^{\parallel} \in \Gamma T^*_{\parallel}\mathcal{M} : \mathcal{P}^{\parallel} \cdot \underline{\boldsymbol{X}}^{\parallel} = \underline{\boldsymbol{X}}^{\parallel}\ ,\ \mathcal{P}^{\perp} \cdot \underline{\boldsymbol{X}}^{\parallel} = \mathbf{0} \tag{2.26a}$$

$$\forall \underline{\boldsymbol{X}}^{\perp} \in \Gamma T^*_{\perp}\mathcal{M} : \mathcal{P}^{\parallel} \cdot \underline{\boldsymbol{X}}^{\perp} = \mathbf{0}\ ,\ \mathcal{P}^{\perp} \cdot \underline{\boldsymbol{X}}^{\perp} = \boldsymbol{X}^{\perp} \tag{2.26b}$$

These projections can be extended to tensor product of cotangent spaces in $\mathcal{M}$ by contracting each covector index in the tensor with the parallel or the normal projectors in (2.22,2.23).

2.1.4 Tensor decomposition

The embedded hyper-surface $S_C \subset \mathcal{M}$ induces a split of the (m,n)-tensor bundle $T^m_n\mathcal{M}$ into a "spatial (m,n)-tensor bundle" $T_{\|}{}^m_n\mathcal{M}$, and "normal (m,n)-tensor" $T_{\perp}{}^m_n\mathcal{M}$:

$$T\mathcal{M} = T_{\|}{}^m_n\mathcal{M} \oplus T_{\perp}{}^m_n\mathcal{M} \tag{2.27}$$

For any (m,n)-tensor field $\boldsymbol{T} \in \Gamma T^m_n\mathcal{M}$, we can uniquely decompose $\boldsymbol{T}$ at $p \in \mathcal{M}$ into a parallel, normal and mixed parts:

$$\boldsymbol{T}_p := \sum_{i_1,\dots,i_{m+n}=\{\|,\perp\}} (\mathcal{P}^{i_1}\cdots\mathcal{P}^{i_{m+n}})\cdot\boldsymbol{T}_p \tag{2.28}$$

2.1.5 Transverse metric (1^{st} fundamental form)

In what follows, we introduce what is called the *transverse metric*.

Definition 3. *the transverse metric or 1^{st} fundamental form is the parallel part of the metric g at each point $p \in \mathcal{M}$ and it is defined by:*

$$h := \mathcal{P}^{\|}\mathcal{P}^{\|}\cdot g = g - \epsilon\underline{\boldsymbol{n}}\otimes\underline{\boldsymbol{n}} \tag{2.29}$$

In terms of components:

$$h_{\mu\nu} := \mathcal{P}^{\|}{}_{\mu}{}^{\rho}\mathcal{P}^{\|}{}_{\nu}{}^{\sigma}g_{\rho\sigma} = g_{\mu\nu} - \epsilon n_\mu n_\nu \tag{2.30}$$

one can see that:

$$\mathcal{P}^{\|\mu}_{\nu} = h^{\mu}_{\nu} \tag{2.31}$$

It satisfies these properties:

$$\forall \boldsymbol{X}^{\|},\ \boldsymbol{Y}^{\|} \in \Gamma T_{\|}\mathcal{M},\ \forall p \in \mathcal{M} : g(\boldsymbol{X}^{\|}_p, \boldsymbol{Y}^{\|}_p) = h(\boldsymbol{X}^{\|}_p, \boldsymbol{Y}^{\|}_p) \tag{2.32a}$$

$$\forall \boldsymbol{X}^{\|} \in \Gamma T_{\|}\mathcal{M},\ \forall p \in \mathcal{M} : h(\boldsymbol{n}, \boldsymbol{X}_p) = 0 \tag{2.32b}$$

The geometrical meaning behind this notion is that if we want to find the scalar product between two tangent vectors of the hyper-surface S_C, the transverse metric is the only part of the metric that contributes in the result, for any $\boldsymbol{X}_p^{\parallel}$, $\boldsymbol{Y}_p^{\parallel} \in T_{\parallel}\mathcal{M}$:

$$\begin{aligned} g(\boldsymbol{X}_p^{\parallel}, \boldsymbol{Y}_p^{\parallel}) &= h(\boldsymbol{X}_p^{\parallel}, \boldsymbol{Y}_p^{\parallel}) + \epsilon \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}}(\boldsymbol{X}_p^{\parallel}, \boldsymbol{Y}_p^{\parallel}) \\ &= h(\boldsymbol{X}_p^{\parallel}, \boldsymbol{Y}_p^{\parallel}) + \epsilon \underbrace{\underline{\boldsymbol{n}}(\boldsymbol{X}_p^{\parallel})}_{0} \underbrace{\underline{\boldsymbol{n}}(\boldsymbol{Y}_p^{\parallel})}_{0} \\ &= h(\boldsymbol{X}_p^{\parallel}, \boldsymbol{Y}_p^{\parallel}) \end{aligned}$$

Accordingly, the transverse metric works as a spatial metric at each hyper-surface S_C. In fact, the transverse metric is not a metric notion, since it degenerates once at each point in the hyper-surface S_C by the spanned space of the unit normal vector field $\boldsymbol{n}$:

$$h_{\mu\nu} n^{\nu} = 0 \tag{2.33}$$

2.1.6 Induced metric

For any point $p \in S_C \subset \mathcal{M}$, the push-forward[4] of the intrinsic induced basis $\{\left(\frac{\partial}{\partial \sigma^a}\right)_{\boldsymbol{\sigma}(\bar{e}^{-1}(p))}\}$ at $\bar{e}^{-1}(p) \in \Sigma$ by the embedding map (2.1) gives $d-1$ vectors:

$$e_* \left(\frac{\partial}{\partial \sigma^a}\right)_{\boldsymbol{\sigma}(\bar{e}^{-1}(p))} = \left.\frac{\partial x^{\mu} \circ e \circ \boldsymbol{\sigma}^{-1}}{\partial \sigma^a}\right|_{\boldsymbol{\sigma}(\bar{e}^{-1}(p))} \left(\frac{\partial}{\partial x^{\mu}}\right)_{\boldsymbol{x}(p)} \equiv e_a^{\mu}(p) \left(\frac{\partial}{\partial x^{\mu}}\right)_{\boldsymbol{x}(p)} \tag{2.34}$$

are tangential to the hyper-surface S_C at the point p, where we have taken:

$$e_a^{\mu}(p) \equiv \left.\frac{\partial x^{\mu} \circ e \circ \boldsymbol{\sigma}^{-1}}{\partial \sigma^a}\right|_{\boldsymbol{\sigma}(\bar{e}^{-1}(p))} \tag{2.35}$$

Since the $d-1$ vectors (2.34) are tangential to the hyper-surface S_C, then one has:

$$n_{\mu} e_a^{\mu} = 0 \tag{2.36}$$

Definition 4. *The induced metric is the pull-back[5] of the metric g by the embedding map (2.1):*

$$q = e^* g \tag{2.37}$$

it is $d-1$ tensor at each point in Σ and its components are written by:

$$q_{ab} = e_a^{\mu} e_b^{\nu} g_{\mu\nu} \tag{2.38}$$

[4]The push-forward map: $\phi_* : T_p\mathcal{M} \to T_{\phi(p)}\mathcal{N}, \boldsymbol{X}_p \mapsto \phi_*(\boldsymbol{X}_p)f := \boldsymbol{X}_p(f \circ \phi)$

[5]The pull-back map: $\phi^* : T^*_{\phi(p)}\mathcal{N} \to T^*_p\mathcal{M}, \underline{\boldsymbol{X}}_{\phi(p)} \mapsto \phi^*(\underline{\boldsymbol{X}}_{\phi(p)})(\boldsymbol{Y}_p) = \underline{\boldsymbol{X}}_{\phi(p)}(\phi_*(\boldsymbol{Y}_p))$ with the map $\phi : \mathcal{M} \to \mathcal{N}$ and for any $C^1(\mathcal{N})$ function $f : \mathcal{N} \to \mathbb{R}$ and for any $\boldsymbol{Y}_p \in T_p\mathcal{M}$.

This acts as a metric tensor on the tangent space of Σ. There are two basic relations between transverse and induced metric:

$$e^\mu_a e^\nu_b h_{\mu\nu} = q_{ab} \tag{2.39a}$$

$$h^{\mu\nu} = e^\mu_a e^\nu_b q^{ab} \tag{2.39b}$$

where q^{ab} is the inverse of q_{ab} and $h^{\mu\nu}$ is the raised transverse metric components by using the inverse metric components $g^{\mu\nu}$ (it is not the inverse of $h_{\mu\nu}$ since the latter is not invertible)

2.1.7 Extrinsic curvature (2^{nd} fundamental form)

The embedded hyper-surface $S_C \subset \mathcal{M}$ induces a split of the covariant derivative of a vector field, which is a section of the tangent bundle $T\mathcal{M}$ (vector field), then from Eq. (2.14) it can be decomposed into parallel and normal parts. In the next, we will study how a spatial tangent field can be affected if we parallelly translate it along a small curve tangent to the hyper-surface S_C. Then from now on, we will just take tangent vector fields to the hyper-surface S_C and label them by bold capital letters $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \ldots$, without putting the parellel sign $\|$, one has:

$$\begin{aligned} \nabla_{\boldsymbol{X}}\boldsymbol{Y} &= \mathcal{P}^{\|} \cdot \nabla_{\boldsymbol{X}}\boldsymbol{Y} + \mathcal{P}^{\perp} \cdot \nabla_{\boldsymbol{X}}\boldsymbol{Y} \\ &= \underbrace{\nabla_{\boldsymbol{X}}\boldsymbol{Y} - \epsilon g(\boldsymbol{n}, \nabla_{\boldsymbol{X}}\boldsymbol{Y})\boldsymbol{n}} + \underbrace{\epsilon g(\boldsymbol{n}, \nabla_{\boldsymbol{X}}\boldsymbol{Y})\boldsymbol{n}} \\ &:= \qquad D_{\boldsymbol{X}}\boldsymbol{Y} \qquad\quad + \; K(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{n} \end{aligned} \tag{2.40}$$

where K is a map of two vector fields $\boldsymbol{X}, \boldsymbol{Y} \in \Gamma T_{\|}\mathcal{M}$. ∇ is the Levi-Civita connection on the tangent bundle $T\mathcal{M}$ with respect to the metric g: the unique, torsion free and metric compatible covariant derivative associated with g.

Lemma 2.1.1. *The following two properties are hold:*

(i) The map:

$$\begin{aligned} K : T_{\|}\mathcal{M} \times T_{\|}\mathcal{M} &\longrightarrow C^\infty(\mathcal{M}) \\ (\boldsymbol{X}, \boldsymbol{Y}) &\longmapsto K(\boldsymbol{X}, \boldsymbol{Y}) := \epsilon g(\boldsymbol{n}, \nabla_{\boldsymbol{X}}\boldsymbol{Y}) \end{aligned} \tag{2.41}$$

is a symmetric tensor field called the extrinsic curvature or 2^{nd} fundamental form of the hyper-surface S_C.

(ii) D is the Levi-Civita connection on the tangent bundle $T_{\|}\mathcal{M}$ with respect to the transverse metric h.

Proof. A detail proof.

(i) We have to show that:

– K is symmetric: $\forall \boldsymbol{X}, \boldsymbol{Y} \in \Gamma T_{\|}\mathcal{M} : K(\boldsymbol{X}, \boldsymbol{Y}) \stackrel{?}{=} K(\boldsymbol{Y}, \boldsymbol{X})$

$$\begin{aligned} K(\boldsymbol{X}, \boldsymbol{Y}) &= \epsilon g(\boldsymbol{n}, \nabla_{\boldsymbol{X}} \boldsymbol{Y}) \\ &= \epsilon g(\boldsymbol{n}, \nabla_{\boldsymbol{Y}} \boldsymbol{X} + [\boldsymbol{X}, \boldsymbol{Y}]) \\ &= \epsilon g(\boldsymbol{n}, \nabla_{\boldsymbol{Y}} \boldsymbol{X}) + \epsilon \underbrace{g(\boldsymbol{n}, [\boldsymbol{X}, \boldsymbol{Y}])}_{0} \\ &= K(\boldsymbol{Y}, \boldsymbol{X}) \ \square \end{aligned}$$

we have used the fact that $[\boldsymbol{X}, \boldsymbol{Y}] \in \Gamma T_{\|}\mathcal{M}$; for any $\boldsymbol{X}$,$\boldsymbol{Y} \in \Gamma T_{\|}\mathcal{M}$:

$$\begin{aligned} g(\boldsymbol{n}, [\boldsymbol{X}, \boldsymbol{Y}]) &= n_\mu (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu) \\ &= \underbrace{X^\nu \partial_\nu (n_\mu Y^\mu)}_{0} - X^\nu Y^\mu \partial_\nu n_\mu - \underbrace{Y^\nu \partial_\nu (n_\mu X^\mu)}_{0} + Y^\nu X^\mu \partial_\nu n_\mu \\ &= 2 X^{[\mu} Y^{\nu]} \partial_\nu \left(\frac{\epsilon \partial_\mu f}{|df|^{1/2}} \right) \\ &= 2 \underbrace{X^{[\mu} Y^{\nu]} \partial_\mu f}_{0} \ \partial_\nu \left(\frac{\epsilon}{|df|^{1/2}} \right) + 2 \left(\frac{\epsilon}{|df|^{1/2}} \right) \underbrace{X^{[\mu} Y^{\nu]} \partial_\nu \partial_\mu f}_{0} \\ &= 0 \ \square \end{aligned}$$

In the first line, we have used the definition of the Lie bracket with torsion free space. In the second line, we have applied the derivative leibniz rule and the fact that $\boldsymbol{n}$ is normal to any tangent vector along the hyper-surface has been taken. In the third line, the normal vector field $\boldsymbol{n}$ is substituted by its definition in Eq. (2.12). In the last line, the first and second term are vanish since the function f is constant along the hyper-surface and there is a symmetric contraction with antisymmetric ones.

- K is $C^\infty(\mathcal{M})$-bilinear: since K is symmetric, we will just prove the linearity of the first argument: $\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \Gamma T_{\|}\mathcal{M},\ \forall f \in C^\infty(\mathcal{M}):$

$$K(f\boldsymbol{X} + \boldsymbol{Z}, \boldsymbol{Y}) \stackrel{?}{=} fK(\boldsymbol{X}, \boldsymbol{Y}) + K(\boldsymbol{Z}, \boldsymbol{Y})$$

$$\begin{aligned} K(f\boldsymbol{X} + \boldsymbol{Z}, \boldsymbol{Y}) &= \epsilon g(\boldsymbol{n}, \nabla_{f\boldsymbol{X}+\boldsymbol{Z}}\boldsymbol{Y}) \\ &= \epsilon g(\boldsymbol{n}, f\nabla_{\boldsymbol{X}}\boldsymbol{Y} + \nabla_{\boldsymbol{Z}}\boldsymbol{Y}) \\ &= \epsilon f g(\boldsymbol{n}, \nabla_{\boldsymbol{X}}\boldsymbol{Y}) + \epsilon g(\boldsymbol{n}, \nabla_{\boldsymbol{Z}}\boldsymbol{Y}) \\ &= fK(\boldsymbol{X}, \boldsymbol{Y}) + K(\boldsymbol{Z}, \boldsymbol{Y}) \ \square \end{aligned}$$

In the seconde step, we have used the linearity of the Levi-Civita connection ∇ and in the third step the linearity of the metric g has been taken.

(ii) We have to show that:

- D is metric preserving connection: $\forall \boldsymbol{X} \in \Gamma T_{\|}\mathcal{M} : D_{\boldsymbol{X}} h \stackrel{?}{=} 0$

$$\begin{aligned} D_{\boldsymbol{X}} h &= \mathcal{P}^{\|}\mathcal{P}^{\|} \cdot \nabla_{\boldsymbol{X}} h \\ &= \mathcal{P}^{\|}\mathcal{P}^{\|} \cdot \nabla_{\boldsymbol{X}}(g - \epsilon \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}}) \\ &= \mathcal{P}^{\|}\mathcal{P}^{\|} \cdot \underbrace{\nabla_{\boldsymbol{X}} g}_{0} - \epsilon \mathcal{P}^{\|} \cdot \nabla_{\boldsymbol{X}}(\underline{\boldsymbol{n}}) \otimes \underbrace{\mathcal{P}^{\|} \cdot \underline{\boldsymbol{n}}}_{0} - \epsilon \underbrace{\mathcal{P}^{\|} \cdot \underline{\boldsymbol{n}}}_{0} \otimes \mathcal{P}^{\|} \cdot \nabla_{\boldsymbol{X}}(\underline{\boldsymbol{n}}) \\ &= 0 \ \square \end{aligned}$$

- D is torsion free connection: $\forall \boldsymbol{X}, \boldsymbol{Y} \in \Gamma T_{\|}\mathcal{M} :^D T(\boldsymbol{X}, \boldsymbol{Y}) \stackrel{?}{=} 0$

$$\begin{aligned} {}^D T(\boldsymbol{X}, \boldsymbol{Y}) &= D_{\boldsymbol{X}}\boldsymbol{Y} - D_{\boldsymbol{Y}}\boldsymbol{X} - [\boldsymbol{X}, \boldsymbol{Y}] \\ &= \mathcal{P}^{\|} \cdot (D_{\boldsymbol{X}}\boldsymbol{Y} - D_{\boldsymbol{Y}}\boldsymbol{X} - [\boldsymbol{X}, \boldsymbol{Y}]) \\ &= \mathcal{P}^{\|} \cdot \underbrace{T(\boldsymbol{X}, \boldsymbol{Y})}_{0} \\ &= 0 \ \square \end{aligned}$$

$\square$

Definition 5. *The covariant derivative D_μ with respect to the transverse metric h on the hyper-surface S_C can be written in terms of the covariant derivative ∇_μ as:*

$$\forall \boldsymbol{T} \in \Gamma T_{\|n}^{\ m}\mathcal{M}:$$
$$D_\gamma T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n} := h_\gamma^{\ \rho}(h^{\alpha_1}_{\ \ \mu_1} \cdots h^{\alpha_m}_{\ \ \mu_m})(h_{\beta_1}^{\ \ \nu_1} \cdots h_{\beta_n}^{\ \ \nu_n}) D_\rho T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} \tag{2.42}$$

Proof. This definition is a natural generalization of the covariant derivative D_μ of a vector field tangent to the slice S_C, we have from Eq. (2.40):

$$\begin{aligned}\forall \boldsymbol{X}, \boldsymbol{Y} \in \Gamma T_\| \mathcal{M}: & D_{\boldsymbol{X}} Y^\mu = h^\mu_{\ \rho} \nabla_{\boldsymbol{X}} Y^\rho \\ & \Leftrightarrow X^\nu D_\nu Y^\mu = h^\mu_{\ \rho} X^\nu \nabla_\nu Y^\rho \\ & \Leftrightarrow D_\nu Y^\mu = h^\mu_{\ \rho} h_\nu^{\ \sigma} \nabla_\sigma Y^\rho\end{aligned}$$

□

Definition 6. *The extrinsic curvature K is a spatial $(0,2)$ symmetric tensor field on the hyper-surface S_C, it is defined by:*

$$K := -\epsilon \mathcal{P}^\| \mathcal{P}^\| \cdot \nabla \underline{\boldsymbol{n}} = -\epsilon \nabla \underline{\boldsymbol{n}} + \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{a}} \tag{2.43}$$

where $\underline{\boldsymbol{a}} := \nabla_{\boldsymbol{n}} \underline{\boldsymbol{n}} \in \Gamma T_\|^* \mathcal{M}$. In terms of components:

$$K_{\mu\nu} := -\epsilon h_\mu^{\ \rho} h_\nu^{\ \sigma} \nabla_\rho n_\sigma = -\epsilon \nabla_\mu n_\nu + n_\mu a_\nu \tag{2.44}$$

with $a_\nu = n^\rho \nabla_\rho n_\nu$ are the components of the covector field $\underline{\boldsymbol{a}}$.

Proof. This definition is coming from the starting point in Eq. (2.40), we have:

$$\begin{aligned}\forall \boldsymbol{X}, \boldsymbol{Y} \in \Gamma T_\| \mathcal{M}: K(\boldsymbol{X}, \boldsymbol{Y}) &= \epsilon g(\boldsymbol{n}, \nabla_{\boldsymbol{X}} \boldsymbol{Y}) = -\epsilon g(\nabla_{\boldsymbol{X}} \boldsymbol{n}, \boldsymbol{Y}) \\ &= -\epsilon (\nabla_{\boldsymbol{X}} \underline{\boldsymbol{n}})(\boldsymbol{Y}) = -\epsilon (\nabla \underline{\boldsymbol{n}})(\boldsymbol{X}, \boldsymbol{Y}) \\ \Leftrightarrow K &= -\epsilon \mathcal{P}^\| \mathcal{P}^\| \cdot \nabla \underline{\boldsymbol{n}} \ \square\end{aligned}$$

and also one can write the extrinsic curvature in terms of the covariant derivative ∇ by following:

$$\begin{aligned}
K_{\mu\nu} &= -\epsilon h_\mu{}^\rho h_\nu{}^\sigma \nabla_\rho n_\sigma \\
&= -\epsilon(\delta_\mu{}^\rho - \epsilon n_\mu n^\rho)(\delta_\nu{}^\sigma - \epsilon n_\nu n^\sigma)\nabla_\rho n_\sigma \\
&= -\epsilon \nabla_\mu n_\nu + n_\nu \underbrace{n^\sigma \nabla_\mu n_\sigma}_{0} + n_\mu n^\rho \nabla_\rho n_\nu - \epsilon n_\mu n^\rho n_\nu \underbrace{n^\sigma \nabla_\rho n_\sigma}_{0} \\
&= -\epsilon \nabla_\mu n_\nu + n_\mu n^\rho \nabla_\rho n_\nu \ \square
\end{aligned}$$

we have used: $n^\sigma \nabla_\mu n_\sigma = \dfrac{1}{2}\nabla_\mu(n_\sigma n^\sigma) = \frac{1}{2}\nabla_\mu(\epsilon) = 0.$ $\square$

The extrinsic curvature measures how much the hyper-surface S_C is curved in the way it sits in the manifold $\mathcal{M}$. It also says how much a vector tangent to S_C fail to be tangent if we parallelly translate it a bit using the Levi-Civita connection ∇ along the hyper-surface on the manifold.

Definition 7. *The Weingarten map W is a spatial $(1,1)$ symmetric tensor field on the hyper-surface S_C, it is defined by:*

$$W := -\epsilon \mathcal{P}^{\parallel}\mathcal{P}^{\parallel} \cdot \nabla \boldsymbol{n} = -\epsilon \nabla \boldsymbol{n} + \underline{\boldsymbol{n}} \otimes \boldsymbol{a} \tag{2.45}$$

where $\boldsymbol{a} = \nabla_{\boldsymbol{n}} \boldsymbol{n} \in \Gamma T_{\parallel}\mathcal{M}$. In terms of components:

$$W_\mu{}^\nu := -\epsilon h_\mu{}^\rho h^\nu{}_\sigma \nabla_\rho n^\sigma = -\epsilon \nabla^\mu n_\nu + n_\mu a^\nu \tag{2.46}$$

with $a^\nu = n^\rho \nabla_\rho n^\nu$ are the components of the vector field $\boldsymbol{a}$.

It is related to the extrinsic curvature by the relation:

$$\forall \boldsymbol{X}, \boldsymbol{Y} \in \Gamma T_{\parallel}\mathcal{M} : K(\boldsymbol{X}, \boldsymbol{Y}) = h(W(\boldsymbol{X}), \boldsymbol{Y}) \tag{2.47}$$

and it has a linear map property: $\forall p \in S_C \subset \mathcal{M}$

$$\begin{aligned}
W : T_{\parallel}\mathcal{M} &\longrightarrow T_{\parallel}\mathcal{M} \\
\boldsymbol{X}_p &\longmapsto W(\boldsymbol{X}_p) := -\epsilon \mathcal{P}^{\parallel} \cdot \nabla_{\boldsymbol{X}_p} \boldsymbol{n}
\end{aligned} \tag{2.48}$$

The Weingarten map K determines how the unit normal vector field $\boldsymbol{n}$ of a hyper-surface fail to be a normal under a parallel transport along any vector field tangent to this hyper-surface.

Lemma 2.1.2. *There is a relation between extrinsic curvature K and transverse metric h, it is written by:*

$$K = \frac{-\epsilon}{2} \mathcal{L}_{\boldsymbol{n}} h \tag{2.49}$$

Proof. In order to prove this lemma, we use the definition of the transverse metric (2.29), we have:

$$h = g - \epsilon \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}} \Rightarrow \mathcal{L}_{\boldsymbol{n}} h = \mathcal{L}_{\boldsymbol{n}} g - \epsilon(\mathcal{L}_{\boldsymbol{n}} \underline{\boldsymbol{n}}) \otimes \underline{\boldsymbol{n}} - \epsilon \underline{\boldsymbol{n}} \otimes (\mathcal{L}_{\boldsymbol{n}} \underline{\boldsymbol{n}})$$

$$\Rightarrow \mathcal{L}_{\boldsymbol{n}} h = 2Sym(\nabla \underline{\boldsymbol{n}}) - \epsilon \underline{\boldsymbol{a}} \otimes \underline{\boldsymbol{n}} - \epsilon \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{a}}$$

$$\Rightarrow \mathcal{L}_{\boldsymbol{n}} h = -2\epsilon Sym(-\epsilon \nabla \underline{\boldsymbol{n}} + \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{a}})$$

$$\Rightarrow \mathcal{L}_{\boldsymbol{n}} h = -2\epsilon Sym(K)$$

$$\Rightarrow \mathcal{L}_{\boldsymbol{n}} h = -2\epsilon K \quad \square$$

In the first line, at the RHS, we have applied the Leibniz rule of the Lie derivative. In the seconde line, we have used the definition of the Lie derivative to compute the follwing: $\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \Gamma T\mathcal{M}$:

$$\mathcal{L}_{\boldsymbol{Z}} g(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{Z}(g(\boldsymbol{X}, \boldsymbol{Y})) - g(\mathcal{L}_{\boldsymbol{Z}} \boldsymbol{X}, \boldsymbol{Y}) - g(\boldsymbol{X}, \mathcal{L}_{\boldsymbol{Z}} \boldsymbol{Y})$$

$$= \nabla_{\boldsymbol{Z}}(g(\boldsymbol{X}, \boldsymbol{Y})) - g([\boldsymbol{Z}, \boldsymbol{X}], \boldsymbol{Y}) - g(\boldsymbol{X}, [\boldsymbol{Z}, \boldsymbol{Y}])$$

$$= g(\nabla_{\boldsymbol{Z}} \boldsymbol{X}, \boldsymbol{Y}) + g(\boldsymbol{X}, \nabla_{\boldsymbol{Z}} \boldsymbol{Y}) - g([\boldsymbol{Z}, \boldsymbol{X}], \boldsymbol{Y}) - g(\boldsymbol{X}, [\boldsymbol{Z}, \boldsymbol{Y}])$$

$$= g(\nabla_{\boldsymbol{Z}} \boldsymbol{X} - [\boldsymbol{Z}, \boldsymbol{X}], \boldsymbol{Y}) + g(\boldsymbol{X}, \nabla_{\boldsymbol{Z}} \boldsymbol{Y} - [\boldsymbol{Z}, \boldsymbol{Y}])$$

$$= g(\nabla_{\boldsymbol{X}} \boldsymbol{Z}, \boldsymbol{Y}) + g(\boldsymbol{X}, \nabla_{\boldsymbol{Y}} \boldsymbol{Z})$$

$$= (\nabla_{\boldsymbol{X}} \underline{\boldsymbol{Z}})(\boldsymbol{Y}) + (\nabla_{\boldsymbol{Y}} \underline{\boldsymbol{Z}})(\boldsymbol{X})$$

$$= (\nabla \underline{\boldsymbol{Z}})(\boldsymbol{X}, \boldsymbol{Y}) + (\nabla \underline{\boldsymbol{Z}})(\boldsymbol{Y}, \boldsymbol{X})$$

$$= 2Sym(\nabla \underline{\boldsymbol{Z}})(\boldsymbol{X}, \boldsymbol{Y})$$

$$\Rightarrow \mathcal{L}_{\boldsymbol{Z}} g = 2Sym(\nabla \underline{\boldsymbol{Z}}) \quad \square$$

and,

$$
\begin{aligned}
\forall \boldsymbol{X} \in \Gamma T\mathcal{M} : \mathcal{L}_n \underline{\boldsymbol{n}}(\boldsymbol{X}) &= \boldsymbol{n}(\underline{\boldsymbol{n}}(\boldsymbol{X})) - \underline{\boldsymbol{n}}(\mathcal{L}_n \boldsymbol{X}) \\
&= \nabla_n(\underline{\boldsymbol{n}}(\boldsymbol{X})) - \underline{\boldsymbol{n}}([\boldsymbol{n}, \boldsymbol{X}]) \\
&= (\nabla_n \underline{\boldsymbol{n}})(\boldsymbol{X}) + \underline{\boldsymbol{n}}(\nabla_n \boldsymbol{X}) - \underline{\boldsymbol{n}}([\boldsymbol{n}, \boldsymbol{X}]) \\
&= \underline{\boldsymbol{a}}(\boldsymbol{X}) + \underline{\boldsymbol{n}}(\nabla_n \boldsymbol{X} - [\boldsymbol{n}, \boldsymbol{X}]) \\
&= \underline{\boldsymbol{a}}(\boldsymbol{X}) + \underbrace{\underline{\boldsymbol{n}}(\nabla_{\boldsymbol{X}} \boldsymbol{n})}_{0} \\
&= \underline{\boldsymbol{a}}(\boldsymbol{X}) \\
\Rightarrow \mathcal{L}_n \underline{\boldsymbol{n}} &= \underline{\boldsymbol{a}} \quad \square
\end{aligned}
$$

We have used $\underline{\boldsymbol{n}}(\nabla_{\boldsymbol{X}} \boldsymbol{n}) = \frac{1}{2} \nabla_{\boldsymbol{X}}(\underline{\boldsymbol{n}}(\boldsymbol{n})) = \frac{1}{2} \nabla_{\boldsymbol{X}}(\epsilon) = 0$. $\square$

2.1.8 Riemannian tensor

We will now study how a spatial tangent field $\boldsymbol{Z} \in \Gamma T_{\|}\mathcal{M}$ can be affected if we parallelly translate it around a ε-small loop tangent to the hyper-surface S_C (this ε-small loop is generated by the flows of two commuting vector fields $\boldsymbol{X}, \boldsymbol{Y} \in \Gamma T_{\|}\mathcal{M}$). The measure of failure to return $\boldsymbol{Z}$ to its original position is defined by the *curvature transformation* linear map ${}^{\nabla}R(\boldsymbol{X}, \boldsymbol{Y})$ as follows:

$$
\boldsymbol{Z} \in \Gamma T_{\|}\mathcal{M} \longmapsto \boldsymbol{Z'} = \boldsymbol{Z} + \varepsilon^2\, {}^{\nabla}R(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z} \in \Gamma T\mathcal{M} \tag{2.50}
$$

where ${}^{\nabla}R$ is the $(1,3)$ *Riemannian tensor* field with respect to the connection ∇_μ on the manifold $\mathcal{M}$, it is defined by:

$$
{}^{\nabla}R(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z} := \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} \in \Gamma T\mathcal{M} \tag{2.51}
$$

In terms of components:

$$
{}^{\nabla}R^{\rho}{}_{\sigma\mu\nu}\partial_\rho := \nabla_\mu\nabla_\nu\partial_\sigma - \nabla_\nu\nabla_\mu\partial_\sigma \tag{2.52}
$$

The last term in Eq. (2.51) does not contribute to Eq. (2.52) since we have $[\partial_\mu, \partial_\nu] = 0$. It is very obvious to see that the new $\boldsymbol{Z'}$ may not be tangent to S_C because of ${}^{\nabla}R(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z}$ has two derivatives, then one has to split it into tangent part and normal proportional to $\boldsymbol{n}$.

Definition 8. *The* $(1,3)$ *spatial Riemannian tensor fields* ${}^{D}R$ *with respect to the connection* D_μ *on the hyper-surface* S_C *is defined by:* $\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \Gamma T_{\parallel}\mathcal{M}$:

$$ {}^{D}R(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} := D_{\boldsymbol{X}} D_{\boldsymbol{Y}} \boldsymbol{Z} - D_{\boldsymbol{Y}} D_{\boldsymbol{X}} \boldsymbol{Z} - D_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} \in \Gamma T_{\parallel}\mathcal{M} \tag{2.53}$$

In terms of components:

$$ {}^{D}R^{\rho}{}_{\sigma\mu\nu}\partial_\rho := D_\mu D_\nu \partial_\sigma - D_\nu D_\mu \partial_\sigma \tag{2.54}$$

Lemma 2.1.3. *The embedded hyper-surfaces* $\{S_C \subset \mathcal{M}, C \in \mathbb{R}\}$ *induces a split of the image space of the curvature transformation map into tangent part and normal proportional to* $\boldsymbol{n}$ *by the following relation:* $\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \Gamma T_{\parallel}\mathcal{M}$:

$$ \begin{aligned} {}^{\nabla}R(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} = {}& {}^{D}R(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} + K(\boldsymbol{Y},\boldsymbol{Z})\nabla_{\boldsymbol{X}}\boldsymbol{n} - K(\boldsymbol{X},\boldsymbol{Z})\nabla_{\boldsymbol{Y}}\boldsymbol{n} \\ & + [(\nabla_{\boldsymbol{X}}K)(\boldsymbol{Y},\boldsymbol{Z}) - (\nabla_{\boldsymbol{Y}}K)(\boldsymbol{X},\boldsymbol{Z})]\boldsymbol{n} \end{aligned} \tag{2.55}$$

Proof. We use the definition of the covariant derivative D_μ in the expression (2.53), we have:

$$ \begin{aligned} {}^{\nabla}R(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} &= \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} \\ &= \nabla_{\boldsymbol{X}}(D_{\boldsymbol{Y}}\boldsymbol{Z} + K(\boldsymbol{Y},\boldsymbol{Z})\boldsymbol{n}) - \nabla_{\boldsymbol{Y}}(D_{\boldsymbol{X}}\boldsymbol{Z} + K(\boldsymbol{X},\boldsymbol{Z})\boldsymbol{n}) - D_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} - K([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z})\boldsymbol{n} \\ &= D_{\boldsymbol{X}}D_{\boldsymbol{Y}}\boldsymbol{Z} - D_{\boldsymbol{Y}}D_{\boldsymbol{X}}\boldsymbol{Z} - D_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} + K(\boldsymbol{Y},\boldsymbol{Z})\nabla_{\boldsymbol{X}}\boldsymbol{n} - K(\boldsymbol{X},\boldsymbol{Z})\nabla_{\boldsymbol{Y}}\boldsymbol{n} \\ &+ [K(\boldsymbol{X},D_{\boldsymbol{Y}}\boldsymbol{Z}) + \nabla_{\boldsymbol{X}}(K(\boldsymbol{Y},\boldsymbol{Z})) - K(\boldsymbol{Y},D_{\boldsymbol{X}}\boldsymbol{Z}) - \nabla_{\boldsymbol{Y}}(K(\boldsymbol{X},\boldsymbol{Z})) - K([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z})]\,\boldsymbol{n} \\ &=^{D} R(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} + K(\boldsymbol{Y},\boldsymbol{Z})\nabla_{\boldsymbol{X}}\boldsymbol{n} - K(\boldsymbol{X},\boldsymbol{Z})\nabla_{\boldsymbol{Y}}\boldsymbol{n} \\ &+ \underbrace{[K(\boldsymbol{X},\nabla_{\boldsymbol{Y}}\boldsymbol{Z}) + \nabla_{\boldsymbol{X}}(K(\boldsymbol{Y},\boldsymbol{Z})) - K(\boldsymbol{Y},\nabla_{\boldsymbol{X}}\boldsymbol{Z}) - \nabla_{\boldsymbol{Y}}(K(\boldsymbol{X},\boldsymbol{Z})) - K([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z})]}_{I}\boldsymbol{n} \end{aligned}$$

where we have used the fact that $K(\boldsymbol{X}, D_{\boldsymbol{Y}}\boldsymbol{Z}) = K(\boldsymbol{X}, \nabla_{\boldsymbol{Y}}\boldsymbol{Z})$ because of the spatial property of the extrinsic curvature. In the last term proportional to $\boldsymbol{n}$, we use the Leibniz rule of the covariant derivative ∇ to get a torsion term:

$$ \begin{aligned} I &= K(\boldsymbol{X},\nabla_{\boldsymbol{Y}}\boldsymbol{Z}) + \nabla_{\boldsymbol{X}}(K(\boldsymbol{Y},\boldsymbol{Z})) - K(\boldsymbol{Y},\nabla_{\boldsymbol{X}}\boldsymbol{Z}) - \nabla_{\boldsymbol{Y}}(K(\boldsymbol{X},\boldsymbol{Z})) - K([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}) \\ &= (\nabla_{\boldsymbol{X}}K)(\boldsymbol{Y},\boldsymbol{Z}) - (\nabla_{\boldsymbol{Y}}K)(\boldsymbol{X},\boldsymbol{Z}) + K(\nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X} - [\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}) \\ &= (\nabla_{\boldsymbol{X}}K)(\boldsymbol{Y},\boldsymbol{Z}) - (\nabla_{\boldsymbol{Y}}K)(\boldsymbol{X},\boldsymbol{Z}) + K(\underbrace{{}^{\nabla}T(X,Y)}_{0},\boldsymbol{Z}) \end{aligned}$$

by substituting I in our relation, the formula (2.55) has been proved. □

Definition 9. *The* $(0,4)$ *Riemannian tensor field* ${}^{\nabla}Riem$ *with respect to the connection* ∇_μ *on the manifold* $\mathcal{M}$ *and is defined by:* $\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{X} \in \Gamma T\mathcal{M}$:

$$ {}^{\nabla}Riem(\boldsymbol{W},\boldsymbol{Z},\boldsymbol{X},\boldsymbol{Y}) := g(\boldsymbol{W},{}^{\nabla}R(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z}) \tag{2.56}$$

In terms of components:

$$ {}^{\nabla}Riem_{\rho\sigma\mu\nu} := g_{\rho\lambda}\ {}^{\nabla}R^{\lambda}{}_{\sigma\mu\nu} = {}^{\nabla}R_{\rho\sigma\mu\nu} \tag{2.57}$$

Definition 10. *The* $(0,4)$ *spatial Riemannian tensor fields* ${}^{D}Riem$ *with respect to the connection* D_μ *on the hyper-surface* S_C *is defined by:* $\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W} \in \Gamma T_{\|}\mathcal{M}$:

$$^{D}Riem(\boldsymbol{W}, \boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{Y}) := g(\boldsymbol{W}, {}^{D}R(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z}) \tag{2.58}$$

In terms of components:

$$^{D}Riem_{\rho\sigma\mu\nu} := g_{\rho\lambda}\ {}^{D}R^{\lambda}{}_{\sigma\mu\nu} =\ {}^{D}R_{\rho\sigma\mu\nu} \tag{2.59}$$

Lemma 2.1.4. *The following are hold:*

(i) Gauss equation: $\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W} \in \Gamma T_{\|}\mathcal{M}$:

$$\begin{aligned} ^{\nabla}Riem(\boldsymbol{W}, \boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{Y}) &= {}^{D}Riem(\boldsymbol{W}, \boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{Y}) \\ &\quad - \epsilon K(\boldsymbol{X}, \boldsymbol{W})K(\boldsymbol{Y}, \boldsymbol{Z}) + \epsilon K(\boldsymbol{Y}, \boldsymbol{W})K(\boldsymbol{X}, \boldsymbol{Z}) \end{aligned} \tag{2.60}$$

In terms of components:

$$h_\mu{}^\alpha h_\nu{}^\beta h_\rho{}^\gamma h_\sigma{}^\delta\ {}^{\nabla}R_{\alpha\beta\gamma\delta} =\ {}^{D}R_{\mu\nu\rho\sigma} - \epsilon K_{\mu\rho}K_{\nu\sigma} + \epsilon K_{\mu\sigma}K_{\nu\rho} \tag{2.61}$$

(ii) Scalar curvature decomposition:

$$^{d}R =\ {}^{d-1}R + \epsilon[(trK)^2 - tr(K \circ K)] + 2\epsilon \boldsymbol{\nabla} \cdot \boldsymbol{v} \in C^\infty(\mathcal{M}) \tag{2.62}$$

where $\boldsymbol{v} \in \Gamma T\mathcal{M}$ *is defined by:*

$$\boldsymbol{v} := \nabla_{\boldsymbol{n}}\boldsymbol{n} - (\boldsymbol{\nabla} \cdot \boldsymbol{n})\boldsymbol{n} = \boldsymbol{a} - (\boldsymbol{\nabla} \cdot \boldsymbol{n})\boldsymbol{n} \tag{2.63}$$

${}^{d}R$ *is the scalar curvature of the d-manifold* $\mathcal{M}$*:*

$$^{d}R = g^{\mu\rho}g^{\nu\sigma}\ {}^{\nabla}R_{\mu\nu\rho\sigma} \tag{2.64}$$

${}^{d-1}R$ *is the spatial scalar curvature of the hyper-surface* S_C*:*

$$^{d-1}R = h^{\mu\rho}h^{\nu\sigma}\ {}^{D}R_{\mu\nu\rho\sigma} \tag{2.65}$$

Proof. A detail proof.

(i) The Gauss equation is the projecting part of the equation (2.55) into hypersurfaces S_C along the $\boldsymbol{W}$ direction, then we just keep the parallel part of (2.55) and use the definition of the extrinsic curvature in (2.41), we have:

$$\begin{aligned}
{}^{\nabla}Riem(\boldsymbol{W},\boldsymbol{Z},\boldsymbol{X},\boldsymbol{Y}) &= g(\boldsymbol{W},{}^{\nabla}R(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z}) \\
&= g(\boldsymbol{W},{}^{D}R(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z}) + g(\boldsymbol{W},\nabla_{\boldsymbol{X}}\boldsymbol{n})K(\boldsymbol{Y},\boldsymbol{Z}) - g(\boldsymbol{W},\nabla_{\boldsymbol{Y}}\boldsymbol{n})K(\boldsymbol{X},\boldsymbol{Z}) \\
&= {}^{D}Riem(\boldsymbol{W},\boldsymbol{Z},\boldsymbol{X},\boldsymbol{Y}) - g(\nabla_{\boldsymbol{X}}\boldsymbol{W},\boldsymbol{n})K(\boldsymbol{Y},\boldsymbol{Z}) + g(\nabla_{\boldsymbol{Y}}\boldsymbol{W},\boldsymbol{n})K(\boldsymbol{X},\boldsymbol{Z}) \\
&= {}^{D}Riem(\boldsymbol{W},\boldsymbol{Z},\boldsymbol{X},\boldsymbol{Y}) - \epsilon K(\boldsymbol{X},\boldsymbol{W})K(\boldsymbol{Y},\boldsymbol{Z}) + \epsilon K(\boldsymbol{Y},\boldsymbol{W})K(\boldsymbol{X},\boldsymbol{Z}) \quad \square
\end{aligned}$$

(ii) The scalar curvature:

$$\begin{aligned}
{}^{d}R &= g^{\mu\nu}g^{\rho\sigma}\ {}^{\nabla}R_{\rho\mu\sigma\nu} \\
&= (h^{\mu\nu} + \epsilon n^{\mu}n^{\nu})(h^{\rho\sigma} + \epsilon n^{\rho}n^{\sigma})\ {}^{\nabla}R_{\rho\mu\sigma\nu} \\
&= \underbrace{h^{\mu\nu}h^{\rho\sigma}\ {}^{\nabla}R_{\rho\mu\sigma\nu}}_{t_1} + 2\epsilon\underbrace{h^{\rho\sigma}n^{\mu}n^{\nu}\ {}^{\nabla}R_{\rho\mu\sigma\nu}}_{t_2}
\end{aligned}$$

where in the third step we used the antisymmetry of the Riemann tensor to eliminate the term quartic in n. Now, we will compute separately each term in the last step:

$$\begin{aligned}
t_1 &= h^{\mu\nu}h^{\rho\sigma}\ {}^{\nabla}R_{\rho\mu\sigma\nu} \\
&= h^{\alpha\beta}h^{\gamma\delta}(h_{\alpha}^{\ \mu}h_{\beta}^{\ \nu}h_{\gamma}^{\ \varrho}h_{\delta}^{\ \sigma}\ {}^{\nabla}R_{\rho\mu\sigma\nu}) \\
&= h^{\alpha\beta}h^{\gamma\delta}(\ {}^{D}R_{\gamma\alpha\delta\beta} - \epsilon K_{\gamma\delta}K_{\alpha\beta} + \epsilon K_{\gamma\beta}K_{\alpha\delta}) \\
&= \ {}^{D}R - \epsilon[(trK)^2 - tr(K\circ K)]
\end{aligned}$$

where we have used the Gauss equation (2.61) in the third step.

$$\begin{aligned}
t_2 &= h^{\rho\sigma}n^{\mu}n^{\nu}\ {}^{\nabla}R_{\rho\mu\sigma\nu} \\
&= g^{\rho\sigma}n^{\mu}n^{\nu}\ {}^{\nabla}R_{\rho\mu\sigma\nu} \\
&= g^{\rho\sigma}n^{\nu}[\nabla_{\sigma},\nabla_{\nu}]n^{\mu} \\
&= (\nabla_{\mu}n^{\mu})(\nabla_{\rho}n^{\rho}) - (\nabla_{\mu}n^{\rho})(\nabla_{\rho}n^{\mu}) + \nabla_{\mu}(n^{\rho}\nabla_{\rho}n^{\mu} - n^{\mu}\nabla_{\rho}n^{\rho}) \\
&= (trK)^2 - tr(K\circ K) + \boldsymbol{\nabla}\cdot\boldsymbol{v}
\end{aligned}$$

In the last step, we have used the fact that:

$$
\begin{aligned}
trK &= g^{\mu\nu}K_{\mu\nu} \\
&= g^{\mu\nu}(h_\mu{}^\rho h_\nu{}^\sigma \nabla_\rho n_\sigma) \\
&= h^{\rho\sigma}\nabla_\rho n_\sigma \\
&= g^{\rho\sigma}\nabla_\rho n_\sigma - \epsilon n^\rho \underbrace{n^\sigma \nabla_\rho n_\sigma}_{0} \\
&= \nabla_\rho n^\rho
\end{aligned}
$$

and,

$$
\begin{aligned}
tr(K\circ K) &= g^{\mu\nu}K_{\mu\rho}g^{\rho\sigma}K_{\sigma\nu} \\
&= g^{\mu\nu}(h_\mu{}^\alpha h_\rho{}^\gamma \nabla_\alpha n_\gamma)g^{\rho\sigma}(h_\sigma{}^\delta h_\nu{}^\beta \nabla_\delta n_\beta) \\
&= h^{\alpha\beta}h^{\gamma\delta}(\nabla_\alpha n_\gamma)(\nabla_\delta n_\beta) \\
&= (h^{\alpha\beta} - \epsilon n^\alpha n^\beta)(h^{\gamma\delta} - \epsilon n^\gamma n^\delta)(\nabla_\alpha n_\gamma)(\nabla_\delta n_\beta) \\
&= h^{\alpha\beta}h^{\gamma\delta}(\nabla_\alpha n_\gamma)(\nabla_\delta n_\beta) - \epsilon h^{\alpha\beta}n^\delta \underbrace{(n^\gamma \nabla_\alpha n_\gamma)}_{0}(\nabla_\delta n_\beta) \\
&- \epsilon n^\alpha h^{\gamma\delta}(\nabla_\alpha n_\gamma)\underbrace{(n^\beta \nabla_\delta n_\beta)}_{0} + n^\alpha n^\delta \underbrace{(n^\gamma \nabla_\alpha n_\gamma)}_{0}\underbrace{(n^\beta \nabla_\delta n_\beta)}_{0} \\
&= (\nabla_\alpha n^\beta)(\nabla_\beta n^\alpha)
\end{aligned}
$$

by substituting t_1 and t_2 in our expression, the formula (2.62) has been proved.

□

2.2 Space-Time Foliation

2.2.1 Space-Time in GR

Definition 11. *A topological space $(\mathcal{M}, \mathcal{J})$ where $\mathcal{M}$ be some set and $\mathcal{J}$ is a collection of subsets of $\mathcal{M}$. Then $\mathcal{J} \subseteq \mathcal{P}(\mathcal{M})$[6] is called a topology if it satisfies the following axioms:*

[6] $\mathcal{P}(\mathcal{M})$ is the power set of $\mathcal{M}$: the set of all subsets of $\mathcal{M}$.

1. *The empty set $\emptyset$ and $\mathcal{M}$ itself belong to $\mathcal{J}$:*

$$\{\emptyset, \mathcal{M}\} \subseteq \mathcal{J} \tag{2.66}$$

2. *Any arbitrary (finite or infinite) union of elements of $\mathcal{J}$ must still in $\mathcal{J}$:*

$$\forall C_\alpha \in \mathcal{J} : \bigcup_\alpha C_\alpha \in \mathcal{J} \tag{2.67}$$

3. *The intersection of any finite number of elements of $\mathcal{J}$ must still in $\mathcal{J}$:*

$$\forall C_i \in \mathcal{J}, i = 1, \ldots, N : \bigcap_{i=1}^{N} C_i \in \mathcal{J} \tag{2.68}$$

Definition 12. *A topological manifold $(\mathcal{M}, \mathcal{J}, \mathcal{A})$ is a topological space $(\mathcal{M}, \mathcal{J})$ that locally similar to (homeomorphic to) Euclidean space near to each point:*

$$\forall p \in \mathcal{M}, \forall U \in \mathcal{J} \wedge p \in U : \exists \boldsymbol{x} \in Homeo(U, \mathbb{R}^d) \tag{2.69}$$

the pair $(U, \boldsymbol{x})$ is called a chart of the manifold $\mathcal{M}$. The collection of all charts $(U, \boldsymbol{x})$ of the manifold is called atlas $\mathcal{A}$. Thus, A topological manifold $(\mathcal{M}, \mathcal{J}, \mathcal{A})$ where $(X, \mathcal{J})$ is a topological space and $\mathcal{A}$ is an atlas.

Definition 13. *A differentiable manifold $(\mathcal{M}, \mathcal{J}, \mathcal{A})$ is a topological manifold $(\mathcal{M}, \mathcal{J})$ equipped with an atlas $\mathcal{A}$ in which the transition maps between their charts are all differentiable. Thus, A differentiable manifold $(\mathcal{M}, \mathcal{J}, \mathcal{A})$ is a topological space $(\mathcal{M}, \mathcal{J})$ that locally diffeomorphic to Euclidean space near to each point:*

$$\forall p \in \mathcal{M}, \forall U \in \mathcal{J} \wedge p \in U : \exists \boldsymbol{x} \in Diff(U, \mathbb{R}^d) \tag{2.70}$$

Definition 14. *A smooth manifold $(\mathcal{M}, \mathcal{J}, \mathcal{A})$ or C^∞-manifold manifold is a differentiable manifold for which all the transition maps between charts of the atlas $\mathcal{A}$ are smooth. That is:*

$$\forall p \in \mathcal{M}, \forall U \in \mathcal{J} \wedge p \in U : \exists \boldsymbol{x} \in C^\infty(U, \mathbb{R}^d) \tag{2.71}$$

Definition 15. *A (smooth) Riemannian manifold* $(\mathcal{M}, \mathcal{J}, \mathcal{A}, g)$ *is a real smooth manifold* $(\mathcal{M}, \mathcal{J}, \mathcal{A})$ *equipped with a smooth inner product* $g_p \in T_2^0\mathcal{M}$ *(symmetric, non-degenerate and positive defined) on each tangent space* $T_p\mathcal{M}$ *at a point* p *in* $\mathcal{M}$. *More precisely,*

$$\forall p \in \mathcal{M}, \forall \boldsymbol{X}, \boldsymbol{Y} \in \Gamma T\mathcal{M} : p \mapsto g_p(\boldsymbol{X}(p), \boldsymbol{Y}(p)) \in C^\infty(\mathcal{M}) \tag{2.72}$$

The family of inner products g_p *at each point* p *in* $\mathcal{M}$ *is called a Riemannian metric.*

Definition 16. *A pseudo-Riemannian manifold* $(\mathcal{M}, \mathcal{J}, \mathcal{A}, g)$ *is a real smooth manifold* $(\mathcal{M}, \mathcal{J}, \mathcal{A})$ *equipped with a smooth, symmetric and non-degenerate metric tensor* g *(it is not necessary positive definite). Such a metric* g *is called a pseudo-Riemannian metric. The signature of a pseudo-Riemannian metric* g *is* (p, q)[7].

Definition 17. *A Lorentzian manifold* $(\mathcal{M}, \mathcal{J}, \mathcal{A}, g)$ *is an important special case of a pseudo-Riemannian manifold in which the signature of the metric* g *is* $(1, d-1)$ *or equivalently* $(d-1, 1)$. *Such a metric* g *is called Lorentzian metric.*

Definition 18. *In General Relativity, a spacetime is a* 4*d-Lorentzian manifold with a metric signature* $(1, 3) \equiv (+, -, -, -)$ *or equivalently* $(3, 1) \equiv (-, +, +, +)$.

2.2.2 Foliation

In what follows, we are going to split the space-time into "space" and "time", that is called a *foliation*. In this framework, a theorem due to Geroch [37] and improved by Bernal and Sanchez [38] says: if the spacetime $\mathcal{M}$ is globally hyperbolic then it is necessarily to admit smooth space-like Cauchy hyper-surfaces and then it is diffeomorphic to $\mathbb{R} \times \Sigma$. This foliation allows us to identify Σ with “space” and the real line $\mathbb{R}$ with “time”. To be more explicit, let us have the following definitions.

Definition 19. *A Cauchy surface is everywhere space-like hyper-surface which is intersected by every inextensible causal (everywhere time-like) curve exactly once.*

[7] p is the number of positive eigenvalues of g, q is the number of negative eigenvalues of g.

its significance in accordance with the determinism of classical physics, giving the initial conditions on this Cauchy hyper-surface determines uniquely the future and the past with respect to this hyper-surface. Then, Cauchy hyper-surfaces are the natural subsets where initial conditions to the differential Einstein's equations are posed.

Definition 20. *In General Relativity, a Globally hyperbolic spacetime is a spacetime with a certain condition on the causal structure; it admits a Cauchy surface in $\mathcal{M}$.*

Theorem 2.2.1. *Geroch's splitting theorem: Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime. Then $(\mathcal{M}, g)$ is strongly causal and there exists a global "time function" on the manifold, i.e. a continuous, surjective map $\tau : \mathcal{M} \to \mathbb{R}$ such that:*

- $\forall C \in \mathbb{R} : S_C = preim_\tau(C) \subset \mathcal{M}$ *is a Cauchy hyper-surface.*
- τ *is strictly increasing on any causal curve.*

Moreover, all Cauchy hyper-surfaces $\{S_C\}$ in $\mathcal{M}$ are homeomorphic, and $\mathcal{M}$ is homeomorphic to $\mathbb{R} \times \Sigma$ where Σ is a 3d-space homeomorphic to any Cauchy hyper-surface S_C of $\mathcal{M}$.

Theorem 2.2.2. *Bernal, Sanchez's splitting theorem: Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime. Then $(\mathcal{M}, g)$ is strongly causal and there exists a global "time function" on the manifold, i.e. a smooth, surjective map $\tau : \mathcal{M} \to \mathbb{R}$ such that:*

- $\forall C \in \mathbb{R} : S_C = preim_\tau(C) \subset \mathcal{M}$ *is a smooth Cauchy surface.*
- τ *is strictly increasing on any causal curve.*

Moreover, all smooth Cauchy hyper-surfaces $\{S_C\}$ in $\mathcal{M}$ are diffeomorphic, and $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times \Sigma$ where Σ is a 3d-space diffeomorphic to any smooth Cauchy hyper-surface S_C of $\mathcal{M}$.

Having made these definitions, one has to consider the spacetime as a $4d$-globaly hyperbolic space $\mathcal{M}$ diffeomorphic to $\mathbb{R}\times\Sigma$, where Σ is a fixed $3d$-manifold of arbitrary topology and positive signature and we write $\mathcal{M}\cong_{Diff}\mathbb{R}\times\Sigma$. We define a diffeomorphism foliation map ϕ by:

$$\begin{aligned}\phi:\mathbb{R}\times\Sigma &\longrightarrow \mathcal{M}\\ (C,\sigma)&\longmapsto \phi(C,\sigma):=e_C(\sigma)\end{aligned} \tag{2.73}$$

where $\{e_C\}$, $C\in\mathbb{R}$ are arbitrary one-parameter embedding maps family defined in Eq. (2.1). We consider the coordinates systems $\{\boldsymbol{x}\}$ and $\{\boldsymbol{y}\}$ as follows:

$$\begin{array}{ccc} \mathbb{R}\times\Sigma & \xrightarrow{\phi} & \mathcal{M}\\ \boldsymbol{y}\equiv(t,\boldsymbol{\sigma})\downarrow & & \downarrow\boldsymbol{x}\\ \mathbb{R}^4 & \underset{\boldsymbol{x}\circ\phi\circ\boldsymbol{y}^{-1}}{\longrightarrow} & \mathbb{R}^4\end{array} \tag{2.74}$$

According to theorem 2.2.2, there exist global smooth time functions t and τ on the manifolds $\mathbb{R}\times\Sigma$ and $\mathcal{M}$ respectively; we have:

$$\begin{array}{ccc} \mathbb{R}\times\Sigma & \xrightarrow{\phi} & \mathcal{M}\\ t\searrow & & \swarrow\tau\\ & \mathbb{R} & \end{array} \tag{2.75}$$

where

$$\tau=t\circ\phi^{-1} \tag{2.76}$$

and the function t is just the standard time coordinate on the foliated manifold $\mathbb{R}\times\Sigma$. It is defined by:

$$\begin{aligned}t:\mathbb{R}\times\Sigma &\longrightarrow \mathbb{R}\\ (C,\sigma)&\longmapsto t(C,\sigma):=C\end{aligned} \tag{2.77}$$

This splitting induces a foliation of the origin manifold $\mathcal{M}$ into Cauchy hypersurfaces (submanifolds) $S_C\subset\mathcal{M}$, $C\in\mathbb{R}$:

$$S_C=\{p\in\mathcal{M}|T(p)=C\Leftrightarrow t(\phi^{-1}(p))=C\}\subset M \tag{2.78}$$

The foliation (2.73) seems to break diffeomorphism invariance of the theory. However, this is not the case because we do not fix the foliation map (2.73), but rather we keep it arbitrary.

Lemma 2.2.3. *The freedom in the choice of the foliation is equivalent to the diffeomorphism group $Diff(\mathcal{M})$.*

Proof. Let $\varphi \in Diff(\mathcal{M})$ be a diffeomorphism between globaly hyperbolic spaces $\mathcal{M}$ and $\mathcal{N}$; we write $\mathcal{M} \cong_{Diff} \mathcal{N}$. Then by theorem 2.2.2, one can find foliations ϕ_1, ϕ_2 of $\mathcal{M}$ and $\mathcal{N}$ respectively to $\mathbb{R} \times \Sigma$:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\varphi} & \mathcal{N} \\ \phi_1 \nwarrow & & \nearrow \phi_2 \\ & \mathbb{R} \times \Sigma & \end{array}$$

then for any diffeomorphism $\varphi \in Diff(\mathcal{M})$, there exist two different foliations ϕ_1, ϕ_2, where:

$$\varphi = \phi_2 \circ \phi_1^{-1}$$

Conversely, for any foliation $\phi : \mathbb{R} \times \Sigma \to \mathcal{M}$, there exists a diffeomorphism $\varphi = \phi$ which is the foliation itself (since any foliation is a diffeomorphism). Thus the lemma has been proved. □

As a conclusion, a splitting of spacetime is an arbitrary choice; there are lots of ways to pick a foliation map in Eq. (2.73). These give different way to define a time function t on the spacetime manifold $\mathcal{M}$. Since the action of general relativity is diffeomorphism-invariant, it does not depend on this auxiliary foliation (time function, unit normal vector field) and varying with respect to it leads to the generators of this invariance group.

The main idea

View $4d$-spacetime $(\mathcal{M}, g)$ as history of $3d$- space-like spaces (Σ, q_t) where t is considered as a time parameter. Mathematically, this means that $\mathcal{M}$ foliates into a one-parameter family of hyper-surfaces $\{S_t = e_t(\Sigma)\}$ by the one-parameter family embedding maps:

$$e_t : \Sigma \hookrightarrow \mathcal{M} \ , \ t \in \mathbb{R} \tag{2.79}$$

We shall only consider a restriction of embeddings such that all $\{S_t = e_t(\Sigma)\}$ are space-like hyper-surfaces in $\mathcal{M}$, this means we can define a unit time-like vector field $\boldsymbol{n} \in \Gamma T\mathcal{M}$ normal to S_t where:

$$g(\boldsymbol{n}, \boldsymbol{n}) = -1 \tag{2.80}$$

In the next, we will keep the ϵ, but remember that we have considered space-like hyper-surfaces, that is $\epsilon = -1$. The induced metric q_t of the $3d$-space Σ at time parameter t is defined by the pull-back map of the metric g by the embedding map e_t:

$$q_t = e_t^* g \tag{2.81}$$

We consider an interpretation of the induced metric q_t as a time-dependent $3d$-tensor field on the family of the slices $\{\Sigma_t\}$ in $\mathbb{R} \times \Sigma$. Then we can store all topological informations of the manifold $(\mathcal{M}, g)$ in the induced metrics $\{q_t\}$. In order to determine the geometry of the full manifold $(\mathcal{M}, g)$ one has to know the geometry of each space-like hyper-surface (S_t, h_t) with respect to an arbitrary time evolution vector field $\boldsymbol{\tau} \in \Gamma T\mathcal{M}$ or equivalently, the dynamics of the $3d$-space (Σ, q_t) with respect to the standard time evolution vector field $\boldsymbol{t} = \frac{\partial}{\partial t} \in \Gamma T(\mathbb{R} \times \Sigma)$. The time-dependent $3d$-metric q_t will play a crucial role as the configuration variables of canonical gravity.

Time vector field

As previously stated, we need to define a direction of time evolution vector field $\boldsymbol{\tau} \in \Gamma T\mathcal{M}$. A natural choice will be the push-forward of the coordinate time vector $\boldsymbol{t} = \frac{\partial}{\partial t} \in \Gamma T(\mathbb{R} \times \Sigma)$ by the foliation map (2.73), for any $p \in \mathcal{M}$, we have:

$$\begin{aligned}\boldsymbol{\tau}_p = \phi_* \boldsymbol{t}_{\phi^{-1}(p)} &= \frac{\partial x^\mu \circ \phi \circ \boldsymbol{y}^{-1}}{\partial t}\bigg|_{\boldsymbol{y}(\phi^{-1}(p))} \left(\frac{\partial}{\partial x^\mu}\right)_{\boldsymbol{x}(p)} \in T_p\mathcal{M} \\ &\equiv \tau^\mu_p \left(\frac{\partial}{\partial x^\mu}\right)_{\boldsymbol{x}(p)} \end{aligned} \tag{2.82}$$

Taking into consideration that the foliation map ϕ is arbitrary, then the choice of the time-evolution vector field $\boldsymbol{\tau}$ is also arbitrary and we conclude that, the parameter t is not a real time; it is just an evolution parameter (auxiliary parameter) for studying the system dynamics[8]. By using the decomposition rule in 2.1.2, the time evolution vector field $\boldsymbol{\tau}$ can be decomposed into a part tangent to S_t and a normal part proportional to the unit normal $\boldsymbol{n}$ as follows:

$$\begin{aligned}\boldsymbol{\tau} :=& \boldsymbol{\tau}^\perp + \boldsymbol{\tau}^\parallel \\ =& N\boldsymbol{n} + \boldsymbol{N}\end{aligned} \tag{2.83}$$

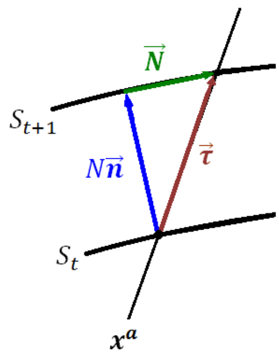


Figure 2.2: Decomposition of the time vector field $\boldsymbol{\tau}$ into parallel $\boldsymbol{N}$ and normal part $N\boldsymbol{n}$ at a given point in the manifold $\mathcal{M}$.

where $N \in C^\infty(\mathcal{M})$ is called the *lapse function* and $\boldsymbol{N} \in \Gamma T_\parallel \mathcal{M}$ is called the *shift vector*. They are given by the following relations:

$$N := \epsilon g(\boldsymbol{\tau}, \boldsymbol{n}) \tag{2.84a}$$

$$\boldsymbol{N} := \mathcal{P}^\parallel \cdot \boldsymbol{\tau} \tag{2.84b}$$

The lapse function (2.84a) is the measured time along the unit normal $\boldsymbol{n}$ elapsed between two points separated by a unit one of proper time. The shift vector (2.84b) measures the shift of the spatial coordinates between two constant hyper-surfaces separated by a unit one

[8] *problem of time*: generally covariant theories do not have a notion of a distinguished physical time. For more details, see ref. [43, 44].

of proper time. Given a time evolution vector field $\boldsymbol{\tau}$, we interpret any tensor field $\boldsymbol{T}(\boldsymbol{x})$ tangent to the hyper-surfaces $\{S_t, t \in \mathbb{R}\}$ as a time-dependent tensor field $(\phi^* \boldsymbol{T})(t, \boldsymbol{\sigma})$ on the $3d$-space Σ with respect to the coordinate time t in the foliated spacetime $\mathbb{R} \times \Sigma$; and we also require the time derivative of a spatial tensor to be a spatial and does not know anything about geometry. Hence, one has the following definition.

Definition 21. *A time derivative of a tensor field $\boldsymbol{T} \in \Gamma T_{\parallel}{}^{m}_{n}\mathcal{M}$ is defined to be spatial part of the Lie derivative along the time-evolution vector field $\boldsymbol{\tau}$:*

$$\dot{\boldsymbol{T}} := (\mathcal{P}^{\parallel} \cdots \mathcal{P}^{\parallel}) \cdot \mathcal{L}_{\boldsymbol{\tau}} \boldsymbol{T} \tag{2.85}$$

Lemma 2.2.4. *There is a relation between extrinsic curvature K and time derivative of transverse metric h, it is written by:*

$$K = \frac{-\epsilon}{2N} \mathcal{L}_{\boldsymbol{\tau} - \boldsymbol{N}} h = \frac{-\epsilon}{2N} (\dot{h} - \mathcal{L}_{\boldsymbol{N}} h) \tag{2.86}$$

Proof. From the lemma 2.1.2, we have:

$$\begin{aligned} K_{\mu\nu} &= \frac{-\epsilon}{2} \mathcal{L}_{\boldsymbol{n}} h_{\mu\nu} \\ &= \frac{-\epsilon}{2} (n^{\rho} \nabla_{\rho} h_{\mu\nu} + h_{\rho\nu} \nabla_{\mu} n^{\rho} + h_{\mu\rho} \nabla_{\nu} n^{\rho}) \\ &= \frac{-\epsilon}{2N} ((N n^{\rho}) \nabla_{\rho} h_{\mu\nu} + h_{\rho\nu} \nabla_{\mu} (N n^{\rho}) + h_{\mu\rho} \nabla_{\nu} (N n^{\rho})) \\ &= \frac{-\epsilon}{2N} \mathcal{L}_{N\boldsymbol{n}} h_{\mu\nu} \\ &= \frac{-\epsilon}{2N} \mathcal{L}_{\boldsymbol{\tau} - \boldsymbol{N}} h_{\mu\nu} \end{aligned}$$

since the extrinsic curvature is a spatial tensor field, then one can write:

$$K = \frac{-\epsilon}{2N} (\dot{h} - \mathcal{L}_{\boldsymbol{N}} h)$$

□

At this point it is useful to pull-back various spatial quantities from the spacetime manifold $\mathcal{M}$ to the abstract space manifold Σ by using the embedding map

(2.1), we have:

$$(e_t^* h)_{ab}(\boldsymbol{\sigma}) = e_a^\mu(t,\boldsymbol{\sigma}) e_b^\nu(t,\boldsymbol{\sigma}) h_{\mu\nu}(\boldsymbol{x}(\boldsymbol{\sigma})) =: q_{ab}(t,\boldsymbol{\sigma}) \tag{2.87a}$$

$$(e_t^* \boldsymbol{N})_a(\boldsymbol{\sigma}) = e_a^\mu(t,\boldsymbol{\sigma}) N_\mu(\boldsymbol{x}(\boldsymbol{\sigma})) =: N_a(t,\boldsymbol{\sigma}) \tag{2.87b}$$

$$(e_t^* N)(\boldsymbol{\sigma}) = N(e_t(\boldsymbol{\sigma})) =: N(t,\boldsymbol{\sigma}) \tag{2.87c}$$

$$(e_t^* K)_{ab}(\boldsymbol{\sigma}) = e_a^\mu(t,\boldsymbol{\sigma}) e_b^\nu(t,\boldsymbol{\sigma}) K_{\mu\nu}(\boldsymbol{x}(\boldsymbol{\sigma})) =: K_{ab}(t,\boldsymbol{\sigma}) \tag{2.87d}$$

$$(e_t^{*D} R)_{abcd}(\boldsymbol{\sigma}) = e_a^\mu(t,\boldsymbol{\sigma}) e_b^\nu(t,\boldsymbol{\sigma}) e_c^\rho(t,\boldsymbol{\sigma}) e_d^\sigma(t,\boldsymbol{\sigma})\, {}^D R_{\mu\nu\rho\sigma}(\boldsymbol{x}(\boldsymbol{\sigma})) =:^3 R_{abcd}(t,\boldsymbol{\sigma}) \tag{2.87e}$$

where e_a^μ is the Jacobian of the embedding map Eq. (2.1), its expression is defined in Eq. (2.35).

Lemma 2.2.5. *The following relations are hold:*

$$(i)\quad K_{ab} = \frac{-\epsilon}{2N}(\dot{q}_{ab} - \mathcal{L}_{\boldsymbol{N}} q_{ab}) \tag{2.88a}$$

$$(ii)\quad g^{\mu\nu} K_{\mu\nu} = q^{ab} K_{ab} =: trK \tag{2.88b}$$

$$(iii)\quad g^{\mu\nu} K_{\mu\rho} g^{\rho\sigma} K_{\sigma\nu} = q^{ab} K_{ac} q^{cd} K_{db} =: tr(K \circ K) \tag{2.88c}$$

$$(iv)\quad h^{\mu\nu} h^{\rho\sigma}\ {}^D R_{\mu\rho\nu\sigma} = q^{ab} q^{cd}\ {}^3 R_{acbd} =:\ {}^3 R \tag{2.88d}$$

Proof. A detail proof.

(i) In order to prove the first relation, one has to prove the next statement: Let $\phi : \mathcal{N} \to \mathcal{M}$ be a diffeomorphism between two smooth manifolds $\mathcal{N}$. Then:

$$\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \Gamma T\mathcal{N},\ \forall \boldsymbol{w} \in \Gamma T_2^0 \mathcal{M} : (\mathcal{L}_{(\phi_* \boldsymbol{Z})} \boldsymbol{w})(\phi_* \boldsymbol{X}, \phi_* \boldsymbol{Y}) = (\mathcal{L}_{\boldsymbol{Z}}(\phi^* \boldsymbol{w}))(\boldsymbol{X}, \boldsymbol{Y})$$

$$\begin{aligned}
&(\mathcal{L}_{(\phi_* \boldsymbol{Z})} \boldsymbol{w})(\phi_* \boldsymbol{X}, \phi_* \boldsymbol{Y}) \\
&= (\phi_* \boldsymbol{Z})(\boldsymbol{w}(\phi_* \boldsymbol{X}, \phi_* \boldsymbol{Y})) - \boldsymbol{w}(\mathcal{L}_{(\phi_* \boldsymbol{Z})}(\phi_* \boldsymbol{X}), \phi_* \boldsymbol{Y}) - \boldsymbol{w}(\phi_* \boldsymbol{X}, \mathcal{L}_{(\phi_* \boldsymbol{Z})}(\phi_* \boldsymbol{Y})) \\
&= (\phi_* \boldsymbol{Z})((\phi^* \boldsymbol{w})(\boldsymbol{X}, \boldsymbol{Y}) \circ \phi^{-1}) - \boldsymbol{w}(\phi_*(\mathcal{L}_{\boldsymbol{Z}} \boldsymbol{X}), \phi_* \boldsymbol{Y}) - \boldsymbol{w}(\phi_* \boldsymbol{X}, \phi_*(\mathcal{L}_{\boldsymbol{Z}} \boldsymbol{Y})) \\
&= \boldsymbol{Z}((\phi^* \boldsymbol{w})(\boldsymbol{X}, \boldsymbol{Y}) \circ \underbrace{\phi^{-1} \circ \phi}_{\mathbb{I}_{\mathcal{N}}}) - (\phi^* \boldsymbol{w})(\mathcal{L}_{\boldsymbol{Z}} \boldsymbol{X}, \boldsymbol{Y}) - (\phi^* \boldsymbol{w})(\boldsymbol{X}, \mathcal{L}_{\boldsymbol{Z}} \boldsymbol{Y}) \\
&= (\mathcal{L}_{\boldsymbol{Z}}(\phi^* \boldsymbol{w}))(\boldsymbol{X}, \boldsymbol{Y})
\end{aligned}$$

In the second step, we have used the relation $\mathcal{L}_{(\phi_* \boldsymbol{Z})}(\phi_* \boldsymbol{X}) = \phi_*(\mathcal{L}_{\boldsymbol{Z}} \boldsymbol{X})$

$$\begin{aligned}
\forall f \in C^\infty(\mathcal{M}) : (\mathcal{L}_{(\phi_* \boldsymbol{Z})}(\phi_* \boldsymbol{X}))f &= [\phi_* \boldsymbol{Z}, \phi_* \boldsymbol{X}] f \\
&= (\phi_* \boldsymbol{Z})((\phi_* \boldsymbol{X})f) - (\phi_* \boldsymbol{X})((\phi_* \boldsymbol{Z})f) \\
&= (\phi_* \boldsymbol{Z})(\boldsymbol{X}(f \circ \phi) \circ \phi^{-1}) - (\phi_* \boldsymbol{X})(\boldsymbol{Z}(f \circ \phi) \circ \phi^{-1}) \\
&= \boldsymbol{Z}(\boldsymbol{X}(f \circ \phi) \circ \underbrace{\phi^{-1} \circ \phi}_{\mathbb{I}_{\mathcal{N}}}) - \boldsymbol{X}(\boldsymbol{Z}(f \circ \phi) \circ \underbrace{\phi^{-1} \circ \phi}_{\mathbb{I}_{\mathcal{N}}}) \\
&= [\boldsymbol{Z}, \boldsymbol{X}](f \circ \phi) \\
&= (\mathcal{L}_{\boldsymbol{Z}} \boldsymbol{X})(f \circ \phi) \\
&= (\phi_*(\mathcal{L}_{\boldsymbol{Z}} \boldsymbol{X}))f
\end{aligned}$$

Using this statement then the formula in Eq. (2.88a) can be easily proved.

□

(ii) Since the extrinsic curvature is a spatial tensor field, then its trace must be invariant under the pull-back map e_t^*:

$$g^{\mu\nu} K_{\mu\nu} = h^{\mu\nu} K_{\mu\nu} = q^{ab} e_a^\mu e_b^\nu K_{\mu\nu} = q^{ab} K_{ab} \quad \square$$

(iii) Likewise, the trace of the K-quadratic term is also invariant under the pull-back map e_t^*:

$$g^{\mu\nu} K_{\mu\rho} g^{\rho\sigma} K_{\sigma\nu} = h^{\mu\nu} K_{\mu\rho} h^{\rho\sigma} K_{\sigma\nu} = q^{ab} e_a^\mu e_b^\nu K_{\mu\rho} q^{cd} e_c^\rho e_d^\sigma K_{\sigma\nu} = q^{ab} K_{ac} q^{cd} K_{db} \quad \square$$

(iv) By the same method, one can prove that the scalar curvature of the embedding hyper-surface (submanifold) (S_t, h) with respect to the connection D_μ is equal to the scalar curvature of the abstract $3d$-space (Σ, q_t) with respect to the $3d$-Levi Civita connection ∇_a.

□

Lemma 2.2.6. *The infinitesimal invariant interval ds^2 can be written in the foliated spacetime $\mathbb{R} \times \Sigma$ as following:*

$$ds^2 = (\epsilon N^2 + N_a N^a)dt^2 + 2N_a dt d\sigma^a + q_{ab} d\sigma^a d\sigma^b \tag{2.89}$$

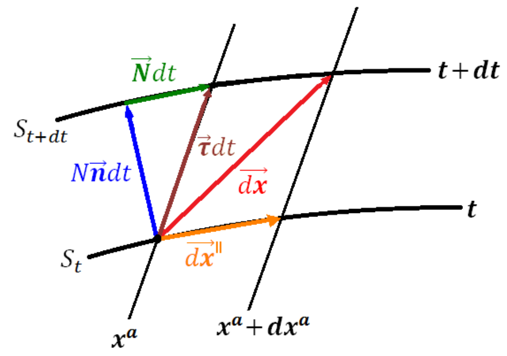


Figure 2.3: Decomposition of the line element $d\boldsymbol{x}$ into parallel $d\boldsymbol{x}^{\parallel}$ and normal part $\boldsymbol{\tau}dt$ at a given point in the manifold $\mathcal{M}$.

Proof. First of all, one has to describe the infinitesimal displacement dx^μ in the $\{\boldsymbol{y}\} = \{t, \sigma^a\}$ coordinates system, we have:

$$\begin{aligned} dx^\mu &= \frac{\partial x^\mu}{\partial t} dt + \frac{\partial x^\mu}{\partial \sigma^a} d\sigma^a \\ &= \ \tau^\mu \ dt + \ e^\mu_a \ d\sigma^a \end{aligned}$$

where τ^μ are the components of the time-evolution vector field (2.82) and e^μ_a is the Jacobian of the embedding map (2.1). Now, we replace it in the infinitesimal invariant interval as follows:

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= g_{\mu\nu}(\tau^\mu dt + e^\mu_a d\sigma^a)(\tau^\nu dt + e^\nu_b d\sigma^b) \\ &= g_{\mu\nu}\tau^\mu\tau^\nu dt^2 + 2g_{\mu\nu}\tau^\mu e^\nu_b dt d\sigma^b + g_{\mu\nu} e^\mu_a e^\nu_b d\sigma^a d\sigma^b \\ &= (\epsilon N^2 + N_a N^a) dt^2 + 2N_a dt d\sigma^a + q_{ab} d\sigma^a d\sigma^b \end{aligned}$$

where in the last step we have used the fact that:

$$\begin{aligned} g_{\mu\nu}\tau^\mu\tau^\nu &= g_{\mu\nu}(Nn^\mu + N^\mu)(Nn^\nu + N^\nu) = \epsilon N^2 + N_a N^a \\ g_{\mu\nu}\tau^\mu e^\nu_b &= h_{\mu\nu}\tau^\mu e^\nu_b = N_\nu e^\nu_b = N_b \end{aligned}$$

□

Therefore, the metric components on the foliated space $\mathbb{R} \times \Sigma$ are given by using the pull-back of the foliation map (2.73):

$$(\phi^* g)_{00} = \epsilon N^2 + N_a N^a \tag{2.90a}$$

$$(\phi^* g)_{0a} = (\phi^* g)_{a0} = N_a \tag{2.90b}$$

$$(\phi^* g)_{ab} = q_{ab} \tag{2.90c}$$

Summarizing the results of the foliation quantities:

$$t_\mu = (\epsilon N^2 + q_{ab} N^a N^b, q_{1a} N^a, q_{2a} N^a, q_{3a} N^a) \qquad t^\mu = (1, 0, 0, 0)$$

$$(\phi^* N)_\mu = (q_{ab} N^a N^b, q_{1a} N^a, q_{2a} N^a, q_{3a} N^a) \qquad (\phi^* N)^\mu = (0, N^1, N^2, N^3)$$

$$(\phi^* n)_\mu = (\epsilon N, 0, 0, 0) \qquad (\phi^* n)^\mu = (\frac{1}{N}, \frac{-N^1}{N}, \frac{-N^2}{N}, \frac{-N^3}{N})$$

$$(\phi^* h)_{\mu\nu} = \begin{bmatrix} q_{ab} N^a N^b & q_{ab} N^b \\ q_{ab} N^a & q_{ab} \end{bmatrix} \qquad (\phi^* h)^{\mu\nu} = \begin{bmatrix} 0 & 0 \\ 0 & q^{ab} \end{bmatrix}$$

$$(\phi^* h)^\mu{}_\nu = \begin{bmatrix} 0 & 0 \\ N^b & \delta^a{}_b \end{bmatrix} \qquad (\phi^* h)_\nu{}^\mu = \begin{bmatrix} 0 & N^a \\ 0 & \delta_b{}^a \end{bmatrix}$$

$$(\phi^* g)_{\mu\nu} = \begin{bmatrix} \epsilon N^2 + q_{ab} N^a N^b & q_{ab} N^b \\ q_{ab} N^a & q_{ab} \end{bmatrix} \qquad (\phi^* g)^{\mu\nu} = \begin{bmatrix} \frac{\epsilon}{N^2} & \frac{-\epsilon N^a}{N^2} \\ \frac{-\epsilon N^b}{N^2} & q^{ab} + \epsilon \frac{N^a N^b}{N^2} \end{bmatrix}$$

Lemma 2.2.7. *The invariant volume measure can be pulled-back into the foliated space as:*

$$\mathrm{d}^4 \boldsymbol{x} \sqrt{|g|} = \mathrm{d}t \mathrm{d}^3 \boldsymbol{\sigma} \sqrt{q} N \tag{2.91}$$

where we have taken that q to be the positive determinant of the induced metric q_{ab}.

Proof. In order to prove this result, one has to use the decomposition of the metric into parallel and normal parts, we have:

$$
\begin{aligned}
g &= \frac{1}{4!}\epsilon^{\mu_0\mu_1\mu_2\mu_3}\epsilon^{\nu_0\nu_1\nu_2\nu_3}g_{\mu_0\nu_0}g_{\mu_1\nu_1}g_{\mu_2\nu_2}g_{\mu_3\nu_3} \\
&= \frac{1}{4!}\epsilon^{\mu_0\mu_1\mu_2\mu_3}\epsilon^{\nu_0\nu_1\nu_2\nu_3}(h_{\mu_0\nu_0}+\epsilon n_{\mu_0}n_{\nu_0})(h_{\mu_1\nu_1}+\epsilon n_{\mu_1}n_{\nu_1}) \\
&\qquad\qquad\qquad\qquad (h_{\mu_2\nu_2}+\epsilon n_{\mu_2}n_{\nu_2})(h_{\mu_3\nu_3}+\epsilon n_{\mu_3}n_{\nu_3}) \\
&= \frac{1}{4!}\epsilon^{\mu_0\mu_1\mu_2\mu_3}\epsilon^{\nu_0\nu_1\nu_2\nu_3}\left(h_{\mu_0\nu_0}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}+4\epsilon n_{\mu_0}n_{\nu_0}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}\right) \\
&= \underbrace{\det(h)}_{0}+\epsilon\frac{1}{3!}\epsilon^{0a_1a_2a_3}\epsilon^{0b_1b_2b_3}(\epsilon N)(\epsilon N)h_{a_1b_1}h_{a_2b_2}h_{a_3b_3} \\
&= \epsilon N^2\frac{1}{3!}\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}q_{a_1b_1}q_{a_2b_2}q_{a_3b_3} \\
&= \epsilon N^2 q
\end{aligned}
$$

□

As a conclusion of this result, the spacetime geometry of the manifold $\mathcal{M}$ that is described by the 10 metric components $g_{\mu\nu}$ can be viewed in a diffeomorphic way as spatial geometry of slices $\{S_t, t\in\mathbb{R}\}$, encoded by the 6 components of the induced metric q_{ab} on the abstract $3d$-space Σ together with deformations of neighboring slices with respect to each other as described by the 4 components N, N^a. The Einstein's field equations are really 10 different equations, since there are 10 independent components in the Einstein tensor. We will rewrite these equations in terms of the induced metric q_{ab} and the extrinsic curvature K_{ab} of the slice. In Eq. (2.88a) we see that the extrinsic curvature can be thought of as representing the time derivative of the induced metric. Then, in what follows, We shall think of (q_{ab}, K_{ab}) as Cauchy data (initial conditions) for the spacetime dynamics, just as we think of the vector potential on space and the electric field as Cauchy data for electromagnetism or the Yang-Mills field. We will see that from the 10 Einstein's equations, 4 are constraint equations that the Cauchy data must satisfy, while 6 are evolutionary equations saying how the induced metric q_{ab} changes with time. This is called the Arnowitt-Deser-Misner, or ADM formulation of general relativity [39].

2.3 ADM Formalism

Since the Einstein-Hilbert action S_{EH} is an invariant integral quantity under diffeomorphism transformation of spacetime, then we are able to write S_{EH} as an integral over the foliated spacetime $\mathbb{R} \times \Sigma$. From Eqs. (2.62,2.91), the resulting ADM action S_{ADM} for matter-free gravity can be written as:

$$\begin{aligned} S_{ADM}[q_{ab}, N, N^a] &= \frac{1}{16\pi G} \int_{\mathbb{R}} dt \int_{\Sigma} \mathrm{d}^3\boldsymbol{\sigma} \sqrt{q} N[{}^3R - (trK)^2 + tr(K \circ K)] \\ &= \int_{\mathbb{R}} dt L[q_{ab}, \dot{q}_{ab}, N, N^a] \end{aligned} \tag{2.92}$$

up to boundary terms which do not affect local field equations. This action is to be varied with respect to the $3d$-Lorentzian metric $q_{ab}(t, \boldsymbol{\sigma})$, lapse function $N(t, \boldsymbol{\sigma})$ and shift vector $N^a(t, \boldsymbol{\sigma})$ where the extrinsic curvature $K_{ab}(t, \boldsymbol{\sigma})$ to be expressed as (2.88a). Due to the diffeomorphism invariant of the action, we will expect to have a *constrained Hamiltonian system* or *gauge system* (a system of phase space includes non-physical variables, *gauge variables*). In order to deal with these kind of systems, one has to follow the Dirac-Bergman algorithm [40, 41] and for more details [42].

2.3.1 Legendre transformation

We now wish to cast this action into canonical form, that is, we would like to perform the Legendre transform from the Lagrangian density appearing in (2.92) to the corresponding Hamiltonian density: one can write down the conjugate momenta $[p^{ab}(\boldsymbol{\sigma}), p(\boldsymbol{\sigma}), p_a(\boldsymbol{\sigma})]$ to the configuration variables $[q_{ab}(\boldsymbol{\sigma}), N(\boldsymbol{\sigma}), N^a(\boldsymbol{\sigma})]$ respectively. The $20 \times \infty^3$ dimensional kinematical (unconstrained) phase space Γ can be then coordinatized as:

$$\Gamma = [q_{ab}(\boldsymbol{\sigma}), p^{ab}(\boldsymbol{\sigma}), N(\boldsymbol{\sigma}), p(\boldsymbol{\sigma}), N^a(\boldsymbol{\sigma}), p_a(\boldsymbol{\sigma})] \tag{2.93}$$

The classical canonical algebra of the system is expressed by the only non-vanishing basic Poisson relations between the configuration variables and their conjugate momenta:

$$\{q_{ab}(\boldsymbol{\sigma}), p^{cd}(\boldsymbol{\sigma'})\} = \delta^c_{(a}\delta^d_{b)}\delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \tag{2.94a}$$

$$\{N(\boldsymbol{\sigma}), p(\boldsymbol{\sigma'})\} = \delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \tag{2.94b}$$

$$\{N^a(\boldsymbol{\sigma}), p_b(\boldsymbol{\sigma'})\} = \delta^a_b\delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \tag{2.94c}$$

Observing that the action depends on $\dot{q}_{ab}$ via K_{ab} while is independent of time derivative of the remaining spacetime metric components N and N^a, we have:

$$p^{ab}(\boldsymbol{\sigma}) := \frac{\delta L}{\delta \dot{q}_{ab}(\boldsymbol{\sigma})} = \frac{\sqrt{q}}{16\pi G}[K^{ab} - tr(K)q^{ab}] \tag{2.95a}$$

$$p(\boldsymbol{\sigma}) := \frac{\delta L}{\delta \dot{N}(\boldsymbol{\sigma})} = 0 \tag{2.95b}$$

$$p_a(\boldsymbol{\sigma}) := \frac{\delta L}{\delta \dot{N}^a(\boldsymbol{\sigma})} = 0 \tag{2.95c}$$

This confirms the status that the lapse function N and shift vector N^a as non-dynamical variables that can be specified as arbitrary functions on $\mathbb{R} \times \Sigma$; they are only Lagrange multipliers (similar to A_0 in electrodynamics). Since we cannot express $\dot{N}$ and $\dot{N}^a$ as functions of their momenta, then we have $4 \times \infty^3$ primary constraints (4 primary constraints for each spatial coordinates points $\boldsymbol{\sigma}$):

$$C(\boldsymbol{\sigma}) := p = 0 \tag{2.96a}$$

$$C_a(\boldsymbol{\sigma}) := p_a = 0 \tag{2.96b}$$

Since the momentum function p^{ab} in Eq. (2.95a) can be inverted for the time derivative of the spatial metric $\dot{q}_{ab}$ as:

$$\dot{q}_{ab}(\boldsymbol{\sigma}) = 32\pi G N \mathcal{G}_{abcd} p^{cd} + 2\nabla_{(a}N_{b)} \tag{2.97}$$

Then, the Eqs. (2.96a,2.96b) are the only primary constraints; They define $16 \times \infty^3$-dimensional primary constrained surface on the kinematical phase space Γ, denoted by Γ_p as:

$$\Gamma_p = \{\Gamma | C(\boldsymbol{\sigma}) = 0, C_a(\boldsymbol{\sigma}) = 0\} \subset \Gamma \tag{2.98}$$

we will asign the equality on Γ_p by $\approx$. The Hamiltonian treatment of systems with constraints has been developed by Dirac [40, 41]. According to that theory, we are supposed to introduce Lagrange multiplier fields $\lambda(t,\boldsymbol{\sigma})$ and $\lambda^a(t,\boldsymbol{\sigma})$ for the primary constraints and to perform the Legendre transform as usual with respect to the remaining velocities which can be solved for. Following the Dirac algorithm for expressing the primary Hamiltonian one has:

$$H_p[q_{ab},p^{ab},N,N^a] := \int_\Sigma \mathrm{d}^3\boldsymbol{\sigma}\Big[\dot{q}_{ab}p^{ab} + \dot{N}\underbrace{p}_{0} + \dot{N}^a\underbrace{p_a}_{0} - \mathcal{L} + \lambda C + \lambda^a C_a\Big]$$

$$= \int_\Sigma \mathrm{d}^3\boldsymbol{\sigma}\Big[N(16\pi G\mathcal{G}_{abcd}p^{ab}p^{cd} - \frac{\sqrt{q}}{16\pi G}\ {}^3R) - 2q_{ac}N^a\nabla_b p^{bc} + \lambda C + \lambda^a C_a\Big] \quad (2.99)$$

where

$$\mathcal{G}_{abcd}(\boldsymbol{\sigma}) := \frac{1}{2\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd}) \quad (2.100)$$

is the (inverse) DeWitt supermetric on the space of $3d$-metrics. The DeWitt supermetric $\mathcal{G}^{abcd}$:

$$\mathcal{G}^{abcd}(\boldsymbol{\sigma}) := \frac{\sqrt{q}}{2}(q^{ac}q^{bd} + q^{ad}q^{bc} - 2q^{ab}q^{cd}) \quad (2.101)$$

can be interpreted as a metric on the space of contravariant Lorentzian metrics and it can define an interval between two infinitesimally separated Lorentzian metric q_{ab} and $q_{ab} + \delta q_{ab}$ as:

$$\langle \delta q_{ab}, \delta q_{ab}\rangle := \int_\Sigma \mathrm{d}^3\boldsymbol{\sigma}\mathcal{G}^{abcd}\delta q_{ab}\delta q_{cd} \quad (2.102)$$

We now have to ensure the consistency of the constraints, i.e. that they are preserved by evolution generated by H_p:

$$0 \approx \{C(\boldsymbol{\sigma}), H_p\} = -\frac{\delta H_p}{\delta N(\boldsymbol{\sigma})} = -16\pi G\mathcal{G}_{abcd}p^{ab}p^{cd} + \frac{\sqrt{q}}{16\pi G}\ {}^3R \quad (2.103a)$$

$$0 \approx \{C_a(\boldsymbol{\sigma}), H_p\} = -\frac{\delta H_p}{\delta N^a(\boldsymbol{\sigma})} = 2q_{ac}\nabla_b p^{bc} \quad (2.103b)$$

Therefore, the primary constraints imply secondary constraints:

$$H(\boldsymbol{\sigma}; q_{ab}, p^{ab}] := 16\pi G\mathcal{G}_{abcd}p^{ab}p^{cd} - \frac{\sqrt{q}}{16\pi G}\ {}^3R \approx 0 \quad (2.104a)$$

$$H_a(\boldsymbol{\sigma}; q_{ab}, p^{ab}] := -2q_{ac}\nabla_b p^{bc} \approx 0 \quad (2.104b)$$

they are $4 \times \infty^3$ constraints on the canonical variables $[q_{ab}(\boldsymbol{\sigma}), p^{ab}(\boldsymbol{\sigma})]$ and it is called *Hamiltonian (or super-Hamiltonian) constraint* $H(\boldsymbol{\sigma}; q_{ab}, p^{ab}]$ and *diffeomorphism (or super-momentum) constraint* $H_a(\boldsymbol{\sigma}; q_{ab}, p^{ab}]$[9]. In fact, these are the Lagrange constraints, and one can obtain it by variation the action with respect to the Lagrange multipliers N and N^a. Now, one has to check consistency for the newly generated secondary constraints, a non-trivial calculation shows that these Poisson brackets generate combinations of the secondary constraints and so we do not have tertiary constraint. The set of $8 \times \infty^3$ constraints (2.96a,2.96b,2.104a,2.104b) define the $12 \times \infty^3$ constrained surface Γ_c on the primary surface Γ_p as:

$$\Gamma_c = \{\Gamma | H(\boldsymbol{\sigma}) = 0, H_a(\boldsymbol{\sigma}) = 0, C(\boldsymbol{\sigma}) = 0, C_a(\boldsymbol{\sigma}) = 0\} \subset \Gamma_p \subset \Gamma \tag{2.105}$$

from now $\approx$ means equality on Γ_c. With these definitions (2.104a,2.104b), we see that the total prime Hamiltonian is a linear combination of constraints

$$H_p[q_{ab}, p^{ab}, N, N^a, \lambda, \lambda^a] = \int_\Sigma \mathrm{d}^3\boldsymbol{\sigma}[NH + N^a H_a + \lambda C + \lambda^a C_a] \tag{2.106}$$

The total Hamiltonian is thus constrained to vanish, this result is in accordance with general reparametrization invariance. This is in agreement with the fact that there is no absolute time in general relativity. In addition to the constraints (2.96a,2.96b,2.104a,2.104b), one has the six dynamical equations; the Hamiltonian equations of motion for the variables q_{ab}, p^{ab}, The first half,

$$\dot{q}_{ab}(\boldsymbol{\sigma}) \approx \{q_{ab}(\boldsymbol{\sigma}), H_p\} = \frac{\delta H_p}{\delta p^{ab}} \tag{2.107}$$

can be determined algebraically and it just reproduces the Eq. (2.97). The second half, yields a lengthy expression:

$$\begin{aligned}\dot{p}^{ab}(\boldsymbol{\sigma}) &\approx \{p^{ab}(\boldsymbol{\sigma}), H_p\} = -\frac{\delta H_p}{\delta q_{ab}} \\ &= -\frac{\sqrt{q}N}{16\pi G} G^{ab} + 8\pi G N \mathcal{G}_{abcd} p^{ab} p^{cd} q^{ab} - \frac{16\pi G N}{\sqrt{q}} (2p^{ac}p^b_c - p^{ab}p^c_c) \\ &\quad + \frac{\sqrt{q}}{16\pi G}(\nabla^a\nabla^b N - q^{ab}\nabla_c\nabla^c N) + \sqrt{q}\nabla_c\left(\frac{p^{ab}N^c}{\sqrt{q}}\right) - 2p^{c(a}\nabla_c N^{b)}\end{aligned} \tag{2.108}$$

[9]Since the constraints expressions are having a locally partial derivative of the canonical fields, we choose the notation $H(\boldsymbol{\sigma}; q_{ab}, p^{ab}]$ and $H_a(\boldsymbol{\sigma}; q_{ab}, p^{ab}]$ to indicate that the constraints are functionals of the canonical fields q_{ab}, p^{ab} and functions of the spatial point $\boldsymbol{\sigma}$.

We write only the final equation here (for details of calculation see [35]). Indeed these equations are not needed for canonical quantization. It is of course needed for applications of the classical canonical formalism such as gravitational-wave emission from compact binary objects.

2.3.2 Discussion of the constraints

1. The Hamiltonian constraints has some similarity to the constraint for the relativistic particle $p^2 + m^2 = 0$. While the diffeomorphism constraints is similar to Gauss's law of electrodynamics $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$.

2. The Hamiltonian of general relativity is not a true Hamiltonian but is a linear combination of constraints. Rather than generating time translations it generates spacetime diffeomorphisms[10]. Since the parameters of these diffeomorphisms N and N^a are completely arbitrary unspecified functions, the corresponding motions on the phase space have to be interpreted as gauge transformations. This is quite similar to the gauge motions generated by the Gauss constraint in Maxwell theory.

3. The Hamilton equations above (2.97) and (2.108) reproduce the projections of the Einstein equations that are tangent to the hyper-surfaces $\{S_t\}$. while the secondary constraints (2.104a) and (2.104b) reproduce the normal projections of these Einstein equations:

4. The last argument is understood in the following sense:

   - If the Lorentzian metric g satisfies the vacuum Einstein equations $G_{\mu\nu}(x; g] = 0$. Then the family of induced metrics $q_t = \phi_t^* g$ and momenta $p_t = \frac{\delta L}{\delta \dot{q}_t}$ must satisfy the equations (2.104a,2.104b,2.97,2.108).
   - Conversely, if ϕ_t is a space-like foliation of $(\mathcal{M}, g)$ such that the evolution and constraint equations above hold, then g (which is defined by (2.90)) satisfies the vacuum field equations.

[10] *problem of time*: generally covariant theories do not have a notion of a distinguished physical time. For more details, see ref. [43, 44].

5. In fact one can further show that: the Hamiltonian and diffeomorphism constraints are satisfied on every Cauchy hype-rsurface S_t if and only if the metric g satisfies the vacuum Einstein equations (the dynamical equations (2.97,2.108) is guaranteed to satisfy when we impose the secondary constraints (2.104a,2.104b) at any slice S_t).

6. In the terminology of Dirac-Bergman algorithm [40, 41], The primary and secondary constraints are first class and thus generate gauge transformations which do not change the physical information in solutions (due to the reparameterization invariance of coordinates in a generally covariant theory). The Hamiltonian constraint does this for time, and the diffeomorphism constraint for spatial coordinates.

7. Counting the number of physical degrees of freedom in the gravitational field: the kinematical phase space Γ has $20 \times \infty^3$ canonical variables. Due to the presence of the eight constraints (2.96a,2.96b,2.104a,2.104b), $8 \times \infty^3$ have to be subtracted. The remaining $12 \times \infty^3$ variables define the constrained surface Γ_c. Since the constraints are first class, then it generates 8-parameter set of gauge transformations on Γ_c, then $8 \times \infty^3$ degrees of freedom must be subtracted in order to 'fix the gauge'. The remaining $4 \times \infty^3$ variables define the reduced phase space $\Gamma_r \equiv \Gamma_{phys}$ and correspond to 2 degrees of freedom at each coordinate point $\boldsymbol{\sigma}$ in configuration space. This result agrees with the linear field analysis, which shows that the gravitational wave propagating on a fixed background spacetime has two degrees of freedom (the two circular polarisations of a gravitational wave; the two helicity states of a graviton).

8. The lapse function N and shift vector N^a play the role of Lagrange multipliers of secondary constraints. Thus, it is therefore completely straightforward to impose the primary first class constraints (2.96a,2.96b), before solving the dynamical equations of motion (2.97,2.108). To do so, we process their gauge orbits by fixing the phase space independent functions N and N^a

on the spatial manifold (the gauge fixing function must has at least one non-vanishing Poisson bracket with the primary constraints), after that, we can say that N and N^a aren't dynamical variables. We define a new constrained Hamiltonian system called the *Arnowitt - Deser - Misner (ADM) system*, with the $12 \times \infty^3$ dimensional phase space Γ_{ADM} that is canonically coordinatized by:

$$\Gamma_{ADM} = [q_{ab}(\boldsymbol{\sigma}), p^{ab}(\boldsymbol{\sigma})] \tag{2.109}$$

and an evolution ADM Hamiltonian H_{ADM} defined by:

$$H_{ADM}[q_{ab}, p^{ab}] := \int_{\Sigma} \mathrm{d}^3\boldsymbol{\sigma}[NH + N^a H_a] \tag{2.110}$$

In this theory the lapse and shift are viewed as fixed phase space independent functions on the spatial manifold. The resulting ADM action S_{ADM} defined by:

$$S_{ADM}[q_{ab}, p^{ab}] := \int_{\Sigma} \mathrm{d}^3\boldsymbol{\sigma}[\dot{q}_{ab}p^{ab} - NH - N^a H_a] \tag{2.111}$$

The secondary constraints (2.104a,2.104b) define a $8 \times \infty^3$ dimensional constrained surface Γ_{ADMc} on the ADM phase space Γ_{ADM} as:

$$\Gamma_{ADMc} = \Gamma_{ADM} \cap \{H(\boldsymbol{\sigma}) = 0, H_a(\boldsymbol{\sigma}) = 0\} \subset \Gamma_{ADM} \tag{2.112}$$

and we assign $\approx$ to be an equality on Γ_{ADMc}. From now on, we work on the constrained surface Γ_{ADMc}.

2.3.3 The Constraints algebra analysis

The classical canonical algebra of the system is expressed by the only non-vanishing basic Poisson brackets:

$$\{q_{ab}(\boldsymbol{\sigma}), p^{cd}(\boldsymbol{\sigma'})\} = \delta^c_{(a}\delta^d_{b)}\delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \tag{2.113a}$$

$$\{q_{cd}(\boldsymbol{\sigma}), q_{ab}(\boldsymbol{\sigma'})\} = 0 \tag{2.113b}$$

$$\{p^{ab}(\boldsymbol{\sigma}), p^{cd}(\boldsymbol{\sigma'})\} = 0 \tag{2.113c}$$

We now state and discuss the Poisson algebra formed by H and H_a, a crucial property of the canonical formalism is the closure of the Poisson brackets of the super-Hamiltonian and supermomentum, by using (2.113) one can show the constraints are first class. Explicit calculations for the Poisson brackets gives the fundamental relations: From there we can evaluate the following brackets among the constraints:

$$\{H_a(\boldsymbol{\sigma}), H_b(\boldsymbol{\sigma'})\} = H_a(\boldsymbol{\sigma'})\partial_b\delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) - H_b(\boldsymbol{\sigma})\partial'_a\delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \tag{2.114a}$$

$$\{H_a(\boldsymbol{\sigma}), H(\boldsymbol{\sigma'})\} = H(\boldsymbol{\sigma})\partial_a\delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \tag{2.114b}$$

$$\{H(\boldsymbol{\sigma}), H(\boldsymbol{\sigma'})\} = q^{ab}(\boldsymbol{\sigma})H_a(\boldsymbol{\sigma})\partial'_b\delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) - q^{ab}(\boldsymbol{\sigma'})H_a(\boldsymbol{\sigma'})\partial_b\delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \tag{2.114c}$$

Notice that the right-hand sides vanish on the constrained surface Γ_{ADMc}. This means that the Poisson flows generated by the constraints tangent to the constrained surface Γ_{ADMc}. Constraints with this characteristic are said to be first class, as opposed to second class constraints whose Poisson brackets do not vanish on-shell. First class constraints generate gauge transformations on the constraint surface: To see what the gauge transformations look like in our case, consider the smearing of the constraints[11]

$$H[\chi] := \int_\Sigma \mathrm{d}^3\boldsymbol{\sigma}\,\chi(\boldsymbol{\sigma})H(\boldsymbol{\sigma}) \tag{2.115a}$$

$$H[\boldsymbol{\chi}] := \int_\Sigma \mathrm{d}^3\boldsymbol{\sigma}\,\chi^a(\boldsymbol{\sigma})H_a(\boldsymbol{\sigma}) \tag{2.115b}$$

where χ and $\boldsymbol{\chi}$ are any scalar function and $3d$-vector field on Σ, respectively. An explicit computation shows that:

$$\{H[\boldsymbol{\chi}], q_{ab}(\boldsymbol{\sigma})\} = \mathcal{L}_{\boldsymbol{\chi}} q_{ab}(\boldsymbol{\sigma}) \tag{2.116a}$$

$$\{H[\boldsymbol{\chi}], p^{ab}(\boldsymbol{\sigma})\} = \mathcal{L}_{\boldsymbol{\chi}} p^{ab}(\boldsymbol{\sigma}) \tag{2.116b}$$

which means that the vector constraint is the generator of space-diffeomorphism on Σ. The situation is somewhat subtler for the Hamiltonian constraint. We have:

$$\{H[\chi], q_{ab}(\boldsymbol{\sigma})\} = \mathcal{L}_{(\chi \boldsymbol{n})} q_{ab}(\boldsymbol{\sigma}) \tag{2.117a}$$

$$\{H[\chi], p^{ab}(\boldsymbol{\sigma})\} = \mathcal{L}_{(\chi \boldsymbol{n})} p^{ab}(\boldsymbol{\sigma}) + \frac{1}{2}q^{ab}\chi H + 2\chi\sqrt{q}q^{a[b}q^{c]d}R_{cd} \tag{2.117b}$$

[11]The integrals are well-defined since both H and H_a are +1 densities on Σ.

The first bracket is the action of time diffeomorphisms on q_{ab}. The second bracket gives the action of time diffeomorphisms on p^{ab}, but contains also two extra pieces. These vanish if $H = 0$ and $R_{cd} = 0$, namely on the constrained surface Γ_{ADMc} and for physical solutions[12]. Therefore, we conclude that the Hamiltonian and diffeomorphic constraints are the generators of the spacetime diffeomorphism group $Diff(\mathcal{M})$ on physical configurations. For general configurations, they define the algebra of hyper-surface deformations, often called *Dirac algebra* or *Bargmann-Komar algebra* is given using the smeared variables:

$$\{H[\boldsymbol{\chi_1}], H[\boldsymbol{\chi_2}]\} = H[\mathcal{L}_{\boldsymbol{\chi_1}}\boldsymbol{\chi_2}] \tag{2.118a}$$

$$\{H[\boldsymbol{\chi}], H[\chi]\} = H[\mathcal{L}_{\boldsymbol{\chi}}\chi] \tag{2.118b}$$

$$\{H[\chi_1], H[\chi_2]\} = H[q^\sharp(\chi_1 d\chi_2 - \chi_2 d\chi_1)] \tag{2.118c}$$

Two important things should be noted about the Dirac algebra:

- The sub-Dirac algebra consists of the Poisson bracket (2.118a) is a Lie algebra and is isomorphic to the Lie algebra of $Diff(\Sigma)$.
- The structure constants on the right-hand side of Eq. (2.118c) contain the sharp map $q^\sharp$ (the inverse of $3d$-metric q^{ab}); hence they are not constants at all. It means that, unlike $Diff(\mathcal{M})$, the Dirach algebra is not a genuine Lie algebra even though this was the invariance group of the original theory.

Even though the Dirac algebra is not a genuine Lie algebra, it still generates *gauge transformations* on the canonical variables $[q_{ab}(\boldsymbol{\sigma}), p^{ab}(\boldsymbol{\sigma})]$ sited on the constrained surface Γ_{ADMc}; it is obtained by integrating the infinitesimal changes of the form:

$$\delta_{\chi,\boldsymbol{\chi}} q_{ab}(\boldsymbol{\sigma}) := \{q_{ab}(\boldsymbol{\sigma}), H[\chi] + H[\boldsymbol{\chi}]\} \tag{2.119a}$$

$$\delta_{\chi,\boldsymbol{\chi}} p^{ab}(\boldsymbol{\sigma}) := \{p^{ab}(\boldsymbol{\sigma}), H[\chi] + H[\boldsymbol{\chi}]\} \tag{2.119b}$$

for arbitrary infinitesimal smearing functions χ and $3d$-vector fields $\boldsymbol{\chi}$, where q_{ab}, p^{ab} must satisfy the secondary constraints (2.104a,2.104b). We shall refer to

[12] recall that in vacuum Einstein's equations for physical solutions read: $R_{\mu\nu} = 0$

the set of all such trajectories generated by (2.119) in the phase space Γ_{ADMc} as the *gauge orbits* of the Dirac algebra. A peculiar feature of general relativity is that an orbit on the constraint surface includes the dynamical evolution of a pair $[q_{ab}(\boldsymbol{\sigma}), p^{ab}(\boldsymbol{\sigma})]$ with respect to any choice of lapse function and shift vector. Indeed, the dynamical equations (2.97) and (2.108) are simply a special case of the transformations above when we consider the functions χ and $\boldsymbol{\chi}$ are the lapse N and the shift vector $\boldsymbol{N}$ respectively.

Definition 22. *The phase space function $\mathcal{O} : \Gamma_{ADMc} \to \mathbb{R}$ is said to be an observable if and only if $\mathcal{O}$ is a gauge invariant, that is:*

$$\{\mathcal{O}, H[\chi] + H[\boldsymbol{\chi}]\} \approx 0 \tag{2.120}$$

for all functions χ and 3d-vector fields $\boldsymbol{\chi}$

Notice that the basic variables of the theory, q_{ab} and p^{ab} are not observables of the theory because they are not gauge invariant.

2.4 Palatini Formulation (First Order Formulation of GR)

For certain purposes, it can be useful to put an action leading to 2^{nd} order differential equations into 1^{st} order form by the introduction of some auxiliary variables, it was first considered by Palatini [45]. It is worth to mention that a metric $g_{\mu\nu}$ and a connection $\tilde{\Gamma}^{\rho}_{\mu\nu}$ are independent concepts, and that the notion of curvature (curvature, Ricci and Riemann tensors) can be defined for an arbitrary connection

$$R^{\rho}{}_{\sigma\mu\nu}[\tilde{\Gamma}] = \partial_{\mu}\tilde{\Gamma}^{\rho}_{\sigma\nu} - \partial_{\nu}\tilde{\Gamma}^{\rho}_{\sigma\mu} + \tilde{\Gamma}^{\rho}_{\lambda\mu}\tilde{\Gamma}^{\lambda}_{\sigma\nu} - \tilde{\Gamma}^{\rho}_{\lambda\nu}\tilde{\Gamma}^{\lambda}_{\sigma\mu} \tag{2.121a}$$

$$R_{\mu\nu}[\tilde{\Gamma}] = \delta^{\sigma}_{\rho} R^{\rho}{}_{\mu\sigma\nu}[\tilde{\Gamma}] = R^{\rho}{}_{\mu\rho\nu}[\tilde{\Gamma}] \tag{2.121b}$$

$$R[g, \tilde{\Gamma}] = g^{\mu\nu} R_{\mu\nu}[\tilde{\Gamma}] \tag{2.121c}$$

General relativity employs and is formulated in terms of the canonical Levi-Civita connection described by the unique Christoffel symbols $\tilde{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu}[g]$ by the fact that the connection is compatible with the metric and has no torsion:

1. Compatibility condition: the metric is covariantly constant: $\nabla_\rho g_{\mu\nu} = 0$

2. Torsion free condition: the torsion tensor is is everywhere zero: $T^\rho_{\mu\nu} = 2\Gamma^\rho_{[\mu\nu]} = 0$

It is thus easy to come up with various generalizations of general relativity in which these requirements are relaxed. It is of course possible to relax either of the conditions (1) or (2), (not both of them!) and nevertheless reproduce general relativity by treating the connection and metric as independent variables. In particular, connections with torsion (relaxation of condition 2) are popular in certain circles and arise naturally in certain generalized gauge theories of gravity and in string theory. To discuss this a bit more systematically, we consider a general connection:

$$\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}[g] + C^\rho_{\mu\nu} \tag{2.122}$$

with $\Gamma^\rho_{\mu\nu}$ is the Levi-Civita connection, and $C^\rho_{\mu\nu}$ is a $(1, 2)$-tensor field (because it is a difference between two connection). We will also use the corresponding $(0, 3)$-tensor field as:

$$C_{\rho\mu\nu} = g_{\rho\sigma} C^\sigma_{\mu\nu} \tag{2.123}$$

Introducing the covariant derivative $\tilde{\nabla}_\nu$ associated with $\tilde{\Gamma}^\rho_{\mu\nu}$. Since $\tilde{\Gamma}^\rho_{\mu\nu}$ will in general not be symmetric in its lower indices, in this section we need to be particularly careful with the ordering of the lower indices in the covariant derivative. We will choose the convention that the last index always refers to the direction along which one is differentiating, i.e

$$\tilde{\nabla}_\nu V^\rho = \partial_\nu V^\rho + \tilde{\Gamma}^\rho_{\mu\nu} V^\mu \tag{2.124}$$

The reason for this choice is that one should think of the collection of connection objects $\tilde{\Gamma}^\rho_{\mu\nu}$ as the coefficients of a matrix-valued 1-form $\tilde{\Gamma}^\rho_\mu = \tilde{\Gamma}^\rho_{\mu\nu} dx^\nu$. The conditions on the arbitrary $C_{\rho\mu\nu}$ must be imposed to satisfy the conditions (1) and (2) are given as follos:

1. The non-metricity tensor $Q_{\rho\mu\nu}$:

$$\tilde{\nabla}_\rho g_{\mu\nu} = -Q_{\mu\nu\rho} \Rightarrow Q_{\mu\nu\rho} = g_{\mu\sigma} C^\sigma_{\nu\rho} + g_{\nu\sigma} C^\sigma_{\mu\rho} = 2C_{(\mu\nu)\rho} \tag{2.125}$$

   The connection is compatible with the metric if and only if $C\mu\nu\rho$ is anti-symmetric in its first two indices.

2. The torsion tensor $T^\rho_{\mu\nu}$:

$$T^\rho_{\mu\nu} = 2\tilde{\Gamma}^\rho_{[\mu\nu]} \Rightarrow T^\rho_{\mu\nu} = 2C^\rho_{[\mu\nu]} \tag{2.126}$$

   The torsion is zero if and only if $C^\rho_{\mu\nu}$ is symmetric in its lower indices. (or equivalently, if $C_{\rho\mu\nu}$ is symmetric in its last two indices).

In particular, if the torsion is zero and the connection is metric-compatible, one has:

$$C_{\rho\mu\nu} = 0 \tag{2.127}$$

and the connection $\tilde{\Gamma}^\rho_{\mu\nu}$ in (2.122) will be then the Levi-Civita connection.

Specifically, we will consider an action of the generalised Einstein-Hilbert like form in the absence of the coupling of the metric (gravity) to other fields:

$$S[g_{\mu\nu}, \tilde{\Gamma}^\rho_{\mu\nu}] = \frac{1}{16\pi G} \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g} R[g, \tilde{\Gamma}] \tag{2.128}$$

From now, we treat $g_{\mu\nu}$ and $\tilde{\Gamma}^\rho_{\mu\nu}$ as independent variables. Since $R[g, \tilde{\Gamma}]$ depend only on first derivative of the connection, then the Lagrangian density depends purely algebraically on the metric and the connection, and on at most 1^{st} derivatives of the connection (it is a first order formulation). It remains to look at the equations of motion imposed by stationarity of the action with respect to variations of $g_{\mu\nu}$ and $\tilde{\Gamma}^\rho_{\mu\nu}$:

Variation the action over the metric $g_{\mu\nu}$:

$$\delta_g S = 0 \Leftrightarrow G_{\mu\nu}[g, \tilde{\Gamma}] = R_{\mu\nu}[\tilde{\Gamma}] - \frac{1}{2} g_{\mu\nu} R[g, \tilde{\Gamma}] = 0 \tag{2.129}$$

These are, however, not yet the vacuum Einstein equations because the independent connection $\tilde{\Gamma}^{\rho}_{\mu\nu}$ is not the Levi-Civita connection.

Variation the action over the connection $\tilde{\Gamma}^{\rho}_{\mu\nu}$:

$$\delta_{\tilde{\Gamma}} S = 0 \Leftrightarrow (g^{\mu\nu} C^{\sigma\rho}{}_{\sigma} - C^{\mu\nu\rho} - C^{\nu\rho\mu} + g^{\nu\rho} C^{\mu\sigma}{}_{\sigma}) \delta C_{\rho\mu\nu} = 0 \tag{2.130}$$

However, these equations do not determine the $C^{\rho}_{\mu\nu}$ uniquely (we will explicitly parameterize this non-uniqueness below), and hence in this case the Einstein-Hilbert-like action (2.129) alone does not give rise to acceptable equations of motion for the fields. The situation changes if one imposes some a priori constraints on the allowed $\tilde{\Gamma}^{\rho}_{\mu\nu}$ and hence on their variations $\delta C_{\rho\mu\nu}$. We now consider separately the two cases mentioned above:

- $\tilde{\Gamma}^{\rho}_{\mu\nu}$ are restricted to be torsion-free: In terms of the coefficients $C^{\rho}_{\mu\nu}$, this amounts to the condition $C_{\rho\mu\nu} = C_{\rho(\mu\nu)}$ and the same condition should be imposed on their variations in (2.130). Thus, by manipulating appropriately the resulted equation (2.130), we obtain the non-metricity tensor to be vanish $Q_{\mu\nu\rho} = 0$ i.e. that the connection is compatible with the metric. Then we started off with a torsion-free connection and the $\tilde{\Gamma}^{\rho}_{\mu\nu}$-equations of motion fix the connection $\tilde{\Gamma}^{\rho}_{\mu\nu}$ to be the Levi-Civita connection.

- $\tilde{\Gamma}^{\rho}_{\mu\nu}$ are restricted to be compatible with the metric: In terms of the coefficients $C^{\rho}_{\mu\nu}$, this amounts to the condition $C_{\rho\mu\nu} = C_{[\rho\mu]\nu}$ the same condition should be imposed on their variations in (2.130). Thus, by manipulating appropriately the resulted equation (2.130), we obtain the torsion-free condition $C_{\rho\mu\nu} = C_{\rho(\mu\nu)}$. Since we started off with metric-compatible connection, this means that the $\tilde{\Gamma}^{\rho}_{\mu\nu}$-equations of motion fix the connection $\tilde{\Gamma}^{\rho}_{\mu\nu}$ to be the Levi-Civita connection.

This concludes the proof of the following assertion:

Theorem 2.4.1. ***Palatini principle:***

- *If we choose the connections to be torsion-free and imposes the* $\tilde{\Gamma}^{\rho}_{\mu\nu}$*-equations of motion, then the connections are forced to be also compatible with the metric and thus* $\tilde{\Gamma}^{\rho}_{\mu\nu}$ *is uniquely determined to be the Levi-Civita connection.*

- *if we choose the connections to be compatible with the metric and imposes the* $\tilde{\Gamma}^{\rho}_{\mu\nu}$*-equations of motion, then the connections are forced to be also torsion-free and thus* $\tilde{\Gamma}^{\rho}_{\mu\nu}$ *is uniquely determined to be the Levi-Civita connection.*

2.5 Tetradic Palatini Formulation

The tetradic Palatini action for general relativity, is simply the Einstein-Hilbert action rewritten so that it is not a function of metric $g_{\mu\nu}$ but instead a function of a connection $so(1,3)$-value connection $w_{\mu}{}^{I}{}_{J}$ and a frame field e_I. This formalism provides first-order field equations for general relativity.

2.5.1 A tetrad (vierbein, frame field)

A *frame field* (*tetrad*) provides a way to specify geometries alternative but equivalent to metrics or line elements. It can be viewed as a set of 4-orthonormal basis vector fields $\{e_I \in \Gamma T\mathcal{M},\ I = 0,1,2,3\}$. At each point $p \in \mathcal{M}$, we have:

$$g(e_I(p), e_J(p)) = \eta_{IJ} \tag{2.131}$$

where $[\eta_{IJ}] = diag[-1,1,1,1]$, and the frame field e_I can be linearly written in terms of the induced coordinate basis ∂_μ as:

$$e_I = e^{\mu}_{I}\partial_\mu \tag{2.132}$$

where e^{μ}_{I} is the tetrad components (it is often called tetrad). Introducing the co-tetrad fields $\{e^I \in \Gamma T^*\mathcal{M}, I = 0,1,2,3|\ e^I(e_J) = \delta^I_J\}$ as the dual of the tetrad e_I as:

$$e^I = e^I_\mu dx^\mu \tag{2.133}$$

where e^I_μ is the co-tetrad components (it is often called co-tetrad). The tetrad can also be physically understood as describing the frame of reference of an inertial observer, who in a sufficiently small region recovers, by the equivalence principle, special relativity. More formally one can define a tetrad as a vector bundle isomorphism e between a local trivialization $\mathcal{M} \times \mathbb{R}^{3,1}$ and the tangent bundle of spacetime $T\mathcal{M}$, i.e., one has to trivialise the tangent bundle into a vector bundle as:

$$e : \mathcal{M} \times \mathbb{R}^{3,1} \longrightarrow T\mathcal{M} \tag{2.134}$$

where $\mathbb{R}^{3,1}$ represents Minkowski spacetime equipped with a metric η. If one chooses a set of 4-orthonormal basis vectors $\{\xi_I \in \mathbb{R}^{3,1}, I = 0, 1, 2, 3\}$ on the Minkowski space $\mathbb{R}^{3,1}$, such that:

$$\eta(\xi_I, \xi_J) = \eta_{IJ} \tag{2.135}$$

At each point $p \in \mathcal{M}$, the trivialization map (2.134) must send the orthonormal basis ξ_I of $\mathbb{R}^{3,1}$ to the orthonormal basis $e_I(p)$ of $T_p\mathcal{M}$ as:

$$e(p, \xi_I) = e_I(p) \tag{2.136}$$

Therefore, the trivialization map (2.134) sends a copy of Minkowski space $\mathbb{R}^{3,1}$ at each point p of $\mathcal{M}$ to tangent space $T_p\mathcal{M}$ at p:

$$\begin{aligned} e : \{p\} \times \mathbb{R}^{3,1} &\longrightarrow T_p\mathcal{M} \\ (p, v_p^I \xi_I) &\longmapsto e(p, v_p^I \xi_I) := v_p^I e_I(p) = v_p^\mu \partial_\mu|_p \end{aligned} \tag{2.137}$$

where

$$v_p^\mu = e_I^\mu(p) v_p^I \tag{2.138}$$

The idea of Palatini formalism is to do a lot of work on the trivial bundle $\mathcal{M} \times \mathbb{R}^{3,1}$, which serves as a kind of substitute for the tangent bundle $T\mathcal{M}$. This approach makes contact with the mathematical formalism of classical gauge and matter fields, which are described by *principal and associated vector fibre bundles*. In this way one views the copy of Minkowski spacetime as an *internal space* in the same way that one views either the gauge group G or its representation space as an internal space in Yang-Mills matter theory. As usual, we shall use Greek letters $\mu, \nu, \rho, \ldots$ to denote spacetime indices and capital Latin letters $I, J, K, \ldots$ to denote the internal Minkowski space indices. All spacetime indices can be lowered or raised only with the spacetime metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ respectively, and similarly one can lower and raise Minkowski (internal space) indices only with η_{IJ} and η^{IJ} respectively.

In terms of components, the orthogonality of the tetrad is written by:

$$g_{\mu\nu} e^{\mu}_{I} e^{\nu}_{J} = \eta_{IJ} \tag{2.139}$$

Since the trivialization map (2.134) has considered to be inverted, one obtains:

$$\eta_{IJ} e^{I}_{\mu} e^{J}_{\nu} = g_{\mu\nu} \tag{2.140}$$

It is clear from (2.140) that the knowledge of the co-tetrad field can determined uniquely the spacetime metric. However, the converse is not true; there are an infinity of co-tetrad fields satisfying (2.140), all related to each other by local Lorentz transformations, i.e for any $\Lambda \in SO(3,1)$ we have:

$$e^{I}_{\mu} \longmapsto e'^{I}_{\mu} = \Lambda^{I}_{K} e^{K}_{\mu} \tag{2.141}$$

if e^{I}_{μ} satisfies the equation (2.140), then e'^{I}_{μ} will also check:

$$e'^{I}_{\mu} e'^{J}_{\nu} \eta_{IJ} = e^{K}_{\mu} e^{L}_{\nu} \eta_{IJ} \left[\Lambda^{T\ I}_{K} \eta_{IJ} \Lambda^{J}_{\ L} \right] = e^{K}_{\mu} e^{L}_{\nu} \eta_{KL} = g_{\mu\nu} \tag{2.142}$$

using the invariance of the Minkowski metric under Lorentz transformations. Thus the local Lorentz transformations are to be interpreted as gauge in this formalism; this can be seen from the number of independent components in the frame field, the spacetime metric has 10 such components whereas the tetrad has 16 components, the difference 6 corresponds to the dimension of the Lorentz group $SO(3,1)$.

2.5.2 Connections via tetrads

Beside the frame fields, the other structure we need in the Palatini formalism is the connection on the local trivialization $\mathcal{M} \times \mathbb{R}^{3,1}$ as with any vector bundle one cannot define differentiation without this additional structure, and in general the connection will be an $so(3,1)$ algebra - value spacetime 1-form $w^I_J = w_\mu{}^I{}_J dx^\mu$, it is called *Lorentz connection*. As in the tangent bundle $T\mathcal{M}$, we define the connection to be the amount of changing the vector basis under a parallel translation, or formally the connection represents the change in the frame axis (horizontal space) of the principal $GL(4,\mathbb{R})$-bundle fiber bundle. Then, define the $gl(4,\mathbb{R})$-value 1-form connection $\tilde{\Gamma}^\rho_\nu = \tilde{\Gamma}^\rho_{\nu\mu} dx^\mu$ in the frame bundle $L\mathcal{M}$ (principal $GL(4,\mathbb{R})$-bundle) and the $so(3,1)$-value 1-form connection $w^I_J = w^I_{J\mu} dx^\mu$ in the principal $SO(3,1)$-bundle (Lorentz bundle). They defined by:

$$\partial_\mu e_\nu = \tilde{\Gamma}_\nu{}^\rho{}_\mu e_\rho \tag{2.143a}$$

$$\partial_\mu e_I = w_I{}^J{}_\mu e_J \tag{2.143b}$$

where $e_\mu = \partial_\mu$ is the induced coordinates basis and e_I is the orthonormal frame basis, we have placed a tilde on the spacetime connection to distinguish it from the Levi-Civita connection. We may define the action of these connections $\tilde{\Gamma}^\rho_\mu$ and w^I_J by defining the covariant derivatives $\tilde{\nabla}_\mu$ and $\tilde{D}_\mu$ respectively,

$$\tilde{\nabla}_\mu v^\rho = \partial_\mu v^\rho + \tilde{\Gamma}^\rho{}_{\nu\mu} v^\nu \tag{2.144a}$$

$$\tilde{D}_\mu v^I = \partial_\mu v^I + w^I{}_{J\mu} v^J \tag{2.144b}$$

They have to be linear by addition over tensors of the same order and satisfy the Leibniz rule. From (2.143b) one can easily get an expression between the two connections:

$$w^I{}_{J\mu} = e^I_\nu \tilde{\nabla}_\mu e^\nu_J = e^I_\rho \tilde{\Gamma}^\rho{}_{\nu\mu} e^\nu_J + e^I_\nu \partial_\mu e^\nu_J \tag{2.145a}$$

Conversely, can also see:

$$\tilde{\Gamma}^\rho{}_{\nu\mu} = e^\rho_I \tilde{D}_\mu e^I_\nu = e^\rho_I w^I{}_{J\mu} e^J_\nu + e^\rho_I \partial_\mu e^I_\nu \tag{2.146}$$

- One can consider the trivialization as a basis transformation by the Jacobian matrix $e = [e_I^\mu]$. Indeed, it is not induced by a coordinates transformation and then the connection $\tilde{\Gamma}_\mu^\rho$ transforms to w_J^I as:

$$w = e^{-1}\tilde{\Gamma}e + e^{-1}de \tag{2.147}$$

- One can easily check the covariance of the covariant derivative $\tilde{D}$ with respect the basis transformation e

$$\tilde{D}_\mu T^{I_1 \ldots I_m}{}_{J_1 \ldots J_n} = e_{\mu_1}^{I_1} \cdots e_{\mu_m}^{I_m} e_{J_1}^{\nu_1} \cdots e_{J_n}^{\nu_n} \tilde{\nabla}_\mu T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} \tag{2.148}$$

where $\boldsymbol{T} \in \Gamma T_n^m \mathcal{M}$ are arbitrary tensor field which can be written in the tangent space basis or in the orthonormal frame basis:

$$T^{I_1 \ldots I_m}{}_{J_1 \ldots J_n} = e_{\mu_1}^{I_1} \cdots e_{\mu_m}^{I_m} e_{J_1}^{\nu_1} \cdots e_{J_n}^{\nu_n} T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} \tag{2.149}$$

- In general, the Lorentz connection w_J^I depends linearly on the connection $\tilde{\Gamma}_\mu^\rho$ and quadratically on the frame field e_I:

$$w = w(\tilde{\Gamma}, e) \tag{2.150}$$

- If we impose the compatibility and the torsion free conditions, then the connection $\tilde{\Gamma}_\mu^\rho$ will be the Levi-Civita connection Γ_μ^ρ that is a function on the metric and then the frame field, $\Gamma_\mu^\rho = \Gamma_\mu^\rho(g) = \Gamma_\mu^\rho(e)$. This induces a connection $w_J^I = \Gamma_J^I$ that is a function only on the frame field, $\Gamma_J^I = \Gamma_J^I(e)$

2.5.3 Spin connection

It is useful to define tetrad-compatible connection called *spin connection* by defining a new covariant derivative $\tilde{\mathcal{D}}_\mu$ on mixed tensor indices (the tangent space and the Minkowski indices) which parallel transport the tangent space component by contracting them with the Γ_ν^ρ connection and the Minkowski component by the w_J^I connection. It has to be linear by addition over tensors of the same order and satisfies the Leibniz rule. One can easily show that $\tilde{\mathcal{D}}_\mu$ is really a tetrad compatible connection and the same thing for the tetrad:

$$\tilde{\mathcal{D}}_\mu e_I^\nu = 0 \tag{2.151}$$

2.5.4 Torsion and Curvature

The torsion components of the induced connection $T^I_{\mu\nu}$ is defined by:

$$T^I_{\mu\nu} := e^I_\rho T^\rho_{\mu\nu} = 2\tilde{D}_{[\mu} e^I_{\nu]} \tag{2.152}$$

where $T^\rho_{\mu\nu}$ is the torsion components tensor and one can then express this using differential forms as:

$$T^I = De^I \tag{2.153}$$

where $T^I = \frac{1}{2} T^I_{\mu\nu} dx^\mu \wedge dx^\nu$ is a torsion represented by an $\mathbb{R}^{3,1}$-valued 2-form, D is the exterior covariant derivative with respect to the Lorentz connection w_{IJ}.
The final result we need action is the relation between the curvature on the Minkowski vector bundle and the spacetime curvature. First the *internal curvature* two form $F^{IJ}_{\mu\nu}[w]$ is defined by:

$$[\tilde{D}_\mu, \tilde{D}_\nu] v^I := F^I{}_{J\mu\nu} v^J \tag{2.154}$$

where v^I is an arbitrary Lorentz vector and $F^{IJ}_{\mu\nu}[w]$ can be expressed in terms of the Lorentz connection coefficients as:

$$F^{IJ}_{\mu\nu}[w] = \partial_\mu w^{IJ}_\nu - \partial_\nu w^{IJ}_\mu + [w_\mu, w_\nu]^{IJ} \tag{2.155}$$

we may express this relation using differential forms as:

$$F^{IJ} = dw^{IJ} + w^I{}_K \wedge w^{KJ} \tag{2.156}$$

where $F^{IJ} = \frac{1}{2} F^{IJ}_{\mu\nu} dx^\mu \wedge dx^\nu$ is a curvature represented by an $so(3,1)$-valued 2-form. Second recall that the spacetime Riemann tensor $R^\rho{}_{\sigma\mu\nu}[e, w]$ of the covariant derivative $\tilde{\nabla}_\mu$ with nonzero torsion is defined by:

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] v^\rho := R^\rho{}_{\sigma\mu\nu} v^\sigma - T^\sigma_{\mu\nu} \tilde{\nabla}_\sigma v^\rho \tag{2.157}$$

for an arbitrary spacetime vector v^ρ. using the abstract definition for the induced connection in (2.148) one can deduce:

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] v^\rho := e^\rho_I F^I{}_{J\mu\nu} e^J_\sigma v^\sigma - T^\sigma_{\mu\nu} \tilde{\nabla}_\sigma v^\rho \tag{2.158}$$

and hence one can get from (2.157) and (2.158):

$$R^{\rho}{}_{\sigma\mu\nu}[e,w] = e^{\rho}_{I} F^{I}{}_{J\mu\nu}[w] e^{J}_{\sigma} \tag{2.159}$$

Using this relation one can express the spacetime curvature Ricci tensor and scalar in terms of contractions of the internal curvature and tetrad as:
The Ricci tensor:

$$R_{\sigma\nu}[e,w] = e^{\rho}_{I} F^{I}{}_{J\rho\nu}[w] e^{J}_{\sigma} \tag{2.160}$$

The scalar curvature:

$$R[e,w] = e^{\mu}_{I} e^{\nu}_{J} F^{IJ}_{\mu\nu}[w] \tag{2.161}$$

2.5.5 The compatibility and the torsion free conditions:

1. Compatibility condition: according to (2.148) and (2.149), we have:

$$\tilde{\nabla}_{\rho} g_{\mu\nu} = 0 \Leftrightarrow \tilde{D}_{\rho} \eta_{IJ} = 0 \tag{2.162}$$

   Using the definition of the covariant derivative $\tilde{D}_{\mu}$ in (2.144b), the last equation becomes:

$$w_{(IJ)\mu} = 0 \tag{2.163}$$

   where $w_{IJ} = \eta_{IK} w^{K}{}_{J}$, then the Lorentz connection w_{IJ} must be antisymmetric in their internal indices which agree with the fact that w_{IJ} is an $so(3,1)$-valued one-form. Therefore, the compatibility condition imposed by itself and doesn't give new constraints, for this reason we will consider the connection in the Palatini formalism to be compatibile with the metric rather than the torsion free condition.

2. Torsion free condition: from Eq. (2.146), one has:

$$T^{\rho}_{\mu\nu} := 2\tilde{\Gamma}^{\rho}_{[\mu\nu]} = 0 \Leftrightarrow T^{I}_{\mu\nu} := 2\tilde{D}_{[\mu} e^{I}_{\nu]} = 0 \tag{2.164}$$

   fact we shall see that in the Palatini action the torsion free condition $T^{I} = 0$ one of the Euler Lagrange equations derived from it.

2.5.6 The tetradic-Palatini action

We can now write down the tetradic-Palatini action, which is just the first order form of Einstein Hilbert action, but with the Lorentz connection and tetrad independent variables; one has from (2.140) that $\sqrt{-g} = |e|$, where $e = det[e^I_\mu]$ is the determinent of the co-tetrad e^I_μ. The tetradic Palatini action is defined by using the scalar curvature in Eq. (2.161) and cancel the sign of the determinant e, we obtain:

$$S_{t-P}[e,w] = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^4x \; e e^\mu_I e^\nu_J F^{IJ}_{\mu\nu}[e,w] \tag{2.165}$$

where the action is a functional of both the co-tetrad e^I_μ and the Lorentz connection $w^I_{J\mu}$. We now compute the equations of motion for this action:

Variation the action over the co-tetrad e^I_μ:

$$\delta_e S_{t-P} = 0 \Leftrightarrow G^I_\mu[e,w] = e^{\nu I}\Big(R_{\mu\nu}[e,w] - \frac{1}{2}g_{\mu\nu}R[e,w]\Big) = 0 \tag{2.166}$$

Then variation of the tetradic Palatini action with respect to the co-tetrad e^I_μ gives the (mixed index) Einstein tensor G^I_μ to be zero. Since the tetrad is invertible, then multiplying both side of the equation (2.166) by the co-tetrad $e_{\rho I}$, we get:

$$G_{\mu\nu}[e,w] = R_{\mu\nu}[e,w] - \frac{1}{2}g_{\mu\nu}R[e,w] = 0 \tag{2.167}$$

which of course would be Einstein's equations if our induced connection were torsion free.

Variation the action over the co-tetrad $w^I_{J\mu}$:

$$\delta_w S_{t-P} = 0 \Leftrightarrow \tilde{D}_\mu\Big(e e^{[\mu}_I e^{\nu]}_J\Big) = 0 \tag{2.168}$$

where we have dropped the boundary term. We can prove that the equation of motion derived from the variation with respect to the connection is equivalent to:

$$\tilde{D}_{[\mu}e^I_{\nu]} = 0 \tag{2.169}$$

which is the torsion free condition, then we conclude that the Lorentz connection can be uniquely determined by the tetrad and we write $w^{IJ}_\mu = \Gamma^{IJ}_\mu$ where Γ^{IJ}_μ is the internal Christoffel symbol associated to D_μ, also the generalized derivative operator $\tilde{D}_\mu$ must agree with D_μ. The torsion free condition now implies the induced spacetime connection is the unique Levi-Civita connection and hence that Eq. (2.167) is now equivalent to Einstein's equations in vacuum.

Before we discuss the Hamiltonian analysis of tetrad Palatini action we should like to make some remarks concerning the tetradic Palatini formalism, described here:

- The Palatini formalism is often called first order because the equations of motion only involve first order derivatives of the dynamical variables in contrast to the Einstein Hilbert action where e.g. the Ricci tensor involves second order derivatives of the metric.

- The following identity:

$$ee^{[\mu}_I e^{\nu]}_J = \frac{1}{4}\epsilon^{\mu\rho\rho\sigma}\epsilon_{IJKL}e^K_\rho e^L_\sigma \tag{2.170}$$

  allows us to re-write the tetradic Palatini action as an integral of a four form as:

$$S_{t-P}[e,w] = \frac{1}{32\pi G}\int_{\mathcal{M}} \epsilon_{IJKL}e^I \wedge e^J \wedge F^{KL}[e,w] \tag{2.171}$$

  the co-tetrad e^I is defined in (2.133) and the 2-form curvature tensor F^{IJ} is written by its spacetime components as:

$$F^{IJ} = \frac{1}{2}F^{IJ}_{\mu\nu}dx^\mu \wedge dx^\nu \tag{2.172}$$

- The third point we wish to make regards matter coupling in the Palatini formalism, all matter types may be coupled to this action including fermionic matter. Indeed as we mentioned earlier in section 3.4 only the tetrad formalism may be used to describe fermionic degrees of freedom. However, one can simply re-write the Einstein Hilbert action directly in terms of a tetrad basis but where the connection is fixed and non-dynamical such that it induces the Levi-Civita connection and in this case one can describe all matter degrees of freedom, i.e one have:

$$S_{EH}[e] = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^4x \; e e^{\mu}_{I} e^{\nu}_{J} F^{IJ}_{\mu\nu}[e] \tag{2.173}$$

However, in the Palatini formalism the connection is dynamical and this leads to a nonequivalence in the dynamics for fermions coupled to gravity. This non-equivalence appears because in order to write down a covariant derivative for fermions one must use the Lorentz connection and then one has, in this formalism, a fermionic standard model action term of the form:

$$\begin{aligned} S_F[e, w, A, \phi, \psi] = \int_{\mathcal{M}} \mathrm{d}^4x \; e\Big[\overline{\psi}\gamma^I e^{\mu}_{I}(\partial_\mu\psi + w^{J}{}_{K\mu} L^{K}_{J}\psi + A^a_\mu L_a\psi) \\ +Y(\phi, \psi, \overline{\psi})\Big] + c.c. \end{aligned} \tag{2.174}$$

where ϕ is scalar field, A^a_μ is a Yang Mills field with gauge group Lie algebra a, ψ is a Dirac spinor,γ^I are the Gamma matrices, L_{IJ}, L_a are representation matrices of the Lorentz and Yang Mills gauge group G up which act upon the representation space ψ, and finally $Y(\phi, \psi, \overline{\psi})$ is a polynomial interaction which will include the mass term for the fermion field after symmetry breaking. The term $\partial_\mu\psi + w^{J}{}_{K\mu} L^{K}_{J}\psi + A^a_\mu L_a\psi$ can be viewed as a generalised covariant derivative acting upon the group $SO(3,1) \times G$. When one performs a variation with respect to the Lorentz connection in this term there will be a non-zero contribution:

$$\frac{\delta S_F}{\delta w^{IJ}_{\mu}} = e\overline{\psi}\gamma^K e^{\mu}_{K} L_{IJ}\psi \tag{2.175}$$

which contributes to the torsion T^I and hence one finds that the spacetime connection on shell is no longer Levi-Civita but will have a non-zero torsion. Hence in the presence of fermions the second order and first order theories are inequivalent; we do not know which one is physically correct, because the effect of gravity on single fermions is hard to measure. Finally we consider the Legendre transform of these tetrad formulations of gravity. The Hamiltonian formulation of the tetrad version of the Einstein Hilbert action , is derived in detail and the result is a first class Hamiltonian system. The configuration variable is a triad e^a_i where $a = 1, 2, 3$ is the spatial index on Σ and $i = 1, 2, 3$ is an $SO(3)$ index, where, in analogy with the tetrad, the triad is an orthonormal frame which satisfies.

2.6 Hamiltonian Analysis of Tetrad Palatini Formulation

For the Hamiltonian formulation we proceed as before, assuming a $3+1$ splitting of the spacetime ($\mathcal{M} \cong \mathbb{R} \times \Sigma$) with coordinates $(t, \boldsymbol{\sigma})$. We introduce the lapse function and the shift vector (N, N^a) as in the ADM decomposition of the metric in (2.84a,2.84b).

2.6.1 triad, co-triad

It is easy to see that the tetrad e^μ_I and the co-tetrad e^I_μ for the ADM metric is projected out to $\mathcal{E}^\mu_I$ and $\mathcal{E}^I_\mu$ respectively by:

$$\mathcal{E}^\mu_I = h^\mu_\nu e^\nu_I = e^\mu_I + n^\mu n_I \tag{2.176a}$$

$$\mathcal{E}^I_\mu = h^\nu_\mu e^I_\nu = e^I_\mu + n^I n_\mu \tag{2.176b}$$

where $n_I = n_\nu e^\nu_I$ and $n^I = e^I_\nu n^\nu$. . Immediately one can deduce that:

$$\mathcal{E}^\mu_I n_\mu = 0 \quad \mathcal{E}^\mu_I n^I = 0 \quad \mathcal{E}^I_\mu n^\mu = 0 \quad \mathcal{E}^I_\mu n_I = 0 \tag{2.177a}$$

$$\mathcal{E}^I_\mu \mathcal{E}^J_\nu \eta_{IJ} = h_{\mu\nu} \quad \mathcal{E}^\mu_I \mathcal{E}^\nu_J \eta^{IJ} = h^{\mu\nu} \quad \mathcal{E}^\mu_I \mathcal{E}^I_\nu = h^\mu_\nu \tag{2.177b}$$

$$\mathcal{E}^I_\mu \mathcal{E}^J_\nu h^{\mu\nu} = h^{IJ} \quad \mathcal{E}^\mu_I \mathcal{E}^\nu_J h_{\mu\nu} = h_{IJ} \quad \mathcal{E}^\mu_I \mathcal{E}^J_\mu = h^J_I \tag{2.177c}$$

where h_{IJ} is the *internal transverse metric*, or *internal projector*; it is defined by:

$$h_{IJ} = \eta_{IJ} + n_I n_J \tag{2.178}$$

and hence we view $\mathcal{E}^\mu_I$ as a degenerate tetrad corresponds to the degenerate transverse metric $h_{\mu\nu}$. If we pull-back these quantities from the tangent bundle spacetime $T\mathcal{M}$ to the tangent bundle of the foliated spactime $T(\mathbb{R}\times\Sigma)$ by using the pull-back of the foliation map (2.73), one can obtain a non-degenerate quantities $\mathcal{E}^a_I, \mathcal{E}^I_a, a = 1,2,3$ called *triad* and *co-triad* respectively, which are,

$$\mathcal{E}^\mu_I = e^\mu_a \mathcal{E}^a_I \qquad , \qquad \mathcal{E}^I_a = e^\mu_a \mathcal{E}^I_\mu \tag{2.179}$$

One can see from that:

$$\mathcal{E}^I_a \mathcal{E}^J_b \eta_{IJ} = q_{ab} \quad \mathcal{E}^a_I \mathcal{E}^b_J \eta^{IJ} = q^{ab} \quad \mathcal{E}^a_I \mathcal{E}^I_b = \delta^a_b \tag{2.180a}$$

$$\mathcal{E}^I_a \mathcal{E}^J_b q^{ab} = h^{IJ} \quad \mathcal{E}^a_I \mathcal{E}^b_J q_{ab} = h_{IJ} \quad \mathcal{E}^a_I \mathcal{E}^J_a = h^J_I \tag{2.180b}$$

and hence we view the triad $\mathcal{E}^a_I$ a non-degenerate tetrad corresponds to the non-degenerate induced metric q_{ab}.

2.6.2 Hamiltonian Analysis

Using the splitting (2.176a) and the definition of time vector field $\boldsymbol{\tau}$ in Eq. (2.83), one can express the tetradic Palatini action (2.165) as:

$$S_{t-P}[\mathcal{E}, w] = \int_{\mathcal{M}} \mathrm{d}^4x \Big[\alpha^\nu_{IJ} \mathcal{L}_{\boldsymbol{\tau}} w^{IJ}_\nu + \tau^\mu w^{IJ}_\mu \tilde{D}_\nu \alpha^\nu_{IJ} - N^\mu \alpha^\nu_{IJ} F^{IJ}_{\mu\nu} + N\frac{16\pi G}{\sqrt{q}} \alpha^{\mu L}_I \alpha^\nu_{LJ} F^{IJ}_{\mu\nu}\Big] \tag{2.181}$$

where

$$\alpha^\nu_{IJ}(\boldsymbol{\sigma}) = \frac{\sqrt{q}\mathcal{E}^\nu_{[I} n_{J]}}{8\pi G} \tag{2.182}$$

and we have used the fact that the co-tetrad determinant $e = N\sqrt{q}$. Since α^ν_{IJ} and N^μ are spatial vector fields, one can hence pull-back all the integral in (2.181) from the our originial spacetime $\mathcal{M}$ to the foliated spacetime $\mathbb{R} \times \Sigma$ as:

$$\begin{aligned} S_{t-P}[\mathcal{E}, w] &= \int_{\mathbb{R}} dt \int_\Sigma \mathrm{d}^3\boldsymbol{\sigma} \Big[\alpha^a_{IJ} \dot{w}^{IJ}_a + w^{IJ}_0 \tilde{D}_a \alpha^a_{IJ} - N^a \alpha^b_{IJ} F^{IJ}_{ab} + \tilde{N} \alpha^{aL}_I \alpha^b_{LJ} F^{IJ}_{ab}\Big] \\ &= \int_{\mathbb{R}} dt \; L_{t-P}[w^{IJ}_a, \dot{w}^{IJ}_a, w^{IJ}_0, \tilde{N}, N^a] \end{aligned} \tag{2.183}$$

where we have taken the change of variable $\tilde{N} = N\frac{16\pi G}{\sqrt{q}}$, we have also introduced:

$$\alpha^a_{IJ}(\boldsymbol{\sigma}) = 2E^a_{[I} n_{J]} \tag{2.184}$$

and the *densitized triad* E^a_I is defined by:

$$E^a_I = \frac{\sqrt{q}\mathcal{E}^a_I}{8\pi G} \tag{2.185}$$

By comparison with the ADM formulation, we expect that the coefficients of $\tilde{N}$, N^a and w^{IJ}_0 will form the Hamiltonian, diffeomorphism and 6 new constraints respectively. We now wish to cast this action into canonical form, that is, we would like to perform the Legendre transform from the Lagrangian density appearing in Eq. (2.238) to the corresponding Hamiltonian density: one can write down the conjugate momenta $[\Pi^a_{IJ}(\boldsymbol{\sigma}), \Pi_{IJ}(\boldsymbol{\sigma}), p(\boldsymbol{\sigma}), p_a(\boldsymbol{\sigma})]$ to the configuration variables $[w^{IJ}_a(\boldsymbol{\sigma}), w^{IJ}_0(\boldsymbol{\sigma}), \tilde{N}(\boldsymbol{\sigma}), N^a(\boldsymbol{\sigma})]$ respectively. The $56 \times \infty^3$ dimensional kinematical (unconstrained) phase space Γ can be then coordinatized as:

$$\Gamma = [w^{IJ}_a(\boldsymbol{\sigma}), \Pi^a_{IJ}(\boldsymbol{\sigma}), w^{IJ}_0(\boldsymbol{\sigma}), \Pi_{IJ}(\boldsymbol{\sigma}), \tilde{N}(\boldsymbol{\sigma}), p(\boldsymbol{\sigma}), N^a(\boldsymbol{\sigma}), p_a(\boldsymbol{\sigma})] \tag{2.186}$$

The symplectic structure is expressed by the only non-vanishing basic Poisson bracket relations between the configuration variables and their conjugate momenta:

$$\{w^{IJ}_a(\boldsymbol{\sigma}), \Pi^b_{KL}(\boldsymbol{\sigma'})\} = \delta^I_{[K}\delta^J_{L]}\delta^b_a\delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \tag{2.187a}$$

$$\{w^{IJ}_0(\boldsymbol{\sigma}), \Pi_{KL}(\boldsymbol{\sigma'})\} = \delta^I_{[K}\delta^J_{L]}\delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \tag{2.187b}$$

$$\{\tilde{N}(\boldsymbol{\sigma}), p(\boldsymbol{\sigma'})\} = \delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \tag{2.187c}$$

$$\{N^a(\boldsymbol{\sigma}), p_b(\boldsymbol{\sigma'})\} = \delta^a_b\delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \tag{2.187d}$$

Observing that the action in (2.238) is in standard canonical form $\int p\dot{q} - H$ and therefore we can read off the conjugate momenta to the configuration variables:

$$\Pi^a_{IJ}(\boldsymbol{\sigma}) := \frac{\delta L_{t-P}}{\delta \dot{w}^{IJ}_a(\boldsymbol{\sigma})} = \alpha^a_{IJ}(\boldsymbol{\sigma}) \tag{2.188a}$$

$$\Pi_{IJ}(\boldsymbol{\sigma}) := \frac{\delta L_{t-P}}{\delta \dot{w}^{IJ}_0(\boldsymbol{\sigma})} = 0 \tag{2.188b}$$

$$p(\boldsymbol{\sigma}) := \frac{\delta L_{t-P}}{\delta \dot{\tilde{N}}(\boldsymbol{\sigma})} = 0 \tag{2.188c}$$

$$p_a(\boldsymbol{\sigma}) := \frac{\delta L_{t-P}}{\delta \dot{N}^a(\boldsymbol{\sigma})} = 0 \tag{2.188d}$$

This confirms the status that the lapse function $\tilde{N}$, shift vector N^a and the time components of the Lorentz connection w_0^{IJ} are non-dynamical variables that can be specified as arbitrary functions on $\mathbb{R}\times\Sigma$; they are only Lagrange multipliers. Since we cannot express $\dot{w}^a_{IJ}$, $\dot{w}_0^{IJ}$, $\dot{\tilde{N}}$ and $\dot{N}^a$ as functions of their momenta, then we have the following constraints in Γ:

$$C^a_{IJ}(\boldsymbol{\sigma}) := \Pi^a_{IJ} - \alpha^a_{IJ} = 0 \tag{2.189a}$$

$$C_{IJ}(\boldsymbol{\sigma}) := \Pi_{IJ} = 0 \tag{2.189b}$$

$$C(\boldsymbol{\sigma}) := p = 0 \tag{2.189c}$$

$$C_a(\boldsymbol{\sigma}) := p_a = 0 \tag{2.189d}$$

Clearly the last three equations being identically zero correspond to primary independent constraints. However, in addition the parameteric equations (2.189a) describes $6\times\infty^3$ constraints because Π^a_{IJ} has $18\times\infty^3$ independent components whereas α^a_{IJ} has $12\times\infty^3$ such components (from Eq. (2.184), α^a_{IJ} contains $3\times\infty^3$ of the unit normal n_I and $9\times\infty^3$ of the triad $\mathcal{E}^a_I$). Hence one expects Eq. (2.189a) to be equivalent to following six constraints dor each coordinates points $\boldsymbol{\sigma}$:

$$C^{ab}(\boldsymbol{\sigma}) := \epsilon^{IJKL}\Pi^a_{IJ}\Pi^b_{KL} = 0 \tag{2.190a}$$

$$tr(\Pi^a\cdot\Pi^b) > 0 \tag{2.190b}$$

Then, the Eqs. (2.189b,2.189c,2.189d,2.190) are the $16\times\infty^3$ primary constraints; They define $40\times\infty^3$ dimensional "primary" constrained surface on the kinematical phase space Γ, denoted by Γ_p as:

$$\Gamma_p := \{\Gamma | C^{ab}(\boldsymbol{\sigma}) = 0, C_{IJ}(\boldsymbol{\sigma}) = 0, C(\boldsymbol{\sigma}) = 0, C_a(\boldsymbol{\sigma}) = 0\} \subset \Gamma \tag{2.191}$$

we will asign the equality on Γ_p by $\approx$. The Hamiltonian treatment of systems with constraints has been developed by the well-known Dirac algorithm. According to that theory, we are supposed to introduce Lagrange multiplier fields $\lambda_{ab}(t,\boldsymbol{\sigma})$, $\lambda^{IJ}(t,\boldsymbol{\sigma})$, $\lambda(t,\boldsymbol{\sigma})$ and $\lambda^a(t,\boldsymbol{\sigma})$ for the primary constraints and to perform the Legendre transform as usual with respect to the remaining velocities which can be solved for. Following the Dirac algorithm for expressing the primary Hamiltonian one has:

$$H_p[w_a^{IJ}, \Pi^a_{IJ}, w_0^{IJ}, \tilde{N}, N^a] := H + \int_\Sigma \mathrm{d}^3\boldsymbol{\sigma}[\lambda_{ab}C^{ab} + \lambda^{IJ}C_{IJ} + \lambda C + \lambda^a C_a] \quad (2.192)$$

where H is the canonical Hamiltonian:

$$H := \int_\Sigma \mathrm{d}^3\boldsymbol{\sigma}\Big[- w_0^{IJ}\tilde{D}_a\Pi^a_{IJ} + N^a\Pi^b_{IJ}F^{IJ}_{ab} - \tilde{N}\Pi^{aL}_I\Pi^b_{LJ}F^{IJ}_{ab}\Big] \quad (2.193)$$

We now have to ensure the consistency of the primary constraints, i.e. that they are preserved by evolution generated by H_p. Therefore, the primary constraints imply $16 \times \infty^3$ secondary constraints:

$$0 \approx \{C^{ab}(\boldsymbol{\sigma}), H_p\} \Leftrightarrow \chi^{ab}(\boldsymbol{\sigma}; w_a^{IJ}, \Pi^a_{IJ}] := \epsilon^{IJKL}\Pi^{cM}_I\Pi^{(a}_{MJ}\tilde{D}_c\Pi^{b)}_{KL} \approx 0 \quad (2.194a)$$

$$0 \approx \{C_{IJ}(\boldsymbol{\sigma}), H_p\} \Leftrightarrow G_{IJ}(\boldsymbol{\sigma}; w_a^{IJ}, \Pi^a_{IJ}] := -\tilde{D}_a\Pi^a_{IJ} \approx 0 \quad (2.194b)$$

$$0 \approx \{C(\boldsymbol{\sigma}), H_p\} \Leftrightarrow H(\boldsymbol{\sigma}; w_a^{IJ}, \Pi^a_{IJ}] := -\Pi^{aL}_I\Pi^b_{LJ}F^{IJ}_{ab} \approx 0 \quad (2.194c)$$

$$0 \approx \{C_a(\boldsymbol{\sigma}), H_p\} \Leftrightarrow H_a(\boldsymbol{\sigma}; w_a^{IJ}, \Pi^a_{IJ}] := \Pi^b_{IJ}F^{IJ}_{ab} \approx 0 \quad (2.194d)$$

they are polynomial constraints on the canonical variables $[w_a^{IJ}(\boldsymbol{\sigma}), \Pi^a_{IJ}(\boldsymbol{\sigma})]$ and they are called *Hamiltonian constraint* H, *diffiomorphism constraint* H_a and the new 6-constraints G_{IJ} are the *Gauss constraint* corresponds to the internal symmetry of Lorentz group $SO(3,1)$; we expect the G_{IJ} to be generators of Lorentz transformations. Indeed this can be confirmed, if one computes the Poisson algebra of the smeared G_{IJ} with any phase space function. One should now check for the consistency of these secondary constraints but fortunately there are no further secondary (tertiary) constraints. The set of $32 \times \infty^3$ independent constraints in Eqs. (2.189b,2.189c,2.189d,2.190,2.194a,2.194b,2.194c,2.194d) defined the $24 \times \infty^3$ constrained surface Γ_c on the primary surface Γ_p as:

$$\Gamma_c = \{\Gamma_p | \chi^{ab}(\boldsymbol{\sigma}) = 0, G_{IJ}(\boldsymbol{\sigma}) = 0, H(\boldsymbol{\sigma}) = 0, H_a(\boldsymbol{\sigma}) = 0\} \subset \Gamma_p \subset \Gamma \quad (2.195)$$

from now $\approx$ means equality on Γ_c. With these definitions, we see that the total prime Hamiltonian is a linear combination of constraints:

$$H_p = \int_\Sigma \mathrm{d}^3\boldsymbol{\sigma}[w_0^{IJ}G_{IJ} + \tilde{N}H + N^a H_a + \lambda_{ab}C^{ab} + \lambda^{IJ}C_{IJ} + \lambda C + \lambda^a C_a] \quad (2.196)$$

as in the ADM dormalism, the total Hamiltonian is thus constrained to vanish, a result that is in accordance with our general discussion of reparametrization invariance. Hence we have completed the Dirac-Bergmann algorithm, all that remains is to classify the constraints we have found into first and second class. We shall see that all constraints are first class except for C^{ab} and χ^{ab}

2.6.3 Constraint Algebra Analysis

By using the Poisson brackets (2.242), one can show all constraints are first class (their Poisson brackets with all constraints are proportional to the secondary class constraints) except for C^{ab} and χ^{ab}. Explicit calculations for the Poisson brackets between the second class constraints C^{ab} and χ^{ab} gives:

$$\{C^{ab}(\boldsymbol{\sigma}), \chi^{ab}(\boldsymbol{\sigma'})\} = 4\Big[tr(\Pi^a \cdot \Pi^b)tr(\Pi^c \cdot \Pi^d) - tr(\Pi^c \cdot \Pi^{(a})tr(\Pi^{b)} \cdot \Pi^d)\Big]\delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \tag{2.197}$$

which is in general not zero on Γ_c. Now we can count how many degrees of freedom in the gravitational field: the constrained phase space Γ_c has $24 \times \infty^3$ degrees of freedom, the first class constraints (2.189b,2.189c,2.189d,2.194b,2.194c,2.194d) generate $20 \times \infty^3$ parameter set of gauge transformations on Γ_c, then $20 \times \infty^3$ degrees of freedom must be subtracted in order to fix the gauge. The remaining $4 \times \infty^3$ variables define the reduced phase space $\Gamma_r \equiv \Gamma_{phys}$ and as expected, they correspond to 2 degrees of freedom at each coordinate point $\boldsymbol{\sigma}$ in configuration space.

2.6.4 Solving Second Class Constraints

The next step in the Dirac procedure is to solve the second class constraints. Since the momentum Π^a_{IJ} is a 2-form in its internal indices, we can decompose it into its *electric* and *magnetic* parts by using the internal projection with respect to the unit normal n_I as:

$$\Pi^a_{IJ} = 2E^a_{[I}n_{J]} + \epsilon_{IJKL}B^{aK}n^L \tag{2.198}$$

The first term is the electric part 9-components of Π^a_{IJ} describes the (boost-spatial) components, where the decond term is the magnetic part 9-components of Π^a_{IJ} describes the (spatial-spatial) components. The electric E^a_I and magnetic B^{aI} fields are defined as:

$$E^a_I := n^K \Pi^a_{KI} \tag{2.199a}$$

$$B^{aI} := \frac{1}{2}\epsilon^{IJKL}\Pi^a_{JK} n_L \tag{2.199b}$$

They are both orthogonal with the internal unit normal n_I:

$$E^a_I n^I = 0 \qquad B^{aI} n_I = 0 \tag{2.200}$$

The following relations are hold:

$$tr(\Pi^a \cdot \Pi^b) = 2(E^a_I E^{bI} - B^a_I B^{bI}) \tag{2.201a}$$

$$\epsilon^{IJKL}\Pi^a_{IJ}\Pi^b_{KL} = 8E^{(a}_I B^{b)I} \tag{2.201b}$$

Now it is time to solve the second class constraints (2.190). Substituting (2.198) in (2.190) and use (2.201b), we obtain:

$$E^{(a}_I B^{b)I} = 0 \tag{2.202}$$

One can use the Lorentz transformation to fix the 3-dimensional freedom in the initial choice of the unit normal n_I to set:

$$E^{[a}_I B^{b]I} = 0 \tag{2.203}$$

this can be done by absorbing 3 components of the momentum Π^a_{IJ} to n_I, Eqs (2.202) and (2.203), imply:

$$E^a_I B^{bI} = 0 \tag{2.204}$$

both the electric E^a_I and magnetic B^{bI} are degenerate one times in the diection of n_I, then one of them must be vanish. Now, let us try to achieve the inequality in Eq. (2.190) by using (2.201a), we get:

$$E^a_I E^{bI} > B^a_I B^{bI} \tag{2.205}$$

From (2.204) and (2.205), the only solution to (2.190) is to have:

$$B^{aI} = 0 \tag{2.206}$$

Then after solving the second class constrains (2.190), the 18-components of the momentum Π^a_{IJ} is constrained to just 12-components:

$$\Pi^a_{IJ} = 2E^a_{[I} n_{J]} \tag{2.207}$$

Next, we have to solve the remaining second class constraint (2.194a). It turns out that the simplest way to solve it is by using part of the Gauss constraint (2.194b) to gauge fix the internal vector n_I. This gauge fixing further reduces the momentum variables to just 9, and the momentum is now fully determined by the nine components of E^a_I. However, since we wish to gauge fix the internal vector n_I, we must also solve the electric part (boost part) of the Gauss constraint (since it will be also second class), i.e., we must solve the 3 independent equations:

$$G^{Boost}_{IJ} = 2n^K G_{K[I} n_{J]} = 0 \tag{2.208}$$

The remaining spactial Gauss constraints stay first class since its generated $SO(3)$ internal rotations will leave the gauge-fixed n^I invariant. The 9 equations (2.194a) and (2.208) reduce the independent components in w^{IJ}_a from $18 \times \infty^3$ to $9 \times \infty^3$. To solve them, let us first define a field K^{IJ}_a as a difference between the general $3d$-Lorentz connection w^{IJ}_a and the torsion free $3d$-Lorentz connection Γ^{IJ}_a, we have:

$$w^{IJ}_a = \Gamma^{IJ}_a[E] + K^{IJ}_a \tag{2.209}$$

where the torsion free $3d$-Lorentz connection Γ^{IJ}_a can be decomposed into electric and magnetic parts via the densitized triad E^a_I by the following relation:

$$\Gamma^{IJ}_a[E] = 2n^K \Gamma^{[I}_{aK} n^{J]} + E^I_b \nabla_a E^{bJ} - n^J \nabla_a n^I \tag{2.210}$$

Let us decompose the internal 2-form indices of K^{IJ}_a in terms of their electric and magnetic parts:

$$K^{IJ}_a = 2K^{[I}_a n^{J]} + \epsilon^{IJMN} \bar{K}_{aM} n_N \tag{2.211}$$

the electric K^I_a and magnetic $\bar{K}_{aM}$ fields are defined by:

$$K^I_a := n_M K^{MI}_a \tag{2.212a}$$

$$\bar{K}_{aI} := \frac{1}{2}\epsilon_{IJMN} K^{JM}_a n^N \tag{2.212b}$$

they are both orthogonal with the internal uniot normal n_I:

$$K^I_a n_I = 0 \qquad \bar{K}_{aI} n^I = 0 \tag{2.213}$$

Substituting this decomposition in Eq. (2.194a) one can show:

$$\epsilon^{MNL[I} n^{J]} n_L \bar{K}_{aM} E^a_N = 0 \tag{2.214}$$

For any fixed indices I, J, the operator $\epsilon^{MNL[I} n^{J]} n_L$ is degenerate one in their free indices M, N and the only zero eigenvector is in the direction of the unit normal n_I, and since E^a_I and $\bar{K}_{aI}$ are orthogonal to n_I, then the only possible case to achieve the Eq. (2.194a) is:

$$\bar{K}_{a[I} E^a_{J]} = 0 \tag{2.215}$$

Also substituting the decomposition (2.211) in Eq. (2.208) one find:

$$E^{(a}_I q^{b)c} \bar{K}^I_c - q^{ab} E^c_I \bar{K}^I_c = 0 \tag{2.216}$$

One can replace q^{ab} by its expression in (2.180a) to get:

$$E^a_I E^b_J (E^{c(I} \bar{K}^{J)}_c - \eta^{IJ} E^c_M \bar{K}^M_c) = 0 \tag{2.217}$$

Since the densitized triad E^a_I is degenerate one by the unit normal n_I, then one can has:

$$E^{c(I} \bar{K}^{J)}_c - \eta^{IJ} E^c_M \bar{K}^M_c = 0 \tag{2.218}$$

Now taking the trace of the this equation with respect to η_{IJ}, one find:

$$E^a_I \bar{K}^I_a = 0 \tag{2.219}$$

putting this in Eq. (2.218), we get:

$$\bar{K}_{a(I} E^a_{J)} = 0 \tag{2.220}$$

Finally, having together the two equation (2.215) and (2.220) we find:

$$\bar{K}_{aI}E^a_J = 0 \tag{2.221}$$

since the densitized triad E^a_I is invertible (and must be non-zero), then the condition for the 9 second class constraints (2.194a) and (2.208) to be satisfied is:

$$\bar{K}_{aI} = 0 \tag{2.222}$$

hence, the condition for the connection K^{IJ}_a has just an electric part (boost):

$$K^{IJ}_a = 2K^{[I}_a n^{J]} \tag{2.223}$$

And finally, after solving the second class constraints (2.190) with (2.194a), the 18-components of the Lorentz connection is constrained to just 9-components per each space point coordinates:

$$w^{IJ}_a = \Gamma^{IJ}_a[E] + 2K^{[I}_a n^{J]} \tag{2.224}$$

To summarize, we have now solved the 12 second class constraints (2.190) and (2.194a) and eliminated the 3 first class constraints of (2.194b) by solving them and fixing its corresponding gauge. The resulted dynamical variables are (E^a_I, K^I_a). Since they are both orthogonal to n^I (which is gauge fixed), their internal indices effectively take only the values $i = 1, 2, 3$ is the internal sub-Minkowski space. Thus, after eliminating the second class constraints, the ADM phase space Γ_{ADM} of the tetradic Palatini formulation is coordinatized by:

$$\Gamma_{ADM} = [E^a_i(\boldsymbol{\sigma}), K^i_a(\boldsymbol{\sigma})] \tag{2.225}$$

The only non-vanishing fundamental Poisson bracket:

$$\{E^a_i(\boldsymbol{\sigma}), K^j_b(\boldsymbol{\sigma}')\} = \delta^a_b \delta^j_i \delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \tag{2.226}$$

The remaining $7 \times \infty^3$ first class constraints functions via the new canonical variables of Γ_{ADM}:

$$G_i(\boldsymbol{\sigma}; E^a_i, K^i_a] := \epsilon_{ijk} K^j_a E^{ak} \approx 0 \tag{2.227a}$$

$$H(\boldsymbol{\sigma}; E^a_i, K^i_a] := -2E^a_{[i}E^b_{j]}K^i_a K^j_b - \frac{q}{(16\pi G)^2}\, {}^3R \approx 0 \tag{2.227b}$$

$$H_a(\boldsymbol{\sigma}; E^a_i, K^i_a] := 4E^b_i \nabla_{[a} K^i_{b]} \approx 0 \tag{2.227c}$$

This result is just the ADM form of the constraints (2.104a,2.104b) for tetrad gravity with three new constraints of Gauss law. Eqs. (2.227) are non-polynomial in the canonical variables and the close relation to Yang-Mills theory is now lost.

2.7 Holst Formulation

The Holst action [46] is an equivalent formulation of the tetradic Palatini action for General Relativity by adding a topological term part in the Lagrangian, it is known by *Nieh-Yan term*:

$$\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}[e, \tilde{\Gamma}] \tag{2.228}$$

which does not affect on the classical equations of motion as long as there is no torsion. We can now write down the Holst action, which is just the sum of half of tetradic-Palatini action and the Nieh-Yan term. The Holst action is defined as:

$$\begin{aligned} S_{Holst}[e, w] &= \frac{1}{32\pi G} \int_{\mathcal{M}} \mathrm{d}^4x \; e e^{\mu}_I e^{\nu}_J \left(\delta^I_{[K} \delta^J_{L]} - \frac{1}{2\gamma} \epsilon^{IJ}{}_{KL} \right) F^{KL}_{\mu\nu}[e, w] \\ &= \frac{1}{2} S_{t-P} - \frac{1}{64\pi G\gamma} \int_{\mathcal{M}} \mathrm{d}^4x \; \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}[e, \tilde{\Gamma}] \end{aligned} \tag{2.229}$$

with a real γ called *Immirzi parameter*. If we impose the torsion free condition, by using the first Bianchi identity and the symmetric property of the Riemann tensor, one has:

$$\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}[e] = 0 \tag{2.230}$$

Thus, the Nieh-Yan term (the last term in Eq. (2.229)) does not alter the classical equations of motion as long as there is no torsion. We now compute the equations of motion for this action:

Variation the action over the co-tetrad $w^I_{J\mu}$:

$$\delta_w S_{Holst} = 0 \Leftrightarrow \left(\delta^I_{[K} \delta^J_{L]} - \frac{1}{2\gamma} \epsilon^{IJ}{}_{KL} \right) \tilde{D}_\mu \left(e e^{[\mu}_I e^{\nu]}_J \right) = 0 \tag{2.231}$$

Up to boundary term and we have used the covariant constancy of the $\epsilon^{IJ}{}_{KL}$. We can prove that the equation of motion derived from the variation with respect to the connection is equivalent to:

$$\tilde{D}_{[\mu}e^I_{\nu]} = 0 \tag{2.232}$$

which is the torsion free condition, then we conclude that the Lorentz connection can be uniquely determined by the tetrad and we write $w^{IJ}_\mu = \Gamma^{IJ}_\mu$ where Γ^{IJ}_μ is the internal Christoffel symbol associated to D_μ.

Variation the action over the co-tetrad e^I_μ:

$$\begin{aligned}\delta_e S_{Holst} = 0 \Leftrightarrow & G^I_\mu[e,w] = e^{\nu I}\left(R_{\mu\nu}[e,w] - \frac{1}{2}g_{\mu\nu}R[e,w]\right) \\ & - \frac{1}{2\gamma}e^{\nu I}\epsilon^{KL}{}_{MN}\left(e_{\mu L}e^\rho_K F^{MN}_{\rho\nu}[w] - \frac{1}{2}g_{\mu\nu}e^\rho_K e^\sigma_L F^{MN}_{\rho\sigma}[w]\right) = 0 \end{aligned} \tag{2.233}$$

when we impose the torsion free condition, the contributions of the Nieh-Yan term in the equation (2.233) is zero by implying the first Bianchi identity of the Riemann tensor. The remaining non-zero terms is just the tetradic Palatini action and hence Eqs. (2.231,2.233) of the Holst formulation are now equivalent to Einstein's equations in vacuum.

2.8 Hamiltonian Analysis of Holst Formulation

Since the Holst action (2.229) differs from the tetradic Palatini action (2.165) just by the term of $(\delta^I_{[K}\delta^J_{L]} - \frac{1}{2\gamma}\epsilon^{IJ}{}_{KL})$ instead of $\delta^I_{[K}\delta^J_{L]}$. then one can perform a canonical analysis step by step (Legendre tansform, classification of constraints, solving SCC) as we did for the tetradic Palatini formulation. However, the difficulties in the presence of second class constraints can get past them by choosing a partial gauge fixing prior to performing the computation. We then fix the boost part of the internal Lorentz $SO(3,1)$ transformations by working on the so-called *time gauge* $n^I = \delta^I_0$. Physically, it means that the 0^{th} frame field e_0 is a unit orthogonal to the spacelike hyper-surface S_t for any foliation parameter $t \in \mathbb{R}$, then one has, $e_0 = \boldsymbol{n}$. With this time gauge fixing, the internal Minkowski symmetry $SO(3,1)$ are reduced to spatial sub-rotations $SO(3)$ which leaves the unit normal n^I invariant. By using this time gauge, the splitting $\mathcal{E}_i^a = e_i^a$ and the definition of time vector field $\boldsymbol{\tau}$ in Eq. (2.83), it is then possible to rewrite the action (2.229) in terms of the new variables [13, 14]:

The densitized triad:

$$E_i^a := e e_i^a = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}e_b^j e_c^k \tag{2.234}$$

The Ashtekar-Barbero connection:

$$A_a^i := \Gamma_a^i + \gamma K_a^i \tag{2.235}$$

where Γ_a^i is the $SO(3)$ connection that induces the Levi-Civita connection:

$$\Gamma_a^i := \frac{1}{2}\epsilon^i{}_{jk}\Gamma_a^{jk} = \frac{1}{2}\epsilon^i{}_{jk}w_a^{jk} \tag{2.236}$$

and the electric part K_a^i is defined in Eq. (2.212a), it can be related to the extrinsic curvature as:

$$K_a^i = K_{ab}e^{ib} \tag{2.237}$$

2.8.1 Hamiltonian analysis

Now, we write the foliated Holst action in terms of the new variables:

$$\begin{aligned} S_{Holst}[A, E] &= \frac{1}{8\pi G\gamma}\int_{\mathbb{R}} dt \int_{\Sigma} \mathrm{d}^3\boldsymbol{\sigma}\big[E_i^a \dot{A}_a^i - A_0^i D_a E_i^a - NH - N^a H_a\big] \\ &= \frac{1}{8\pi G\gamma}\int_{\mathbb{R}} dt\ L_{Holst}[A_a^i, \dot{A}_a^i, E_i^a] \end{aligned} \tag{2.238}$$

Observing first that the new variables A_a^i and E_i^a are canonically conjugate with each other. Second, the Holst Lagrangian does not depend on the time derivative of N, N^a and A_0^i. This confirms the status that the lapse function N, shift vector N^a and the time components of the Ashtekar-Barbero connection A_0^i are non-dynamical variables, they can be specified as arbitrary functions on $\mathbb{R}\times\Sigma$; they are only Lagrange multipliers. By comparison with the ADM formulation, we expect that the coefficients of N, N^a and A_0^i will form the *Hamiltonian*, *diffeomorphism* and

Gauss constraints respectively, they are polynomial constraints on the canonical variables $[A^i_a(\boldsymbol{\sigma}), E^a_i(\boldsymbol{\sigma})]$ as follows:

$$G_i(\boldsymbol{\sigma}; A^i_a, E^a_i] := D_a E^a_i = \partial_a E^a_i + \epsilon_{ijk} A^j_a E^{ak} \approx 0 \tag{2.239a}$$

$$H_a(\boldsymbol{\sigma}; A^i_a, E^a_i] := F^i_{ab} E^b_i \approx 0 \tag{2.239b}$$

$$H(\boldsymbol{\sigma}; A^i_a, E^a_i] := \epsilon^{ij}{}_k \frac{E^a_i E^b_j}{\sqrt{\det(E)}} \left(F^k_{ab} - 2\frac{1+\gamma^2}{\gamma^2} K^i_{[a} K^j_{b]} \right) \approx 0 \tag{2.239c}$$

where we introduce the covariant derivative D_a associated with the Ashtekar-Barbero connection A^i_a and its associated curvature F^i_{ab} as:

$$F^i_{ab} := 2\partial_{[a} A^i_{b]} + \epsilon^i{}_{jk} A^j_a A^k_b \tag{2.240}$$

The $18 \times \infty^3$ dimensional kinematical (unconstrained) phase space Γ can be then coordinatized as:

$$\Gamma = [A^i_a(\boldsymbol{\sigma}), E^a_i(\boldsymbol{\sigma})] \tag{2.241}$$

The symplectic structure is expressed by the basic Poisson bracket relations between the configuration variables and their conjugate momenta:

$$\{A^i_a(\boldsymbol{\sigma}), E^b_j(\boldsymbol{\sigma'})\} = 8\pi G\gamma \delta^i_j \delta^b_a \delta^3(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) \tag{2.242a}$$

$$\{A^i_a(\boldsymbol{\sigma}), A^j_b(\boldsymbol{\sigma'})\} = 0 \tag{2.242b}$$

$$\{E^a_i(\boldsymbol{\sigma}), E^b_j(\boldsymbol{\sigma'})\} = 0 \tag{2.242c}$$

Then, the Eqs. (2.239a,2.239b,2.239c), are the $7 \times \infty^3$ first class constraints; They define $11 \times \infty^3$ dimensional constrained surface on the kinematical phase space Γ, denoted by Γ_c as:

$$\Gamma_c := \{\Gamma | G_i(\boldsymbol{\sigma}) = 0, H_a(\boldsymbol{\sigma}) = 0, H(\boldsymbol{\sigma}) = 0\} \subset \Gamma \tag{2.243}$$

The Gauss constraint G_i are corresponds to the internal symmetry of $SO(3)$ group; we expect that G_i to be generators of $SO(3)$ transformations. Indeed this can be confirmed, if one computes the Poisson algebra of the smeared G_i with any phase space function.

2.8.2 Constraint Algebra Analysis

By using the Poisson brackets 2.242, one can show all constraints are first class. First class constraints generate gauge transformations on the constraint surface: To see what the gauge transformations look like in our case, consider the smearing of the Gauss constraints:

$$G[\Lambda] := \int_\Sigma \mathrm{d}^3\boldsymbol{\sigma}\Lambda^i(\boldsymbol{\sigma})G_i(\boldsymbol{\sigma}) \tag{2.244}$$

where Λ^i is an arbitrary $3d$- internal vector field on Σ. An explicit computation shows that:

$$\{G[\Lambda], A^i_a(\boldsymbol{\sigma})\} = \gamma D_a\Lambda^i \tag{2.245a}$$

$$\{G[\Lambda], E^a_i(\boldsymbol{\sigma})\} = \gamma\epsilon_{ij}{}^k\Lambda^j E^a_k \tag{2.245b}$$

which means that the Gauss constraint (2.239a) is the generator of $SO(3)$ internal symmetry. To see it clearly, one has:

$$\{G[\Lambda_1], G[\Lambda_2]\} = \frac{\gamma}{2}G\Big[[\Lambda_1, \Lambda_2]\Big] \tag{2.246}$$

which is the structure algebra of $su(2)$. The same thing can be done with the diffeomorphic and Hamiltonian constraints, they generate the $Diff(\Sigma)$ group and the time reparametrization respectively. Now we can count how many degrees of freedom in the gravitational field: the constrained phase space Γ_c has $11 \times \infty^3$ degrees of freedom, the first class constraints (2.239a,2.239b,2.239c) generate $7\times\infty^3$ parameter set of gauge transformations on Γ_c, then $7 \times \infty^3$ degrees of freedom must be subtracted in order to fix the gauge. The remaining $4 \times \infty^3$ variables define the reduced phase space $\Gamma_r \equiv \Gamma_{phys}$ and as expected, they correspond to 2 degrees of freedom at each coordinate point $\boldsymbol{\sigma}$ in configuration space. Therefore, the Hamiltonian analysis of the Holst action lead to an emergence of Gauss law's constraints associated to $SO(3) \cong SU(2)/_{Z_2}$ internal symmetry like Yang-Mills theories. Due to the algebra-isomorphism $so(3) \cong_{alg} su(2)$, one can use the generators basis of $su(2)$-algebra rather than $so(3)$-algebra. This can be achieved by equipping the internal indices of the connection with the Pauli matrices as,

$$A_a := A_a^i \tau_i \tag{2.247}$$

where,

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.248}$$

A suitable Legendre transformation of the phase space variables $[A_a^i(\boldsymbol{\sigma}), E_i^a(\boldsymbol{\sigma})]$ to another canonical variables $[h_\gamma[A], E_i(S)]$, where $h_\gamma[A] \in SU(2)$ is the holonomy of the connection A_a along a curve γ and $E_i(S) \in su(2)$ is the flux of the densitized triad E_i^a through a surface S. This step is very crucial to jump on the quantum world, since the new phase space $TSU(2) = SU(2) \times su(2)$ for each curve γ and surface S is easy to quantize and we have already know how its starting Hilbert space can be constructed from a well-defined measure on $SU(2)$. In the next, we will discuss in detail these canonical transformation in order to develop the theory of Loop Quantum Gravity.

Chapter 3

Loop Quantum Gravity

This chapter is based on papers in refs. [47, 48, 49, 50, 51, 52, 53] and textbooks in refs. [4, 5, 6, 7, 8, 9, 10].

Loop Quantum Gravity is based on the formulation of general relativity in terms of the Ashtekar-Barbero connection and the densitized triad in the language of the intenral symmetry $SU(2)$ (like Yang-Mills gauge theory), with Poisson brackets (2.242) and the three sets of constraints:

$G_i = 0$	Gauss law
$H_a = 0$	Spatial diffeomorphism invariance
$H = 0$	Hamiltonian constraint

The difference with a Yang-Mills theory is of course in the dynamics: In gauge theory, after imposing the Gauss law, we have a physical Hamiltonian. Here instead we still have a fully constrained system.

3.1 Quantization of the New Variables

We would like now to focus on quantating general relativity. There are two ways of doing that: the first is to solve the classical constraints to obtain the reduced phase space Γ_{red} and then quantizing the result space by finding a representation of the algebra of the observables which describes their dynamics. This procedure is usually called *reduced quantization*, but it is very complicated to apply in GR since the constraints algebra is quite difficult. The second way is known as *Dirac quantization procedure* [11, 12], it consists on quantizing the whole kinematical phase space Γ_{kin} by promoting the canonical variables to operators and also introducing a suitable Hilbert space $\mathcal{H}_{kin}$. Subsequently, one has to find the states $\Psi \in \mathcal{H}_{kin}$ that are annihilated by the constraint opertors; they form the physical Hilbert space space $\mathcal{H}_{phys}$. This is precisely what Wheeler and DeWitt did in quantizing ADM formalism. In the next, we will follow the Dirac quantization procedure A formal quantization of general relativity theory can be obtained following the basic steps:

- In order to introduce a suitable Hilbert space $\mathcal{H}_{kin}$, one has to define a physical inner product, i.e., we need an invariant measure δA on the space of smooth connections $\mathcal{A}$ modulo $SU(2) \times Diff(\Sigma)$ transformations. Our Hilbert space is the space of square integrable functionals:

$$\mathcal{H}_{kin} = L_2[\mathcal{A}, \delta A] \tag{3.1}$$

- Promoting the canonical variable of the phase space Γ_{kin} to operators acting on the Hilbert space $\mathcal{H}_{kin}$, with the Schrödinger representation:

$$\hat{A}^i_a(\boldsymbol{x})\Psi[A] = A^i_a(\boldsymbol{x})\Psi[A], \tag{3.2a}$$

$$\hat{E}^a_i(\boldsymbol{x})\Psi[A] = -i8\pi G\hbar\gamma\frac{\delta\Psi[A]}{\delta A^i_a(\boldsymbol{x})}, \tag{3.2b}$$

  which satisfies the canonical commutation relation,

$$\left[\hat{A}^i_a(\boldsymbol{x}), \hat{E}^b_j(\boldsymbol{x'})\right] = i8\pi G\hbar\gamma\delta^b_a\delta^i_j\delta^3(\boldsymbol{x}, \boldsymbol{x'}). \tag{3.3}$$

- Impose the Gauss law constraint,

$$\hat{G}_i\Big(\boldsymbol{x}; A, -i8\pi G\hbar\gamma\frac{\delta}{\delta A}\Big]\Psi[A] = 0 \tag{3.4}$$

  which selects the $SU(2)$-invariant states.

- Impose the Diffeomorphism invariance constraint,

$$\hat{H}_a\Big(\boldsymbol{x}; A, -i8\pi G\hbar\gamma\frac{\delta}{\delta A}\Big]\Psi[A] = 0 \tag{3.5}$$

  which selects the $Diff(\Sigma)$-invariant states.

- Impose the Hamiltonian constraint,

$$\hat{H}\Big(\boldsymbol{x}; A, -i8\pi G\hbar\gamma\frac{\delta}{\delta A}\Big]\Psi[A] = 0 \tag{3.6}$$

which selects the final physical Hilbert states $\mathcal{H}_{phys}$.

For our case, however, step 1 of the above procedure poses a problem, since we do not have a background metric (the metric is a fully dynamical quantity) at disposal to define the integration measure δA (due to the background independent nature of GR). Hence, we need to define a measure on the space of connections without having to resort to a fixed background. The key to do this is the notion of Holonomy Flux variables, which we introduce next.

Commutator.

$$\begin{aligned}\Big[\hat{A}^i_a(\boldsymbol{x}), \hat{E}^b_j(\boldsymbol{x'})\Big]\,\Psi[A] &= -i8\pi G\hbar\gamma A^i_a(\boldsymbol{x})\frac{\delta\Psi[A]}{\delta A^j_b(\boldsymbol{x'})} + i8\pi G\hbar\gamma\frac{\delta}{\delta A^j_b(\boldsymbol{x'})}\Big(A^i_a(\boldsymbol{x})\Psi[A]\Big) \\ &= i8\pi G\hbar\gamma\frac{\delta A^i_a(\boldsymbol{x})}{\delta A^j_b(\boldsymbol{x'})}\Psi[A] = i8\pi G\hbar\gamma\delta^b_a\delta^i_j\delta^3(\boldsymbol{x},\boldsymbol{x'})\Psi[A]\end{aligned}$$

3.2 Holonomy, Flux variables

In this section we will take the main step needed to prepare loop quantum gravity; one has to regularize the resulted Poisson algebra (3.3) using paths and surfaces integrals (removing delta functions), as we did previously with the ADM variables. This is necessary in order to proceed with the quantization. At this stage, the different tensorial nature of A^i_a and E^a_i plays a key role. Indeed, a brief look at Eq. (2.234) shows that the densitized triad E^a_i is really a 2-form:

$$E^a_i = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e^j_b e^k_c \tag{3.7}$$

Hence, one may smear it over a surface S to obtain the flux E_i over S (we will name the result as electric flux):

$$E_i^a \longrightarrow E_i(S) := \int_S d\sigma^1 d\sigma^2 \; n_a E_i^a \tag{3.8}$$

in which $n_a := \epsilon_{abc}\frac{\partial x^b}{\partial \sigma_1}\frac{\partial x^c}{\partial \sigma_2}$ is the normal to the surface and (σ_1, σ_2) are the parametrized coordinates of the surface. The connection on the other hand is a 1-form, so it is natural to smear it along a 1-dimensional path. Recall that the connection A_a^i defines the notion of infinitesimally parallel transport in the principal $SU(2)$-bundle over the base 3d-space Σ.

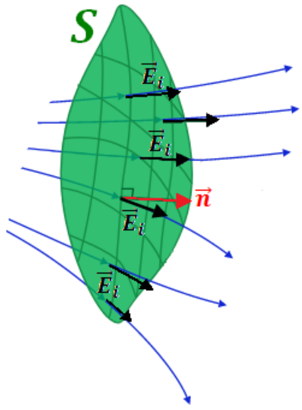


Figure 3.1: The flux of the densitized triad E_i^a through a given surface S.

Consider a path $\gamma : [0,1] \longrightarrow \Sigma$ and given a connection A_a^i we can associate to it an element of $su(2)$ as: $A_a = A_a^i \tau_i$ where τ_i are the generator of $SU(2)$ (Pauli matrices). Then we can integrate A_a along γ as a line integral,

$$A_a^i \longrightarrow \int_\gamma A \equiv \int_0^1 ds \; \frac{dx^a(\gamma(s))}{ds} A_a^i(\boldsymbol{x}(\gamma(s))\tau_i \tag{3.9}$$

Next, we define the holonomy[1] of the connection A_a along γ to be:

$$h_\gamma[A] := \mathcal{P}exp\left(\int_\gamma A\right) \in SU(2) \tag{3.10}$$

where $\mathcal{P}$ is the path-ordered product. That is,

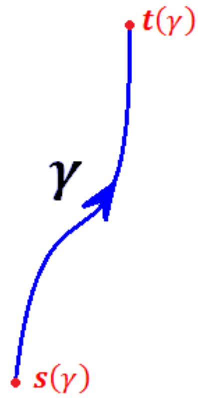


Figure 3.2: The line integral of the connection A along a given curve γ.

$$h_\gamma[A] = \sum_{n=0}^{\infty} \int_0^1 ds_1 \cdots \int_0^{s_n} ds_{n+1} A(\boldsymbol{\gamma}(s_1)) \cdots A(\boldsymbol{\gamma}(s_{n+1})) \tag{3.11}$$

where $\{s_n \in [0,1] | n \in \mathbb{N}, s_0 = 1\}$ is a decreasing sequence, we have used the notation:

$$\boldsymbol{x}(\gamma(s)) \equiv \boldsymbol{\gamma}(s) \tag{3.12}$$

$$A(\boldsymbol{\gamma}(s)) \equiv \frac{dx^a(\gamma(s))}{ds} A_a^i(\boldsymbol{\gamma}(s))\tau_i \tag{3.13}$$

[1]In the mathematical terminology, "holonomy" is often indicated by a parallel transport map, while the "holonomy" name is used for describing a parallel transport map along a closed curve (loop).

Geometrical interpretation of connection and holonomy. If we consider an internal vector field V^i, then one can parallelly transport it from a given coordinates point $\boldsymbol{x}$ to another one $\boldsymbol{x}+\boldsymbol{\varepsilon}$ along an infinitesimal space shift $\boldsymbol{\varepsilon}$ by using the connection as:

$$T_{\boldsymbol{x}\to\boldsymbol{x}+\boldsymbol{\varepsilon}}V^i = V^i(\boldsymbol{x}) - \epsilon^i{}_{jk}\boldsymbol{\varepsilon}^a A^j_a(\boldsymbol{x})V^k(\boldsymbol{x}) \tag{3.14}$$

Whereas the parallel transport of an internal vector V^i from a given point $\boldsymbol{x}$ to another arbitrary point $\boldsymbol{y}$ (they are not necessarily close to each other) through the curve $\gamma : [0,1] \to \Sigma$ where $\boldsymbol{\gamma}(0) = \boldsymbol{x}, \boldsymbol{\gamma}(1) = \boldsymbol{y}$ can be done by using the holonomy map:

$$T_{\boldsymbol{x}\to\boldsymbol{y}}V^i = (h_\gamma[A])^i{}_j\, V^j \tag{3.15}$$

More precisely, holonomy is the solution of the differential equation:

$$\frac{d}{ds}h_\gamma(s) - A(\boldsymbol{\gamma}(s))h_\gamma(s) = 0, \qquad h_\gamma(0) = \mathbb{I}.$$

where $\mathbb{I}$ is the unit element of $SU(2)$. integrating the equation by iteration we have:

$$\begin{aligned}
h_\gamma(s) =& \mathbb{I} + \int_0^s ds_1 A(\boldsymbol{\gamma}(s_1))h_\gamma(s_1)\\
=& \mathbb{I} + \int_0^s ds_1 A(\boldsymbol{\gamma}(s_1))\Big[\mathbb{I} + \int_0^{s_1} ds_2 A(\boldsymbol{\gamma}(s_2))h_\gamma(s_2)\Big]\\
=& \mathbb{I} + \int_0^s ds_1 A(\boldsymbol{\gamma}(s_1)) + \int_0^s ds_1 \int_0^{s_1} ds_2 A(\boldsymbol{\gamma}(s_1))A(\boldsymbol{\gamma}(s_2))h_\gamma(\boldsymbol{\gamma}(s_2))\\
&\vdots\\
=& \sum_{n=0}^{N} \int_0^s ds_1 \cdots \int_0^{s_n} ds_{n+1} A(\boldsymbol{\gamma}(s_1)) \cdots A(\boldsymbol{\gamma}(s_{n+1}))\\
&+ \int_0^s ds_1 \cdots \int_0^{s_{N+1}} ds_{N+2} A(\boldsymbol{\gamma}(s_1)) \cdots A(\boldsymbol{\gamma}(s_{N+2}))h_\gamma(\boldsymbol{\gamma}(s_{N+2}))\\
=& h_\gamma(s;N) + R_\gamma(s;N)
\end{aligned}$$

where $\{s_n \in [0,1] | n \in \mathbb{N}, s_0 = s\}$ is a decreasing sequence. $h_\gamma(s;N)$ is the Nth of the $h_\gamma(s)$ series, $R_\gamma(s;N)$ is the Nth-rest To complete the proof, one needs to show that the series $h_\gamma(s)$ is well defined. Indeed, it converges where $N \to \infty$

$$\begin{aligned}
&\lim_{N\to\infty} R_\gamma(s;N) = 0\\
&h_\gamma(s) = \sum_{n=0}^{\infty} \int_0^s ds_1 \cdots \int_0^{s_n} ds_{n+1} A(\boldsymbol{\gamma}(s_1)) \cdots A(\boldsymbol{\gamma}(s_{n+1}))
\end{aligned}$$

For further reference, let us also notice that the terms of the series can be written as integrals over square domains $(s_1, \ldots, s_n, s_{n+1}) \in [0,t]^{n+1}$, instead triangle domains $s < s_1 < \cdots < s_n < s_{n+1} < 0$. This gives

$$\begin{aligned}
=& \sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{P}\Big[\int_0^s dt A(\boldsymbol{\gamma}(t))\Big]^n\\
=& \mathcal{P}exp\Big(\int_0^s dt A(\boldsymbol{\gamma}(t))\Big)
\end{aligned}$$

the integral is expressed in terms of a *path ordered product*, denoted $\mathcal{P}(\cdots)$ is defined such that the quantities with larger values of s_n appear on the left of quantities with smaller values of s_n.

Let us list some of the main properties of the holonomy.

- The holonomy of the composition of two paths is the product of the holonomies of each path,

$$h_{\gamma_1}[A]h_{\gamma_2}[A] = h_{\gamma_1\sharp\gamma_2}[A] \tag{3.16}$$

  One can show from this:

$$h_{\gamma^{-1}}[A] = h_{\gamma}^{-1}[A] \tag{3.17}$$

- Under a local gauge transformations $g(x)$ ∈SU(2), the holonomy transforms as:

$$h_{\gamma}^{g}[A] = g(\boldsymbol{s}(\gamma))h_{\gamma}[A]g^{-1}(\boldsymbol{t}(\gamma)) \tag{3.18}$$

  where $\boldsymbol{s}(\gamma)$ and $\boldsymbol{t}(\gamma)$ are respectively the source and target points of the line γ, respectively:

$$\boldsymbol{s}(\gamma) \equiv \boldsymbol{\gamma}(0) \tag{3.19a}$$

$$\boldsymbol{t}(\gamma) \equiv \boldsymbol{\gamma}(1) \tag{3.19b}$$

- Under the action of diffeomorphism, the holonomy transforms as:

$$h_{\gamma}[\phi^{*}A] = h_{\phi\circ\gamma}[A] \tag{3.20}$$

In what follows, we are going to build a nice quantum representation of the kinematical Hilbert space. To do that we introduce in first place the notion of cylindrical functionals.

3.3 The kinematical state space $\mathcal{H}_{kin}$

3.3.1 Cylindrical functionals

A *cylindrical functional* is a functional of a field which only depend on cetain components of the fields. In the case at hand, the field is the Ashtekar connection A^i_a, and the cylindrical functions would be functionals that depend on the connection only through holonomies $h_e[A] = \mathcal{P}\exp\left(\int_e A\right)$ along some finite set of paths $\{e_l\}$. Consider a graph Γ, defined as a collection of oriented paths $e \subset \Sigma$, called, *links* of graph meeting at most at their endpoints, called, *nodes* of graph. We denote by L the total number of links in the graph. Therefore, a cylindrical functional is a couple $(\Gamma; f)$ of:

- a graph Γ:

$$\Gamma := \{e_l : [0,1] \to \Sigma | l = 1, \ldots, L\} \tag{3.21}$$

- a smooth complex-valued function f:

$$\begin{aligned} f \,:\, SU(2)^L &\longrightarrow \mathbb{C} \\ (g_1, \ldots, g_L) &\longmapsto f(g_1, \ldots, g_L) \end{aligned} \tag{3.22}$$

and it is given by a functional of the connection through holonomy:

$$\Psi_{(\Gamma;f)}[A] := f(h_{e_1}[A], \ldots, h_{e_L}[A]) \equiv \langle A|\Gamma; f\rangle \in Cyl_\Gamma \tag{3.23}$$

where Cyl_Γ is the collection set of all cylindrical functionals through the graph Γ. Notice that, the function is a functional of the connection only on a subset points $\Gamma \subset \Sigma$ through holonomies as stated previously. Figure 3.3 provides a description of a given graphs involved in the cylindrical functions.

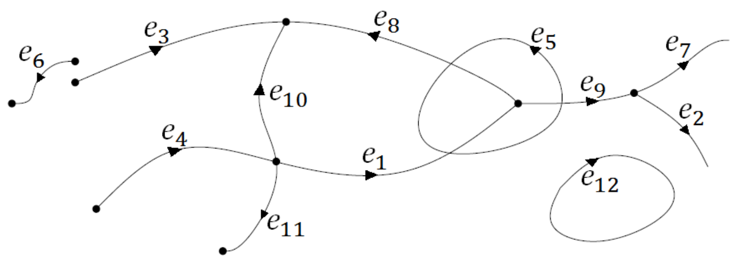


Figure 3.3: A particular graph: a collection of 12 ordered and oriented curves $\Gamma = \{e_1, \ldots, e_{12}\}$.

3.3.2 Kinematical space

Having introduced the cylindrical functionals, we want to define an inner product on it. Since the holonomy is an element of $SU(2)$, the space of cylindrical functionals Cyl_Γ can be converted into a Hilbert space if we equip it with an inner product over the $SU(2)$ space. As we already know, the integration over $SU(2)$ is well-defined, there is a unique gauge-invariant and normalized measure $\mathrm{d}\mu_{Haar}$, called the *Haar measure* on $SU(2)$ [54, 55]. Using L copies of the Haar measure. Thus, The switch from connection to holonomy variable is crucial in this respect. We define the scalar product on Cyl_Γ as:

$$\langle \Gamma; f_1 | \Gamma; f_2 \rangle := \int_{SU(2)} \mathrm{d}^L \mu_{Haar} \, \overline{f_1(h_1, \ldots, h_L)} f_2(h_1, \ldots, h_L) \tag{3.24}$$

where the bar sign denotes complex conjugation. With this scalar product, Cyl_Γ turns into a Hilbert space $\mathcal{H}_\Gamma$ associated to the graph Γ. we can now define the kinematical Hilbert space as the direct sum over all such Hilbert spaces for all possible graph,

$$\mathcal{H}_{kin} = \bigoplus_{\Gamma \subset \Sigma} \mathcal{H}_\Gamma. \tag{3.25}$$

where

$$\mathcal{H}_\Gamma = L_2[SU(2)^L, \mathrm{d}^L \mu_{Haar}] \tag{3.26}$$

The Haar measure on $SU(2)$.: Since the topology of $SU(2)$ is isomorphic to the 3-sphere S^3, then the Haar measure on $SU(2)$ is defined to be the restricted Euclidean measure of $\mathbb{R}^4$ on the hyper-surface $S^3 \subset \mathbb{R}^4$. it defines a unique gauge-invariant and normalized measure $\mathrm{d}\mu_{Haar}$ in which:

- right and left invariant: $\forall g \in SU(2) : \mathrm{d}\mu_{Haar} = \mathrm{d}(g \cdot \mu_{Haar}) = \mathrm{d}(\mu_{Haar} \cdot g)$
- normalized: $\int_{SU(2)} \mathrm{d}\mu_{Haar} = 1$

The Euler angle parametrisation of $SU(2)$ is defined by:

$$D^{(j)}(\psi, \theta, \phi) := e^{\psi J_3^{(j)}} e^{\theta J_2^{(j)}} e^{\phi J_3^{(j)}}$$

where

$$\psi \in [0, 2\pi[\qquad \theta \in [0, \pi[\qquad \phi \in [0, 4\pi[$$

In terms of these coordinates, the Haar measure reads:

$$\int_{SU(2)} d\mu_{Haar} := \frac{1}{16\pi^2} \int_0^{2\pi} d\psi \int_0^{\pi} d\theta \int_0^{4\pi} d\phi$$

The scalar product on $\mathcal{H}_{kin}$ is easily induced from (3.24) in the following manner: if f_1 and f_2 share the same graph, then (3.24) immediately applies. If they have different graphs, say Γ_1 and Γ_2:

$$\Gamma_1 := \{e_l : [0, 1] \to \Sigma | l = 1, \dots, L_1\} \tag{3.27a}$$

$$\Gamma_2 := \{e_l : [0, 1] \to \Sigma | l = L_1 + 1, \dots, L_1 + L_2\} \tag{3.27b}$$

we consider a further graph $\Gamma_1 \cup \Gamma_2$, then we extend f_1 and f_2 trivially on $\Gamma_1 \cup \Gamma_2$

$$\begin{aligned} SU(2)^{L_1+L_2} &\longrightarrow \mathbb{C} \\ (g_1, \dots, g_{L_1+L_2}) &\longmapsto f_1^*(g_1, \dots, g_{L_1+L_2}) := f_1(g_1, \dots, g_{L_1}) \\ (g_1, \dots, g_{L_1+L_2}) &\longmapsto f_2^*(g_1, \dots, g_{L_1+L_2}) := f_2(g_{L_1+1}, \dots, g_{L_1+L_2}) \end{aligned} \tag{3.28}$$

Hence, the inner product between functionals of two different graphs on $\mathcal{H}_{kin}$ are given by:

$$\langle \Gamma_1; f_1 | \Gamma_2; f_2 \rangle := \langle \Gamma_1 \cup \Gamma_2; f_1^* | \Gamma_1 \cup \Gamma_2; f_2^* \rangle \tag{3.29}$$

Equipped with such a scalar product, one is able to obtain the kinematical Hilbert space $\mathcal{H}_{kin}$. The key result, due to Ashtekar and Lewandowski, is that one can see the kinematical space as a Hilbert space of square integrable functionals on the connection (the original canonical variable). To do that, we must extend $\mathcal{A}$ to $\bar{\mathcal{A}}$ such that $\mathcal{H}_{kin}$ is isomorphic to some square integrable space over $\bar{\mathcal{A}}$.

Therefore, (3.25) defines a Hilbert space over gauge connections A on Σ, i.e.(see [56, 57] for details)

$$\mathcal{H}_{kin} = L_2[\bar{\mathcal{A}}, \mathrm{d}\mu_{AL}] \tag{3.30}$$

where $\bar{\mathcal{A}}$ is the extended space that contains distributions (it not necessarily smooth connections) to our classical smooth connection space $\mathcal{A}$. The integration measure $\mathrm{d}\mu_{AL}$ over the extended space of connections is called the *Ashtekar-Lewandowski measure*. What (3.30) means is that (3.29) can be seen as an inner product between cylindrical functionals of the connection with respect to the Ashtekar-Lewandowski measure:

$$\langle \Gamma_1; f_1 | \Gamma_2; f_2 \rangle = \int_{\bar{\mathcal{A}}} \mathrm{d}\mu_{AL}[A] \, \overline{\Psi_{(\Gamma_1;f_1)}[A]} \Psi_{(\Gamma_2;f_2)}[A] \tag{3.31}$$

Until now, the kinematical Hilbert space has been constructed without requiring a background metric. As a conclusion, two important points need to be discussed:

- Loop quantum gravity is a *continuous theory* whose kinematical Hilbert space is the direct continuous sum (3.25) of spaces (3.26) on a single graph Γ. This continuous sum gives rise to the problem of non-separability of $\mathcal{H}_{kin}$, which in turn does not allow us to define a countable quantum state basis for the kinematical space $\mathcal{H}_{kin}$[2]. The huge size of the starting Hilbert space $\mathcal{H}_{kin}$ will turn out to be just a gauge: thanks to diffeomorphism invariance, the physical Hilbert space will be separable.

- Due to the line functional property of the holonomy, each Hilbert space $\mathcal{H}_\Gamma$ on a given graph Γ captures only a finite number of degrees of freedom of the theory.

Remarkably, the configuration space of $\mathcal{H}_\Gamma$ corresponds to $SU(2)^L$ which is L-copies of compact lie groups, then the spectrum functional of $\mathcal{H}_\Gamma$ would be discrete. The next step is to introduce a discrete orthogonal basis in the space $\mathcal{H}_\Gamma$ of a given graph Γ.

[2]A Hilbert space admits a countable orthonormal basis if and only if it is separable. Therefore, Hilbert spaces are mostly assumed to be separable. a topological space is called *separable* if it contains a countable, dense subset.

3.3.3 An orthonormal basis

Thanks to the Peter-Weyl theorem [58], which states that a basis on the Hilbert space $L_2[G, \mathrm{d}\mu_{Haar}]$ of square integrable functions on a compact group G is given by the matrix elements of the unitary irreducible representation of the group. For the case of $SU(2)$, it can be easily find an orthonormal basis in $\mathcal{H}_\Gamma$, denote by $D^{(j)}_{mn}(g)$, called the *Wigner D-matrices*, they give the spin-(j) irreducible representation of the group element g as well as it measures how the magnetic quantum direction can be affected under a given rotation $g \in SU(2)$:

$$D^{(j)m}{}_{n}(g) \equiv \langle g|j,m,n\rangle := \langle j,m|D^{(j)}(g)|j,n\rangle \tag{3.32}$$

This will allow us to define an inner product, making use of the Haar measure $\mathrm{d}\mu_{Haar}$:

$$\langle j,m,n|j',m',n'\rangle := \int_{SU(2)} \mathrm{d}\mu_{Haar}\; D^{*(j)m}{}_{n}(g) D^{(j')m'}{}_{n'}(g) = \frac{\delta^{jj'}\delta^{mm'}\delta_{nn'}}{2j+1} \tag{3.33}$$

where the complex conjugate of the Wigner D-matrix

$$D^{*(j)m}{}_{n}(g) = (-1)^{m-n} D^{(j)-m}{}_{-n}(g) \tag{3.34}$$

The completeness relation is satisfied

$$\sum_{j,m,n} (2j+1)|j,m,n\rangle\langle j,m,n| = \mathbb{I} \tag{3.35}$$

given a function $f \in L_2[SU(2), \mathrm{d}\mu_{Haar}]$, one can decompose f in terms of the orthogonal basis (3.34) as:

$$\begin{aligned} f(g) &= \langle g|f\rangle \\ &= \sum_{j,m,n} f_{jm}^{\;n} D^{(j)m}{}_{n}(g) \\ \text{for } j = 0,&\frac{1}{2},1,\frac{3}{2},\dots \qquad m,n = -j,\dots,j \end{aligned} \tag{3.36}$$

The inverse transform gives the coefficients $f_{jm}^{\;n}$ by the relation:

$$\begin{aligned} f_{jm}^{\;n} &\equiv (2j+1)\langle j,m,n|f\rangle \\ &= (2j+1)\int_{SU(2)} \mathrm{d}\mu_{Haar} D^{*(j)m}{}_{n}(g) f(g) \end{aligned} \tag{3.37}$$

Then any state vector $|f\rangle$ can be written in terms of the orthogonal basis elements $|j, m, n\rangle$ as:

$$|f\rangle = \sum_{j,m,n} f_{jm}^{\ \ n} |j, m, n\rangle \tag{3.38}$$

with taking summation over all repeated indices. This immediately can be applied for the kinematical space $\mathcal{H}_\Gamma$ on fixed graph Γ, since the latter is just a tensor product of $L_2[SU(2), d\mu_{Haar}]$. The orthogonal basis elements $|\Gamma; j_l, m_l, n_l\rangle$ is a tensor product of the states $|j, m, n\rangle$ over each path of the graph Γ:

$$\begin{aligned} |\Gamma; j_l, m_l, n_l\rangle &:= |e_1; j_1, m_1, n_1\rangle \otimes \cdots \otimes |e_L; j_L, m_L, n_L\rangle \\ &\equiv |e_1, \ldots, e_L; j_1, \ldots, j_L, m_l, \ldots, m_L, n_l, \ldots, n_L\rangle \end{aligned} \tag{3.39}$$

It can be written in the connection representation $|A\rangle$ via the holonomy as:

$$\begin{aligned} \langle A|\Gamma; j_l, m_l, n_l\rangle &\equiv \langle A|e_1, \ldots, e_L; j_1, \ldots, j_L, m_l, \ldots, m_L, n_l, \ldots, n_L\rangle \\ &\equiv \langle h_\Gamma[A]|e_1, \ldots, e_L; j_1, \ldots, j_L, m_l, \ldots, m_L, n_l, \ldots, n_L\rangle \\ &= \langle h_{e_1}[A]|e_1; j_1, m_1, n_1\rangle \cdots \langle h_{e_1}[A]|e_L; j_L, m_L, n_L\rangle \\ &= D^{(j_1)m_1}{}_{n_1}(h_{e_1}[A]) \cdots D^{(j_L)m_L}{}_{n_L}(h_{e_L}[A]) \end{aligned} \tag{3.40}$$

with tensor product of Wigner matrices. Then, any function $\Psi_{(\Gamma;f)}[A] \in \mathcal{H}_\Gamma$ can be decomposed as:

$$\begin{aligned} \Psi_{(\Gamma;f)}[A] &= \langle A|\Gamma; f\rangle \\ &= \sum_{\substack{j_1,\ldots,j_L \\ m_1,\ldots,m_L \\ n_1,\ldots,n_L}} f_{j_1\ldots j_L m_1 \ldots m_L}^{\qquad\quad n_1\ldots n_L} \langle A|\Gamma; j_l, m_l, n_l\rangle \\ &= \sum_{\substack{j_1,\ldots,j_L \\ m_1,\ldots,m_L \\ n_1,\ldots,n_L}} f_{j_1\ldots j_L m_1 \ldots m_L}^{\qquad\quad n_1\ldots n_L} D^{(j_1)m_1}{}_{n_1}(h_{e_1}[A]) \cdots D^{(j_L)m_L}{}_{n_L}(h_{e_L}[A]) \end{aligned} \tag{3.41}$$

where the inverse transform gives the coefficients $f_{j_1\ldots j_L m_1 \ldots m_L}^{\qquad\quad n_1\ldots n_L}$ by the relation:

$$\begin{aligned} f_{j_1\ldots j_L m_1 \ldots m_L}^{\qquad\quad n_1\ldots n_L} &\equiv \prod_{i=1}^{L} (2j_i + 1) \langle \Gamma; j_l, m_l, n_l|\Gamma; f\rangle \\ &= \int_{SU(2)} \prod_{i=1}^{L} \Big[(2j_i + 1) \mathrm{d}h_i \ D^{*(j_i)m_i}{}_{n_i}(h_i) \Big] f(h_1, \ldots, h_L) \end{aligned} \tag{3.42}$$

Then any state vector $|\Gamma; f\rangle$ can be written in terms of the orthogonal basis elements $|\Gamma; j_l, m_l, n_l\rangle$ as:

$$|\Gamma; f\rangle = \sum_{\substack{j_1,\ldots,j_L \\ m_1,\ldots,m_L \\ n_1,\ldots,n_L}} f_{j_1\ldots j_L m_1\ldots m_L}{}^{n_1\ldots n_L} |\Gamma; j_l, m_l, n_l\rangle \tag{3.43}$$

Accordingly with the orthonormal basis (3.40), the kinematical Hilbert space on a fixed graphcan be decomposed into a tensor product between irreduicible representation and its dual for each link as:

$$\mathcal{H}_\Gamma = \bigoplus_{j_l} \bigotimes_{l=1}^{L_\Gamma} \left(V_\Gamma^{*(j_l)} \otimes V_\Gamma^{(j_l)} \right) \tag{3.44}$$

3.3.4 Holonomy-flux algebra

On this orthogonal basis (3.40), one can give a Schrödinger representation as (3.2) for the holonomy-flux variables. The holonomy acts by multiplication:

$$\hat{h}_\gamma[A]\Psi_{(\Gamma;f)}[A] = h_\gamma[A]\Psi_{(\Gamma;f)}[A], \tag{3.45a}$$

and the flux acts through the functional derivative:

$$\hat{E}_i(S)\Psi_{(\Gamma;f)}[A] = -i8\pi G\hbar\gamma \int_S \mathrm{d}\sigma^1 \mathrm{d}\sigma^2 n_a \frac{\delta \Psi_{(\Gamma;f)}[A]}{\delta A_a^i(\boldsymbol{x}(\boldsymbol{\sigma}))} \tag{3.45b}$$

For simplicity, let us take the case of one smooth curve graph $\Gamma = \{\gamma : [0,1] \to \Sigma\}$ with the wave functional $\Psi_{(\gamma;f)}[A] = f^j_{mn} D^{(j)}_{mn}(h_\gamma)$, and consider a given $2d$-surface S intersects at most once with the curve γ then the fluxes acts trivialy as:

$$\hat{E}_i(S)\Psi_{(\gamma;f)}[A] = -i8\pi G\hbar\gamma \begin{cases} \frac{\kappa_S(\gamma)}{2} J_i^{(j)} \Psi_{(\gamma;f)}[A], & S \cap \mathrm{Im}\gamma = \boldsymbol{\gamma}(0) \\ \frac{\kappa_S(\gamma)}{2} \Psi_{(\gamma;f)}[A] J_i^{(j)}, & S \cap \mathrm{Im}\gamma = \boldsymbol{\gamma}(1) \\ \kappa_S(\gamma) \Psi_{(\gamma_1;f)}[A] J_i^{(j)} \Psi_{(\gamma_2;f)}[A], & S \cap \mathrm{Im}\gamma \in \gamma(]0,1[) \\ 0, & S \cap \mathrm{Im}\gamma = \emptyset \end{cases} \tag{3.46}$$

Here $J_i^{(j)}$ is the j-representation angular momentum of $SU(2)$ symmetry. γ_1 and γ_2 are the two new curves defined by the point at which the densitized triad acts and the sign $\kappa_S(\gamma)$ is defined by:

$$\kappa_S(\gamma) := \mathrm{sign}(n_a \dot{\gamma}^a) = \pm 1, 0 \tag{3.47}$$

it depends on the relative orientation of γ and S and it vanishes if γ is tangential to S at the intersection point. Now, it is time to write the resulting smeared algebra between the two operator quantities $\hat{E}_i(S)$ and $\hat{h}_\gamma[A]$, that is called *holonomy-flux algebra*: we will take the assumption that the surface S intersects with the curve γ only at one point inside the curve, as well as the surface S and the curve γ has the same orientation ($\kappa_S(\gamma) = 1$), we get:

$$[\hat{E}_i(S), \hat{E}_k(S)]\Psi_{(\gamma;f)}[A] = -(8\pi G\hbar\gamma)^2 \ \Psi_{(\gamma_1;f)}[A]\left[J_i^{(j)}, J_k^{(j)}\right]\Psi_{(\gamma_2;f)}[A] \tag{3.48a}$$

$$[\hat{h}_{\gamma'}[A], \hat{E}_j(S)]\Psi_{(\gamma;f)}[A] = i8\pi G\hbar\gamma \ h_{\gamma_1'}\tau_i h_{\gamma_2'}\Psi_{(\gamma;f)}[A] \tag{3.48b}$$

$$[\hat{h}_{\gamma'}[A], \hat{h}_{\gamma'}[A]]\Psi_{(\gamma;f)}[A] = 0 \tag{3.48c}$$

From (3.48a), we immediately find that two fluxes operators *do not commute*,

$$\begin{aligned}[\hat{E}_i(S), \hat{E}_k(S)]\Psi_{(\gamma;f)}[A] &= -(8\pi G\hbar\gamma)^2 \ \Psi_{(\gamma_1;f)}[A]\epsilon_{ik}{}^l J_l^{(j)}\Psi_{(\gamma_2;f)}[A] \\ &= -i8\pi G\hbar\gamma \ \epsilon_{ik}{}^l \hat{E}_l(S)\Psi_{(\gamma;f)}[A]\end{aligned} \tag{3.49}$$

Then one can write,

$$\left[\frac{i\hat{E}_i(S)}{8\pi G\hbar\gamma}, \frac{i\hat{E}_k(S)}{8\pi G\hbar\gamma}\right] = \epsilon_{ik}{}^l \frac{i\hat{E}_l(S)}{8\pi G\hbar\gamma} \tag{3.50}$$

which indicates that $\frac{i\hat{E}_i(S)}{8\pi G\hbar\gamma}$ is an $su(2)$ generator algebra. Consider now the action of the scalar product of two fluxes acting inside the link,

$$\begin{aligned}\hat{E}_i(S)\hat{E}^i(S)\Psi_{(\gamma;f)}[A] &= -(8\pi G\hbar\gamma)^2 \ (\gamma, S)\Psi_{(\gamma_1;f)}[A]\delta^{ik}J_i^{(j)}J_k^{(j)}\Psi_{(\gamma_2;f)}[A] \\ &= -(8\pi G\hbar\gamma)^2 \ \Psi_{(\gamma_1;f)}[A][-j(j+1)\mathbb{I}_{2j+1}]\Psi_{(\gamma_2;f)}[A] \\ &= (8\pi G\hbar\gamma)^2 j(j+1) \ \Psi_{(\gamma;f)}[A]\end{aligned} \tag{3.51}$$

On the right hand side, we see the appearance of the scalar contraction of algebra generators, $\delta^{ik}J_i^{(j)}J_k^{(j)} \equiv -j(j+1)\mathbb{I}_{2j+1}$. This scalar product is known as the *Casimir operator* of the algebra[3]. Now, we have reached a stage of determining the dynamics of the theory, one has to solve the quantum Einstein equations of

[3]The Casimir clearly commutes with all group elements.

LQG (3.4,3.5,3.6) with the starting kinematical state $\Psi \in \mathcal{H}_{kin}$; the quantum reduction is consist of the following steps:

$$\mathcal{H}_{kin} \xrightarrow{\hat{G}_i\Psi = 0} \mathcal{H}_0 \xrightarrow{\hat{H}_a\Psi = 0} \mathcal{H}_{Diff} \xrightarrow{\hat{H}\Psi = 0} \mathcal{H}_{phys}. \tag{3.52}$$

as we saw that the physical Hilbert space $\mathcal{H}_{phys}$ is given by those states that are annihilated by all Gauss, diffeomorphic and Hamiltonian constraints. In the next, we will provide the procedure to solve these constraints.

3.4 Gauge invariance state space $\mathcal{H}_0$

The first step is to solve the quantum Gauss constraint, which are those states in $\mathcal{H}_{kin}$ that are invariant under the action of $SU(2)$ gauge transformation. The space of all solutions define a new Hilbert space called the *gauge invariance space*, denoted by $\mathcal{H}_0$. Recalling from (3.18) how the holonmy transforms under $SU(2)$ gauge transformations:

$$h_e \longrightarrow h'_e = g_s\ h_e\ g_t^{-1}. \tag{3.53}$$

Similarly, the j-irreducible representation transforms as:

$$\begin{aligned} D^{(j)}(h_e) \longrightarrow D^{(j)}(h'_e) &= D^{(j)}\Big(g_s\ h_e\ g_t^{-1}\Big) \\ &= D^{(j)}\Big(g_s\Big)D^{(j)}\Big(h_e\Big)D^{(j)}\big(g_t^{-1}\Big). \end{aligned} \tag{3.54}$$

From this it follows that, the $SU(2)$ gauge transformations act only on the source and target points of the links, namely on the *nodes* of a graph. Given a graph $\Gamma = \{e_l\}$ with L links and N nodes. We say that a cylindrical functional f_0 is gauge invariant under the action of $SU(2)$ group at the nodes of the graph Γ if and only if:

$$f_0(h_1, \dots, h_L) = f_0(g_{s_1}h_1g_{t_1}^{-1}, \dots, g_{s_L}h_Lg_{t_L}^{-1}) \in Cyl_\Gamma/SU(2) \tag{3.55}$$

This property can be easily achieved by using the so-called *group averaging* method.

3.4.1 Group averaging method

Given an arbitrary cylindrical functional $f \in Cyl_\Gamma$, since the Haar measure is right and left multiplication invariant by any element of the group $SU(2)$, then the function:

$$f_0(h_1, \ldots, h_L) = \int_{SU(2)} \prod_{n=1}^{N} \mathrm{d}g_n \, f(g_{s_1} h_1 g_{t_1}^{-1}, \ldots, g_{s_L} h_L g_{t_L}^{-1}) \tag{3.56}$$

clearly satisfies the gauge-invariance condition (3.55). Before using this method to solve the Gauss constraints, let us agree on some useful notations: for each node n, we associates a valency number V_n; the number of the links whose intersect with the node n at their endpoints where, $V_n = O_n + I_n$, it is the sum number of the outgoing O_n and the incoming I_n links on the node n. There are two equivalent ways to select the irreducible representation for the graph:

- We select a quantum number j_l, $l = 1, \ldots, L$ for each link index l.
- We select a quantum number $j_{(n,i)}$, $n = 1, \ldots, N$, $i = 1, \ldots, V_n$ for each link index i intersects with the node index n.

one can see the equivalence by:

$$\{j_l, j_l |\ l = 1, \ldots, L\} = \{j_{(n,i)} |\ n = 1, \ldots, N \ , \ i = 1, \ldots, V_n\} \tag{3.57}$$

and

$$\sum_{n=1}^{N} V_n = 2L \tag{3.58}$$

Having made this notation, we write down the gauge-invariant wave functional in the orthonormal basis (3.40) via the invariant cylindrical functional (3.56), one has:

$$\Psi_{(\Gamma;f_0)}[A] = \sum_{\substack{j_1,\ldots,j_L \\ m_1,\ldots,m_L \\ n_1,\ldots,n_L}} f^{j_1\cdots j_L} \cdot \prod_{n=1}^{N} \mathcal{P}^{j_{(n,1)}\cdots j_{(n,V_n)}} \cdot \prod_{l=1}^{L} D^{(j_l)}(h_{e_l}[A]) \tag{3.59}$$

where the sum over the magnetic number m_l, n_l is implicitly implied. The projector $\mathcal{P}_n$ for each node n is define by the following integral:

$$\mathcal{P}^{j_{(n,1)}\cdots j_{(n,V_n)}} = \int_{SU(2)} \mathrm{d}g_n \prod_{o=1}^{O_n} D^{(j_{n,o})}(g_n) \prod_{i=1}^{I_n} D^{(j_{n,O_n+i})}(g_n^{-1}) \tag{3.60}$$

3.4.2 Spin network state

Since each node n of the graph intersects with V_n- links of irreducible representation, then the wave functional (3.59) must have a tensor product of V_n-copies of $SU(2)$-representation at the node,

$$\prod_{i=1}^{V_n} D^{(j_{(n,i)})} \in \bigotimes_{i=1}^{V_n} V^{(j_{(n,i)})}. \tag{3.61}$$

transforms non-trivially under gauge transformation and it is in general reducible, one can decompose it into irreducible representations $V^{(J_{n,\alpha})}$ as:

$$\bigotimes_{i=1}^{V_n} V^{(j_{(n,i)})} = \bigoplus_{\alpha} V^{(J_{n,\alpha})} \tag{3.62}$$

Then, the projector in (3.60) selects the gauge-invariant part of (3.41), namely it gives the *singlet* space for 0-total irr-representations $V^{(0_{n,k})}$ for some indices k at each node n. That is,

$$V^{(0_n)} := \bigoplus_{k} V^{(0_{n,k})} \subseteq \bigoplus_{\alpha} V^{(J_{n,\alpha})} \tag{3.63}$$

Since $\mathcal{P}_n$ is a projector from the space (3.62) to (3.63) at the node n, one can decompose it in terms of a basis of $V^{(0_n)}$ as:

$$\mathcal{P}_n = \sum_k i_{n,k}\, i^*_{n,k} \tag{3.64}$$

where $\{i_{n,k}\}$ is the basis vector of $V^{(0_n)}$ and its $\{i^*_{n,k}\}$ dual basis at the node n. Then the invariant wave functional (3.59) can be written as a linear combination of products of representation matrices $D^{(j)}(h_e)$ contracted with the basis $i_{n,k}$. These invariants vectors $i_n = v^k i_{n,k} \in V^{(0_n)}$ are called *intertwiners* and the signlet space $V^{(0_n)}$ for the node n called *the V_n-valent intertwiner space*. As a final result, the basis of $\mathcal{H}_0$ are the quantum states labelled by a graph Γ, a spin-j_l of the holonomy along each link e_l and an intertwiner i_n for each node n, are called *spin network states*, and are given by:

$$\begin{aligned}\Psi_{(\Gamma;j_l,i_n)}[A] &:= \langle A|\Gamma; j_l, i_n\rangle \\ &= \prod_{l=1}^{L} D^{(j_l)}(h_{e_l}[A]) \cdot \prod_{n=1}^{N} i_n\end{aligned} \tag{3.65}$$

For simplicity of notation, the indices of the irr-matrices and of the interwiners are hidden, their contraction can be easily reconstructed from the connectivity of the graph. Then for each node n of valency V_n, the intertwiner state corresponds to a singlet state which has a zero total irr-representation:

$$\sum_{l=1}^{V_n} J_i^{(j_l)} = 0 \tag{3.66}$$

which is known by *the closure relation* for each node n: the sum of all $su(2)$ irr-representation vectors sharing the same node must be vanish. We will take advantage of this result later in section 4.3, which is similar to the closure condition of the area-norm vectos of a any convex Euclidean polyedron with V_n faces. Then we obtain the important result that the spin network states (3.65) form a complete basis of the gauge invariance Hilbert space $\mathcal{H}_0$ and the quantum numbers of a given spin network are $(\Gamma; j_l, i_n)$. They define the notion of *quantum geometry.*

3.4.3 Intertwiner space $\mathcal{H}_{V_n}$

The gauge invariant Hilbert space on a fixed graph can be decomposed into a sum of the intertwiner spaces,

$$\mathcal{H}_\Gamma^0 = L_2 \left[SU(2)^L / SU(2)^N, \mathrm{d}^L \mu_{Haar} \right] \tag{3.67}$$

$$= \bigoplus_{j_l} \bigotimes_n V^{(0_n)} \tag{3.68}$$

In other words, intertwiners are the building blocks of spin network states. We will refer to them by $\mathcal{H}_{V_n}$. Thus, the intertwiner space corresponds to the node n of valency V_n, is the singlet space of 0-total irr-representation:

$$\mathcal{H}_{V_n} := \mathrm{inv} \left[\bigotimes_{i=1}^{V_n} V^{(j_{(n,i)})} \right] = V^{(0_n)} \tag{3.69}$$

As before, different graphs Γ select different orthonormal basis $\Psi_{(\Gamma; j_l, i_n)}[A]$ for each node n, thus $\mathcal{H}_0$ can be decomposed into a sum over spaces on a fixed graph as:

$$\mathcal{H}_0 = \bigoplus_{\Gamma \subset \Sigma} \bigoplus_{j_l} \bigotimes_n V^{(0_n)}. \tag{3.70}$$

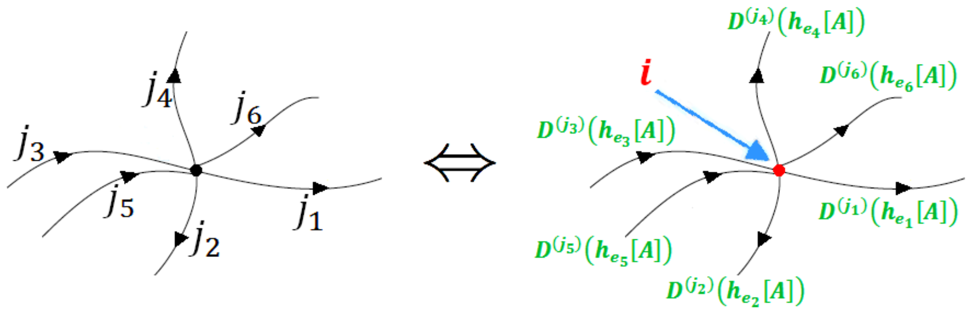


Figure 3.4: Description of an intertwiner state corresponds to a 6-valent node.

3.4.4 Loop Representation of $\mathcal{H}_0$

A particular class of gauge-invariant wave functionals are the Wilson loop variables [17], which are the trace of holonomies (3.10) through given closed curves (loops)[4]. If we consider a graph system contains only one loop $\alpha : [0, 1] \to \Sigma, \alpha(0) = \alpha(1)$, the gauge-invariant Wilson loop is defined by the the cylindrical functional $(\Gamma; f) = (\alpha; \mathrm{Tr})$. That is,

$$W_\alpha[A] = \langle A|\alpha; \mathrm{Tr}\rangle \tag{3.71}$$

$$= \mathrm{Tr}\left[(h_\alpha[A])^m{}_n\right] \tag{3.72}$$

It is clear that the Wilson loop solves the Gauss constraints (3.4). In terms of the orthonormal basis (3.40), the Wilson loop is defined by the the pair $(\Gamma; j) = (\alpha; \frac{1}{2})$ with an invariant intertwiner δ^n_m. That is,

$$W_\alpha[A] = \delta^n_m \langle A|\alpha; \frac{1}{2}, m, n\rangle \tag{3.73}$$

$$= \delta^n_m (h_\alpha[A])^m{}_n \tag{3.74}$$

Checking the gauge invariance of the Wilson loops. Under local $SU(2)$ gauge transformation $g : \Sigma \to SU(2)$ where $g(\alpha(0)) = g(\alpha(1)) = g_0$. According to the transformation

[4]For this reason, the name of loop in LQG theory has been taken [51].

property (3.53), the Wilson loop transforms as:

$$
\begin{aligned}
W'_\alpha[A] &= \mathrm{Tr}\left[(h'_\alpha[A])^m{}_n\right] \\
&= \delta^n_m \left(g_0 h_\alpha[A] g_0^{-1}\right)^m{}_n \\
&= \delta^n{}_m (g_0)^m{}_{m'} (h_\alpha[A])^{m'}{}_{n'} (g_0^{-1})^{n'}{}_n \\
&= \underbrace{(g_0^{-1})^{n'}{}_n \delta^n{}_m (g_0)^m{}_{m'}}_{\delta^{n'}{}_{m'}} (h_\alpha[A])^{m'}{}_{n'} \\
&= \delta^{n'}{}_{m'} (h_\alpha[A])^{m'}{}_{n'} \\
&= W_\alpha[A]
\end{aligned}
$$

The important remark that we want to refer is that the spin network state (3.65) can be decomposed into a finite linear combination of Wilson loop states representations, and it can form a basis for the gauge invariant space $\mathcal{H}_0$ which minimize the degree of completeness of the loop basis. The key idea of switching the basis from spin network to loop state is coming from a corollary in a representation theory of Lie group, any irr-representation j can be writen as a symmetrized tensor product of $2j$ fundamental representations as:

$$\left[\bigotimes_{i=1}^{2j} \frac{1}{2}\right]_{sym} = j \tag{3.75}$$

Therefore, any elements of $V^{(j)}$ can be written as symmetric complex tensors with $2j$ spinor indices $0, 1$. The representation matrices in this basis can be taken in the following simple form:

$$D^{(j)A_1\ldots A_{2j}}{}_{B_1\ldots B_{2j}} = D^{(\frac{1}{2})(A_1}{}_{(B_1}(h) \cdots D^{(\frac{1}{2})A_{2j})}{}_{B_{2j}1)}(h) \tag{3.76}$$

We have used the parentheses to indicate the complete symmetrization. Moreover, in this basis the intertwiners are the combination of the two $SU(2)$-invariant tensors δ^A_B and ϵ_{AB}. i.e., invariant under any $g \in SU(2)$ transformation:

$$(g^{-1})^M_K \; \delta^K_L \; g^L{}_N = \delta^M_N \tag{3.77a}$$

$$g^K{}_M \; \epsilon_{KL} \; g^L{}_N = \epsilon_{MN} \tag{3.77b}$$

$$(g^{-1})^M_K \; \epsilon^{KL} \; (g^{-1})^N_L = \epsilon^{MN} \tag{3.77c}$$

From these, one can easily check:

$$g^K_{\ M}\ \epsilon_{KL} = \epsilon_{MN}\ (g^{-1})^N_{\ L} \tag{3.78a}$$

$$\epsilon^{KL}\ (g^{-1})^N_{\ L} = g^K_{\ L}\ \epsilon^{LN} \tag{3.78b}$$

The basic role of the invariants δ^A_B, ϵ_{AB} and ϵ^{AB} are to link two curves state together in just one curve state. For more detail, we shall consider the three cases of linking two curves:

1^{st} **case:** If we consider $(\gamma_1; \frac{1}{2})$ as incoming representation and $(\gamma_2; \frac{1}{2})$ as outgoing representation. The graph: $\Gamma = \{\gamma_1, \gamma_2\}$. The irr-representations: $j_1 = j_2 = \frac{1}{2}$. The intertwiner matrice of incoming-outgoing indices: $i^M_{\ N}$. The wave functional is given by:

$$\Psi_{(\Gamma;j_l,i)}[A]^m_{\ n} = \langle A|\Gamma; j_l, i\rangle^m_{\ n} = h_{\gamma_1}[A]^m_{\ M}\ i^M_{\ N}\ h_{\gamma_2}[A]^N_{\ n} \tag{3.79}$$

Recall that $\Psi_{(\Gamma;j_l,i)}[A]^m_{\ n}$ is invariant under $SU(2)$ transformation i.e., $\Psi_{(\Gamma;j_l,i)}[A]^m_{\ n} = \Psi_{(\Gamma;j_l,i)}[A']^m_{\ n}$, we focus just on the transformation act at the node of magnetic numbers (M, N) by an element $g \in SU(2)$ as:

$$\begin{aligned}\Psi_{(\Gamma;j_l,i)}[A']^m_{\ n} &= h_{\gamma_1}[A']^m_{\ M}\ i^M_{\ N}\ h_{\gamma_2}[A']^N_{\ n} \\ &= h_{\gamma_1}[A]^m_{\ K}\underbrace{\left(D^{(\frac{1}{2})}(g^{-1})\right)^K_{\ M}\ i^M_{\ N}\ \left(D^{(\frac{1}{2})}(g)\right)^N_{\ L}}_{(*)}h_{\gamma_2}[A]^L_{\ n}\end{aligned} \tag{3.80}$$

In order to have an invariant wave functional, then $(*)$ in Eq. (3.80) must be equal to $i^K_{\ L}$. The only intertwiner that satisfies this invariant property is $i^M_{\ N} = \delta^M_N$ (see that from Eq. (3.77a)). Therefore, the wave functional can be finally written as:

$$\begin{aligned}\Psi_{(\Gamma;j_l,i)}[A]^m_{\ n} &= h_{\gamma_1}[A]^m_{\ M}\ \delta^M_{\ N}\ h_{\gamma_2}[A]^N_{\ n} \\ &= [h_{\gamma_1}[A] \cdot h_{\gamma_2}[A]]^m_{\ n} \\ &= h_{\gamma_1 \sharp \gamma_2}[A]^m_{\ n}\end{aligned} \tag{3.81}$$

Graphically,this result can be seen as follows:

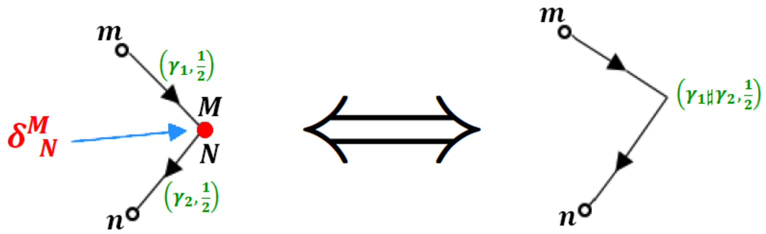


Figure 3.5: Description of how δ^M_N can eliminate a 2−valent node.

2^{nd} **case:** If we consider $(\gamma_1; \frac{1}{2})$ and $(\gamma_2; \frac{1}{2})$ are incoming representations. The graph: $\Gamma = \{\gamma_1, \gamma_2\}$. The irr-representation: $j_1 = j_2 = \frac{1}{2}$. The intertwiner matrice of two incoming indices: i^{MN}. The wave functional is:

$$\Psi_{(\Gamma;j_l,i)}[A]^{mn} = \langle A|\Gamma; j_l, i\rangle^{mn} = h_{\gamma_1}[A]^m{}_M \, i^{MN} \, h_{\gamma_2}[A]^n{}_N \tag{3.82}$$

Recall that $\Psi_{(\Gamma;j_l,i)}[A]^{mn}$ is invariant under $SU(2)$ transformation i.e., $\Psi_{(\Gamma;j_l,i)}[A]^{mn} = \Psi_{(\Gamma;j_l,i)}[A']^{mn}$, we focus just on the transformation act at the node of magnetic numbers (M, N) by an element $g \in SU(2)$ as:

$$\begin{aligned} \Psi_{(\Gamma;j_l,i)}[A']^{mn} &= h_{\gamma_1}[A']^m{}_M \, i^{MN} \, h_{\gamma_2}[A']^n{}_N \\ &= h_{\gamma_1}[A]^m{}_K \underbrace{\left(D^{(\frac{1}{2})}(g^{-1})\right)^K{}_M \, i^{MN} \, \left(D^{(\frac{1}{2})}(g^{-1})\right)^L{}_N}_{(*)} h_{\gamma_2}[A]^n{}_L \end{aligned} \tag{3.83}$$

In order to have an invariant wave functional, then $(*)$ in Eq. (3.83) must be equal to i^{KL}. The only intertwiner that satisfies this invariant property is $i^{MN} = \delta^{MN}$ (see that from Eq. (3.77b)). Therefore, the wave functional can be finally written as:

$$\begin{aligned} \Psi_{(\Gamma;j_l,i)}[A]^{mn} &= h_{\gamma_1}[A]^m{}_M \, \epsilon^{MN} \, h_{\gamma_2}[A]^n{}_N \\ &= h_{\gamma_1}[A]^m{}_M \, h_{\gamma_2^{-1}}[A]^M{}_N \, \epsilon^{Nn} \\ &= h_{\gamma_1 \sharp \gamma_2^{-1}}[A]^m{}_N \, \epsilon^{Nn} \end{aligned} \tag{3.84}$$

Graphically,this result can be seen as follows:

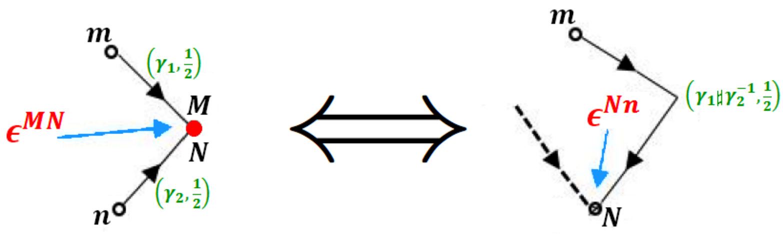


Figure 3.6: Description of how ϵ^{MN} can eliminate a 2−valent node.

3^{rd} **case:** If we consider $(\gamma_1; \frac{1}{2})$ and $(\gamma_2; \frac{1}{2})$ are outgoing representations. The graph: $\Gamma = \{\gamma_1, \gamma_2\}$. Te irr-representation: $j_1 = j_2 = \frac{1}{2}$. The intertwiner matrice of two outgoing indices: i_{MN}. The wave functional is:

$$\Psi_{(\Gamma;j_l,i)}[A]_{mn} = \langle A|\Gamma; j_l, i\rangle_{mn} = h_{\gamma_1}[A]^M{}_m \; i_{MN} \; h_{\gamma_2}[A]^N{}_n \tag{3.85}$$

Recall that $\Psi_{(\Gamma;j_l,i)}[A]_{mn}$ is invariant under $SU(2)$ transformation i.e., $\Psi_{(\Gamma;j_l,i)}[A]_{mn} = \Psi_{(\Gamma;j_l,i)}[A']_{mn}$, we focus just on the transformation act at the node of magnetic numbers (M, N) by an element $g \in SU(2)$ as:

$$\begin{aligned}\Psi_{(\Gamma;j_l,i)}[A']_{mn} = h_{\gamma_1}[A']^M{}_m \; i_{MN} \; h_{\gamma_2}[A']^N{}_n \\ h_{\gamma_1}[A]^K{}_m \underbrace{\left(D^{(\frac{1}{2})}(g)\right)^M{}_K \; i^{MN} \; \left(D^{(\frac{1}{2})}(g)\right)^N{}_L}_{(*)} h_{\gamma_2}[A]^L{}_n\end{aligned} \tag{3.86}$$

In order to have an invariant wave functional, then $(*)$ in Eq. (3.86) must be equal to i_{KL}. The only intertwiner that satisfies this invariant property is $i_{MN} = \epsilon_{MN}$ (see that from Eq. (3.77c)). Therefore, the wave functional can be finally written as:

$$\begin{aligned}\Psi_{(\Gamma;j_l,i)}[A]^{mn} &= h_{\gamma_1}[A]^M{}_m \; \epsilon_{MN} \; h_{\gamma_2}[A]^N{}_n \\ &= \epsilon_{mM} \; h_{\gamma_1^{-1}}[A]^M_N \; h_{\gamma_2}[A]^N{}_n \\ &= \epsilon_{mM} \; h_{\gamma_1^{-1}\sharp\gamma_2}[A]^M{}_n\end{aligned} \tag{3.87}$$

Graphically,this result can be seen as follows:

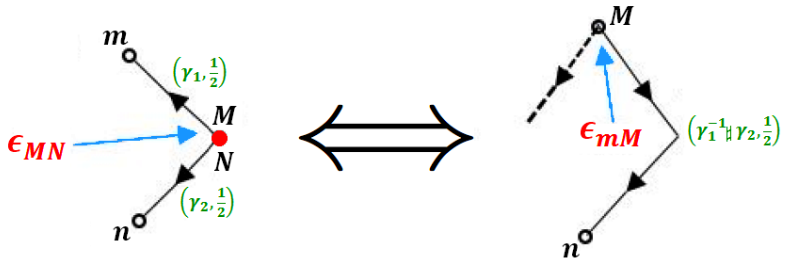


Figure 3.7: Description of how ϵ_{MN} can eliminate a 2−valent node.

An immediate consequence of these results, two holonomies of two continuous links are joined by the invariant tensors δ and ϵ depending on the orientations of the two links. Then if we take for example the case of a three-valent node with incoming irr-representations j, j' and outgoing j'' (See Fig. 3.8), thus the intretwiner i_n can be decomposed into $\frac{1}{2}$ irr-representations as follows:

$$i_n{}^{(M_1 \dots M_{2j})(N_1 \dots N_{2j'})}_{(L_1 \dots L_{2j''})} = \underset{(M)(N)(L)}{\text{sym}} \left[\epsilon^{M_1 N_1} \cdots \epsilon^{M_c N_c} \delta^{M_{c+1}}_{L_1} \cdots \delta^{M_{2j}}_{L_a} \delta^{N_{c+1}}_{L_{a+1}} \cdots \delta^{N_{2j'}}_{L_{2j''}} \right] \tag{3.88}$$

where

$$a + b = 2j'' \qquad a + c = 2j \qquad b + c = 2j' \tag{3.89}$$

After we knew that the invariant intertwiners of the $\frac{1}{2}$ irr-representation can cancel the nodes of the graph by gluing together the links, then finally, we will obtain a graph without any nodes, that means with loops. Therefore, the spin network is equal to a linear combination of loop states that warp along the graph.

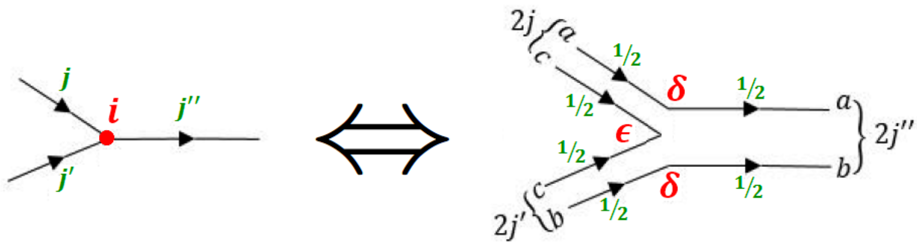


Figure 3.8: Decomposition of the 3−valent node of representations j, j' and j'' to 0−valent node.

3.5 Applications: 3-valent and 4-valent nodes

Before we solve the broblem of 3-valent and 4-valent nodes, we are going to briefly discuss the notion of Clebsch–Gordan coefficients.

3.5.1 Clebsch–Gordan coefficients

Recalling that, from Clebsch–Gordan theory [59] (or Addition of Angular Momentum in some textbook [2]): the tensor product $V^{(j_1)} \otimes V^{(j_2)}$ of two irr-representations of the $su(2)$-lie algebra is reducible, one has to decompose it into a sum of irreducible sub-spaces $V^{(J)}$ as:

$$V^{(j_1)} \otimes V^{(j_2)} = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} V^{(J)} \tag{3.90}$$

We will write the angular momentum state $|J, M\rangle^{j_1 j_2} \in V^{(J)}$ corresponds to the total angular momentum operator $\vec{J} = \vec{j}_1 \otimes 1 + 1 \otimes \vec{j}_2$ as a linear combination of reducible states $\{|j_1, m_1; j_2, m_2\rangle \in V^{(j_1)} \otimes V^{(j_2)}|\ m_i = -j_i, \ldots, j_i; i = 1, 2\}$ by using the well-known Clebsch–Gordan coefficients:

$$|J, M\rangle^{j_1 j_2} = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1, m_1; j_2, m_2 | J, M\rangle |j_1, m_1; j_2, m_2\rangle \tag{3.91}$$

where

$$C^{j_1 m_1 j_2 m_2}_{JM} = \langle j_1, m_1; j_2, m_2 | J, M\rangle \tag{3.92}$$

are the *Clebsch–Gordan coefficients*; they can only be nonzero when:

$$M = m_1 + m_2 \tag{3.93}$$

and the Clebsch-Gordan conditions must hold (the triangle inequality):

$$J \in \{|j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2 - 1, j_1 + j_2\} \tag{3.94}$$

An important relation which is frequently used for $J = 0$ is:

$$\langle j_1, m_1; j_2, m_2 | 0, 0\rangle = \delta_{j_1 j_2} \delta_{m_1, -m_2} \frac{(-1)^{j_1 - m_1}}{\sqrt{2j_1 + 1}} \tag{3.95}$$

Here, $|0, 0\rangle$ is the zero angular momentum of $J = M = 0$ (singlet irreducible-state).

3.5.2 Construction of the intertwiner basis of $\mathcal{H}_3$

Wigner 3j-symbols: the *Wigner 3j-symbols* [60] are the coefficients in which the addition of three angular momenta must give the singlet state. Equivalently, the Wigner 3j-symbols are the unique intertwiner corresponds to the gauge-invariant state of 3-valent node. To be more explicit, let us consider a graph of 3-valent node: $\Gamma = \{(e_1; j_1), (e_2; j_2), (e_3; j_3)\}$, the intertwiner space $\mathcal{H}_3 \equiv V^{(0)}$ at the gauge-invariant node is written by:

$$\mathcal{H}_3 = \text{inv}\left[V^{(j_1)} \otimes V^{(j_2)} \otimes V^{(j_3)}\right] \tag{3.96}$$

is non-empty only if the Clebsch-Gordan conditions hold:

$$j_3 \in \{|j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2 + 1, j_1 + j_2\} \tag{3.97}$$

It is obvious to see that $\dim(\mathcal{H}_3) = 1$ and the unique intertwiner can be found by writting the irreducible singlet $|0,0\rangle^{j_1 j_2 j_3} \in \mathcal{H}_3$ as a linear combination of the reducible states $\{|j_1, m_1; j_2, m_2; j_3, m_3\rangle \in \bigotimes_{l=1}^{3} V^{(j_l)}|\ m_i = -j_i, \ldots, j_i, i = 1, 2, 3\}$, one has first:

$$\begin{aligned} |0,0\rangle^{j_1 j_2 j_3} &= \sum_{M=-J}^{J} \sum_{m_3=-j_3}^{j_3} \langle J, M; j_3, m_3 | 0, 0\rangle |J, M\rangle^{j_1 j_2} \otimes |j_3, m_3\rangle \\ &= \sum_{M=-J}^{J} \sum_{m_3=-j_3}^{j_3} \delta_{J j_3} \delta_{M,-m_3} \frac{(-1)^{J-M}}{\sqrt{2J+1}} |J, M\rangle^{j_1 j_2} \otimes |j_3, m_3\rangle \\ &= \sum_{m_3=-j_3}^{j_3} \frac{(-1)^{j_3+m_3}}{\sqrt{2j_3+1}} |j_3, -m_3\rangle^{j_1 j_2} \otimes |j_3, m_3\rangle \end{aligned} \tag{3.98}$$

where in the first step we have applied (3.91), and in the second step we have used (3.95). Now, we will repeat the same thing for the state $|j_3, -m_3\rangle^{j_1 j_2} \in V^{(j_3)}$, one has:

$$|j_3, -m_3\rangle^{j_1 j_2} = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1, m_1; j_2, m_2 | j_3, -m_3\rangle |j_1, m_1; j_2, m_2\rangle \tag{3.99}$$

Substituting this into Eq. (3.98), we finally get:

$$|0,0\rangle^{j_1 j_2 j_3} = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \frac{(-1)^{j_3+m_3}}{\sqrt{2j_3+1}} \langle j_1, m_1; j_2, m_2 | j_3, -m_3\rangle |j_1, m_1; j_2, m_2; j_3, m_3\rangle \tag{3.100}$$

Since we define the Wigner 3j-symbols to be the unique intertwiner corresponds to the gauge-invariant state of 3-valent node, then we define:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_3+m_3}}{\sqrt{2j_3+1}} \langle j_1, m_1; j_2, m_2 | j_3, -m_3 \rangle \tag{3.101}$$

to be the Wigner 3j-symbols, It can be easily shown that the Wigner 3j-symbols satisies the following permutation properties:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} \tag{3.102a}$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} \tag{3.102b}$$

Then the irreducible singlet can be written as:

$$|0,0\rangle^{j_1 j_2 j_3} = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} |j_1, m_1; j_2, m_2; j_3, m_3\rangle \tag{3.103}$$

projecting this basis into the connection representation to get tha gauge-invariant wave functional. as a final conclusion of this result, the Wigner 3j-symbols is only intertwiner can connect three irr-representation (See Fig. 3.9) if and only if the triangle inequality (3.97) holds.

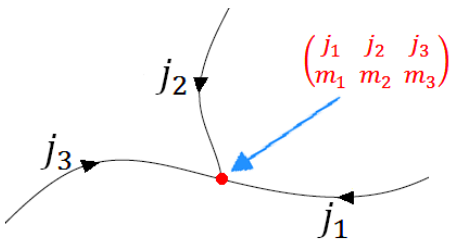


Figure 3.9: A 3-valent intertwiner state.

3.5.3 Construction of the intertwiner basis of $\mathcal{H}_4$

For the case of 4-valent node. Let us consider a graph of 4-valent node: $\Gamma = \{(e_1; j_1), (e_2; j_2), (e_3; j_3), (e_4; j_4)\}$, the intertwiner space $\mathcal{H}_4$ at the gauge-invariant node is written by:

$$\mathcal{H}_4 = \text{inv}\left[V^{(j_1)} \otimes V^{(j_2)} \otimes V^{(j_3)} \otimes V^{(j_4)}\right] \tag{3.104}$$

one has to decompose the reducible representations into irreducible ones, and then take the singlet irreducible spaces $V^{(0_k)}$ to be the generator spaces of $\mathcal{H}_4$ as:

$$\mathcal{H}_4 = \bigoplus_{k=k_{min}}^{k_{max}} V^{(0_k)} \tag{3.105}$$

If we use the recoupling channel $\{(j_1, j_2), (j_3, j_4)\}$, one has then the index k starts from k_{min} to k_{max} in integer steps with,

$$k_{min} = \max(|j_1 - j_2|, |j_3 - j_4|) \tag{3.106}$$

$$k_{max} = \min(j_1 + j_2, j_3 + j_4) \tag{3.107}$$

It is obvious to see that the dimension d of the Hilbert space $\mathcal{H}_4$ is finite and given by:

$$d = k_{max} - k_{min} + 1 \tag{3.108}$$

In order to determine the intertwiner corresponds to the gauge-invariant state, one has to write the irreducible-singlet state $|0,0\rangle_k^{j_1j_2j_3j_4} \in \mathcal{H}_4$ as a linear combination of the reducible states$\{|j_1, m_1; j_2, m_2; j_3, m_3; j_4, m_4\rangle \in \bigotimes_{l=1}^4 V^{(j_l)} |\ m_i = -j_i, \ldots, j_i, i = 1, 2, 3, 4\}$, one has first:

$$\begin{aligned}
|k\rangle &= |0,0\rangle_k^{j_1j_2j_3j_4} \\
&= \sum_{m=-k}^{k} \sum_{n=-k}^{k} \langle k, m; k, n|00\rangle_k |k, m\rangle^{j_1j_2} \otimes |k, n\rangle^{j_3j_4} \\
&= \sum_{m=-k}^{k} \sum_{n=-k}^{k} \delta_{m,-n} \frac{(-1)^{k-m}}{\sqrt{2k+1}} |k, m\rangle^{j_1j_2} \otimes |k, n\rangle^{j_3j_4} \\
&= \sum_{m=-k}^{k} \frac{(-1)^{k-m}}{\sqrt{2k+1}} |k, m\rangle^{j_1j_2} \otimes |k, -m\rangle^{j_3j_4}
\end{aligned} \tag{3.109}$$

Now, we write

$$|k, m\rangle^{j_1j_2} = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1, m_1; j_2, m_2|k, m\rangle |j_1, m_1; j_2, m_2\rangle \tag{3.110a}$$

$$|k, -m\rangle^{j_3j_4} = \sum_{m_3=-j_3}^{j_3} \sum_{m_4=-j_4}^{j_4} \langle j_3, m_3; j_4, m_4|k, -m\rangle |j_3, m_3; j_4, m_4\rangle \tag{3.110b}$$

substituting these in Eq. (3.109), we get:

$$\begin{aligned}
|k\rangle = \sum_{m_1,m_2,m_3,m_4} \sum_{m=-k}^{k} \frac{(-1)^{k-m}}{\sqrt{2k+1}} &\langle j_1, m_1; j_2, m_2|k, m\rangle \langle j_3, m_3; j_4, m_4|k, -m\rangle \\
&|j_1, m_1\rangle \otimes |j_2, m_2\rangle \otimes |j_3, m_3\rangle \otimes |j_4, m_4\rangle
\end{aligned} \tag{3.111}$$

Using the definition (3.101), one has:

$$\langle j_1, m_1; j_2, m_2|k, m\rangle = (-1)^{-k+m}\sqrt{2k+1}\begin{pmatrix} j_1 & j_2 & k \\ m_1 & m_2 & -m \end{pmatrix} \tag{3.112a}$$

$$\langle j_3, m_3; j_4, m_4|k, -m\rangle = (-1)^{-k-m}\sqrt{2k+1}\begin{pmatrix} j_3 & j_4 & k \\ m_3 & m_4 & m \end{pmatrix} \tag{3.112b}$$

substituting this in Eq. (3.111), we finally get the intertwiner basis state for the 4-valent node as:

$$|k\rangle = \sum_{m_1,m_2,m_3,m_4}\sum_{m=-k}^{k}(-1)^{k+m}\sqrt{2k+1}\begin{pmatrix} j_1 & j_2 & k \\ m_1 & m_2 & -m \end{pmatrix}\begin{pmatrix} k & j_3 & j_4 \\ m & m_3 & m_4 \end{pmatrix}$$
$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle \otimes |j_3, m_3\rangle \otimes |j_4, m_4\rangle \tag{3.113}$$

where we have used the symmetry under even column-permutation of the Wigner $3j$-symbols (3.102a). projecting this basis into the connection representation to get tha gauge-invariant wave functional. as a final conclusion of this result, the 4-valent node can be docomposed into 3-valent 2 nodes (See Fig. 3.10) where the virtual link of irr-representation k has been considered.

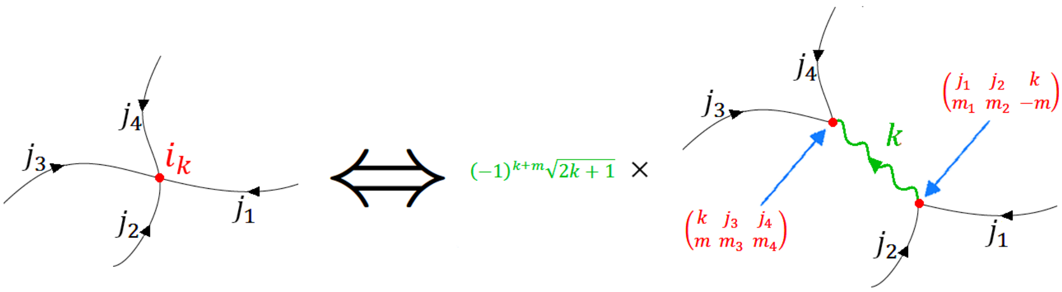


Figure 3.10: Decomposition of 4-valent intertwiner state.

3.6 Diff-invariance state space $\mathcal{H}_{Diff}$

The next step in the Dirac program is to solve the spatial diffeomorphism constraints, which are those states that are invariant under the action of Diff(Σ), The space of all solutions define a new Hilbert space called a *Diff-invariance space*, denoted by $\mathcal{H}_{Diff}$. To that end, recalling from (3.20) how the holonomy transforms under $\phi \in$ Diff(Σ) diffeomorphism:

$$h_e[A] \longrightarrow h'_e[A] \equiv h_e[\phi^* A] = h_{\phi\circ e}[A] \tag{3.114}$$

Since the Haar measure (also the Ashtekar-Lewandowski measure) is diffeomorphism invariant then its action is well-defined and unitary. Since diffeomorphism group $\text{Diff}(\Sigma)$ is a non-compact group, then the diff-invariant states are not a subspace in $\mathcal{H}_0$. Think for instance the simple case of ordinary (free) quantum mechanics, the wave function $\psi \in L_2[\mathbb{R}, \mathrm{d}x]$ required to be invariant under translations which is a non-compact group. Therefore, the result solution to Schrödinger equation is a plane wave $\psi_k(x) = Ae^{ikx}$ that is not a square integrable function for any arbitrary constants k and A, then $\psi_k \notin L_2[\mathbb{R}, \mathrm{d}x]$. It provide however a linear functional $\psi_k \longmapsto \mathcal{F}_k$ as following

$$\begin{aligned} \mathcal{F}_k : L_2[\mathbb{R}, \mathrm{d}x] &\longrightarrow \mathbb{C} \\ \psi &\longmapsto \mathcal{F}_{k=0}(\psi) := A \int_{\mathbb{R}} dx\; e^{ikx}\psi(x)\Big|_{k=0} \end{aligned} \tag{3.115}$$

is the Fourier transform of ψ evaluated in $k = 0$. Similarly, we do the same thing to solve the problem for the diffeomorphism constraint. The solution states can be described in terms of linear functionals on $\mathcal{H}_0$, and the scalar product must be extended to the space of the solutions, denote $\mathcal{H}_0^*$, the space of all linear functionals on $\mathcal{H}_0$ (the dual space of $\mathcal{H}_0$) and then $\mathcal{H}_{Diff}$ is the space of theses diff-invariant elements of $\mathcal{H}_0^*$. It can be defined by the projection map $\mathcal{P}_{Diff}$ as follows:

$$\begin{aligned} \mathcal{P}_{Diff} : \mathcal{H}_0 &\longrightarrow \mathcal{H}_0^* \\ \Psi &\longmapsto \mathcal{P}_{Diff}\Psi \end{aligned} \tag{3.116}$$

where for any $\Psi' \in \mathcal{H}_0$, one has:

$$(\mathcal{P}_{Diff}\Psi)(\Psi') := \sum_{\phi \in \text{Diff}(\Sigma)} \langle U_\phi \Psi | \Psi' \rangle \tag{3.117}$$

The sum is over all diffeomorphism $\phi \in \text{Diff}(\Sigma)$ and it is always well-defined (finite). This is the same technique (group averaging) as we done in Gauss constraints. To be more explicit, we consider the case of states with fixed graphs. One can expand the wave functionals into a finite linear combination of a spin networ basis states:

$$|\Psi\rangle := f^{(j_l)} \cdot |\Gamma; j_l, i_n\rangle \in \mathcal{H}_\Gamma \Rightarrow |\hat{U}_\phi \Psi\rangle := f^{(j_l)} \cdot |\phi \circ \Gamma; j_l^\phi, i_n^\phi\rangle \in \mathcal{H}_{\phi\circ\Gamma} \tag{3.118a}$$

$$|\Psi'\rangle := f'^{(j'_{l'})} \cdot |\Gamma'; j'_{l'}, i'_{n'}\rangle \in \mathcal{H}_{\Gamma'} \tag{3.118b}$$

Then (3.117) will become

$$(\mathcal{P}_{Diff}\Psi)(\Psi') = f^{(j_l)} f'^{(j'_{l'})} \sum_{\phi \in \text{Diff}(\Sigma)} \langle \phi \circ \Gamma; j_l^{\phi}, i_n^{\phi} | \Gamma'; j'_{l'}, i'_{n'} \rangle \tag{3.119}$$

We divide the diffeomorphism into two different cases:

- The case of $\phi \circ \Gamma \neq \Gamma'$: its has no contribution to the sum (3.119) since the $\mathcal{H}_{\phi\circ\Gamma}$ and $\mathcal{H}_{\Gamma'}$ are orthogonal and all inner product of that kind will vanish.

- The case of $\phi \circ \Gamma = \Gamma'$: the diffeomorphism does not change the graph. In general, tere are differences of orientation or ordering of links in the graph. The (finite) discrete group of such symmetries, labelled by $G_\Gamma = \{g_k, k = 1, \ldots, K\}$, its cardinality K depend on the number of links and nodes and their connection with each other. then the sum in (3.119) gives at most finite and discrete multplicity of the group G_Γ, that is:

  $$(\mathcal{P}_{Diff}\Psi)(\Psi') = f^{(j_l)} f'^{(j_l)} \sum_k g_k < \infty \tag{3.120}$$

  it is worth to mention that each class of orientation and ordering can be obtained via infinite number of different diffeomorphisms, the group of such bad symmetries, labelled by TDiff_Γ, one has to take a representative for each equivalence class. That means, to avoid infinities in the sum (3.119), we will restrict our diff-invariance symmetry $\text{Diff}(\Sigma)$ to $\text{Diff}(\Sigma)/\text{TDiff}_\Gamma$

Finally, the Diff-invariance Hilbert space $\mathcal{H}_{Diff}$ is defined to be the image of the projector (3.116) with taking a sum over the restricted symmetry $\text{Diff}(\Sigma)/\text{TDiff}_\Gamma$ rather than $\text{Diff}(\Sigma)$ itself:

$$\mathcal{H}_{Diff} := \mathcal{P}_{Diff}(\mathcal{H}_0) \tag{3.121}$$

The result of this procedure are spin network states defined on *equivalence classes of graphs under diffeomorphism symmetries*, labelled by K,

$$K := \{\phi \circ \Gamma \mid \phi \in \text{Diff}(\Sigma)/\text{TDiff}_\Gamma\} \tag{3.122}$$

These equivalence classes K are called *knots*, (See Fig. 3.11). The study of knots is an interesting branch of mathematics. The Diff-invariant Hilbert space of loop quantum gravity is spanned by *knotted spin networks* or *s-knot states*. Since the space of knots is extremely discrete, then one can decompose our Diff-invariance Hilbert space into a discrete sum over knots as following:

$$\mathcal{H}_{Diff} = \bigoplus_{K} \mathcal{H}_K \tag{3.123}$$

where,

$$\mathcal{H}_K := \mathcal{P}_{Diff}(\mathcal{H}_\Gamma) \tag{3.124}$$

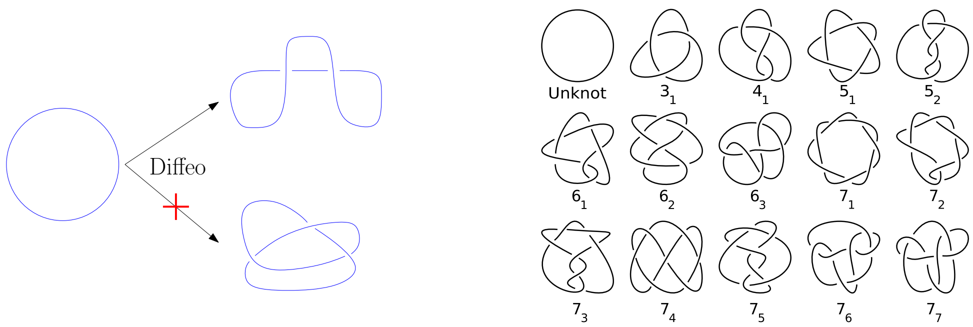


Figure 3.11: *in the left.* A diffeomorphism preserves the knot-class of loops. *in the right.* Classification of th first knots basis (without nodes), taken from Wikipedia.

Now, we want to show how the diffeomorphism invariance immediatly leads to solve the diffeomorphic constraint, $\hat{H}_a\Psi = 0$. One can show: for any $\Psi \in \mathcal{H}_0$

$$H[\boldsymbol{\lambda}](\mathcal{P}_{Diff}\Psi) = \hat{U}_{\phi_{\boldsymbol{\lambda}}}(\mathcal{P}_{Diff}\Psi) - \mathcal{P}_{Diff}\Psi = 0 \tag{3.125}$$

where $\hat{U}_{\phi_{\boldsymbol{\lambda}}}$ is the diffeomorphism generated by the vector field $\boldsymbol{\lambda}$. We conclude that the knotted spin network states solve the diffeomorphic constraints as well as they satisfy the Gauss law.

3.7 Dynamics state space $\mathcal{H}_{phys}$

Finally, the last step of Dirac program is to solve the Hamiltonian constraint, which are those states in $\mathcal{H}_{Diff} \subset \mathcal{H}_0^*$ that are invariant under the gauge transformation generated by the Hamiltonian constraints. Recalling that in real Ashtekar variables, the smearing classical Hamiltonian constraint over a function N is given by:

$$\begin{aligned} H(N) &= \int \mathrm{d}^3 x N \epsilon^{ij}{}_k \frac{E^a_i E^b_j}{\sqrt{\det(E)}} \left(F^k_{ab} - 2\frac{1+\gamma^2}{\gamma^2} K^i_{[a} K^j_{b]} \right) \\ &= H^E(N) - 2\left(1+\gamma^2\right) T(N), \end{aligned} \tag{3.126}$$

where we introduced the shorthand notation $H^E(N)$ and $T(N)$. As with the ADM Hamiltonian constraint, this expression is non-linear, we still have the problem of the non-polynomial term $\frac{1}{\det(E)}$. However, a trick due to Thiemann [61, 62] allows us to rewrite the Hamiltonian constraints in terms of well-defined Poisson brackets. Defining the volume V of the 3d-space Σ by:

$$V = \int_\Sigma \mathrm{d}^3 \boldsymbol{x} \sqrt{\det(E)} \tag{3.127}$$

One introduces the Gauss gauge invariant quantity:

$$\bar{K} = \int_\Sigma \mathrm{d}^3 \boldsymbol{x}\ K^i_a E^a_i, \tag{3.128}$$

we can use the classical brackets (3.3) to establish the following identities,

$$\epsilon^{ijk} \frac{E^a_i E^b_j}{\sqrt{\det(E)}} = \frac{2}{\gamma} \epsilon^{abc} \left\{ A^k_c, V \right\}, \tag{3.129a}$$

$$K^i_a = \frac{1}{\gamma} \left\{ A^i_a, \bar{K} \right\}, \tag{3.129b}$$

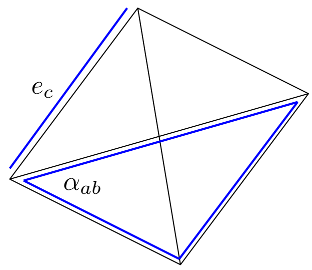
$$\bar{K} = \frac{1}{\gamma^{1/2}} \left\{ \mathcal{H}^E(1), V \right\}. \tag{3.129c}$$

Using these relations, one can rewrite the two terms in (3.126) in terms Poison brackets as:

$$H^E(N) = \frac{2}{\gamma} \int \mathrm{d}^3 x\ N \epsilon^{abc} \delta_{ij} F^i_{ab} \left\{ A^j_c, V \right\}, \tag{3.130a}$$

$$T(N) = \frac{2}{\gamma^4} \int \mathrm{d}^3 x\ N \epsilon^{abc} \epsilon_{ijk} \left\{ A^i_a, \left\{ H^E(1), V \right\} \right\} \left\{ A^j_b, \left\{ H^E(1), V \right\} \right\} \left\{ A^k_c, V \right\}. \tag{3.130b}$$

Since our goal is to quantize the Hamiltonian constraints, one has to rewrite the connection and curvature in the expressions (3.130) in terms of holonomies and fluxes, so that we can turn them into operators. This requires a regularization procedure as follows: we introduce a lattice regularization procedure by assuming that the 3d-space Σ has been divided into infinitesimal tetrahedra Δ_I and regularize the integrals as Riemann sum over the cells Δ_I. For each tetrahedron Δ pick a vertex and call $v(\Delta)$. Let $e_A(\Delta)$, $A = 1, 2, 3$ be three edges started at $v(\Delta)$ and let $e_{AB}(\Delta) = e_B(\Delta) - e_A(\Delta)$. We now construct a loop $\alpha_{ij}(\Delta) = e_A(\Delta)\sharp e_{AB}(\Delta)\sharp e_B^{-1}(\Delta)$. See the following figure.



Along the edge e_C, the holonomies can be easily expressed in terms of connections as following:

$$h_{e_C} \approx \mathbb{I} + A_c e_C^c \tag{3.131}$$

Along the loop α_{AB}, the holonomies can be easily expressed in terms of curvature as following:

$$h_{\alpha_{AB}} \approx \mathbb{I} + \frac{1}{2} F_{ab} e_A^a e_B^b \tag{3.132}$$

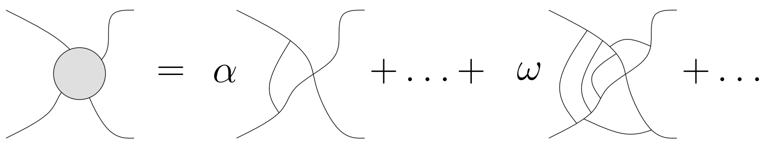
The integrals can then be regularized by a Riemann sum as:

$$\begin{aligned} H^E =& \frac{2}{\gamma} \lim_{\Delta \to 0} \sum_I N_I \epsilon^{abc} \mathrm{Tr} \left(F_{ab} \{A_c, V\} \right) \\ =& \frac{2}{\gamma} \lim_{\Delta \to 0} \sum_I N_I \epsilon^{ABC} \mathrm{Tr} \left(\left(h_{\alpha_{AB}} - h_{\alpha_{AB}}^{-1} \right) h_{e_C}^{-1} \{h_{e_C}, V\} \right). \end{aligned} \tag{3.133}$$

Doing same thing for the second term (3.130b), but the regularization is so complicated, we refer to the paper ref. [62]. Now, the quantization process is very simple: by promoting holonomies and fluxes to opeartors and the Poisson brackets to commutators, we get:

$$\hat{H}^E[N] = \frac{2}{\gamma} \lim_{\Delta \to 0} \sum_I N_I \epsilon^{ABC} \mathrm{Tr} \left(\left(\hat{h}_{\alpha_{AB}} - \hat{h}_{\alpha_{AB}}^{-1} \right) \hat{h}_{e_C}^{-1} \left[\hat{h}_{e_C}, \hat{V} \right] \right). \tag{3.134}$$

This is a well-defined quantum operator whose action is complicated. Notice that in the next, we will discuss the volume operator and its spectrum. We will make it clear that the volume operator acts only on the nodes of the spin network states and gives non-trivial spectrum only for a node of valency 4 and higher. Then from the holonomy operators in $\hat{H}^E[N]$, they modify the spin network by creating new links of $\frac{1}{2}$-spin around the node in such a way the new nodes (intersecton of the new links with the graph) are three-valents in order to have volume invariant state. See the literature [4, 5] for more details. Generally, a formal solution of the Hamiltonian constraint is a linear combination of spin networks with an arbitrary number of links intersect with the links of the graph, and it has coefficients depend on the details of the Hamiltonian constrant opertaor $\hat{H}$ and on the spins carried by the state j_l,



This construction provides a new spin network state as the solution of the Hamiltonian constraint.

Chapter 4

Geometrical applications of LQG

This chapter is based on refs. [18, 19, 20, 21, 22, 23, 24, 25, 26, 27] and [4, 47, 63, 64, 65].

Our next step is to develop a well-defined operators on the gauge invariant Hilbert space $\mathcal{H}_0$ for promoting some geometrical quantity to the quantum level. It is natural to ask what the quantum version of the geometric variables such as length, area, volume, angle, etc. Such operators exist and are called *geometrical operators*. In the next, we will provide the area and volume operator in LQG and their spectra through a spin network state.

4.1 The area operator

The simplest geometric operator that can be constructed in loop quantum gravity is the area operator, which is as manifested by its name, the operator that measures quanta of area. Consider a surface $S \subset \Sigma$ that can be represented by the coordinates pair (σ_1, σ_2):

$$(\sigma_1, \sigma_2) \longmapsto (x^a(\sigma_1, \sigma_2)) \tag{4.1}$$

The area of the surface S can be given in terms of its normal n_a and the densitized triad E^a_i,

$$A(S) := \int_S \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \sqrt{E^a_i E^{bi} n_a n_b} \tag{4.2}$$

where

$$n_a = \epsilon_{abc} \frac{\partial x^b}{\partial \sigma_1} \frac{\partial x^c}{\partial \sigma_2} \qquad a, b = 1, 2, 3 \tag{4.3}$$

The area at classical level. We start from the 2-dimensional indeuced metric ${}^{(2)}q_{\alpha\beta}$ on the surface S in terms of the 3-dimensional metric q_{ab} on Σ

$${}^{(2)}q_{\alpha\beta} := q_{ab} \frac{\partial x^b}{\partial \sigma_\alpha} \frac{\partial x^c}{\partial \sigma_\beta} \qquad \alpha, \beta = 1, 2$$

The standard definition of area in terms of the metric,

$$\begin{aligned} A(S) &:= \int_S \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \sqrt{\det({}^{(2)}q_{\alpha\beta})} \\ &= \int_S \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \sqrt{\det\left(q_{ab} \frac{\partial x^a}{\partial \sigma^\alpha} \frac{\partial x^b}{\partial \sigma^\beta}\right)} \qquad \alpha, \beta = 1, 2 \end{aligned}$$

we have

$$\begin{aligned} \det\left(q_{ab} \frac{\partial x^a}{\partial \sigma^\alpha} \frac{\partial x^b}{\partial \sigma^\beta}\right) &= q_{ab} q_{cd} \left[\frac{\partial x^a}{\partial \sigma^1} \frac{\partial x^b}{\partial \sigma^1} \frac{\partial x^c}{\partial \sigma^2} \frac{\partial x^d}{\partial \sigma^2} - \frac{\partial x^a}{\partial \sigma^1} \frac{\partial x^b}{\partial \sigma^2} \frac{\partial x^c}{\partial \sigma^2} \frac{\partial x^d}{\partial \sigma^1}\right] \\ &= 2 q_{ab} q_{cd} \frac{\partial x^a}{\partial \sigma^1} \frac{\partial x^{[b}}{\partial \sigma^1} \frac{\partial x^{d]}}{\partial \sigma^2} \frac{\partial x^c}{\partial \sigma^2} \\ &= 2 q_{a[b} q_{d]c} \frac{\partial x^a}{\partial \sigma^1} \frac{\partial x^b}{\partial \sigma^1} \frac{\partial x^d}{\partial \sigma^2} \frac{\partial x^c}{\partial \sigma^2} \\ &= q q^{ef} \epsilon_{eac} \epsilon_{fbd} \frac{\partial x^a}{\partial \sigma^1} \frac{\partial x^b}{\partial \sigma^1} \frac{\partial x^d}{\partial \sigma^2} \frac{\partial x^c}{\partial \sigma^2} \\ &= q q^{ef} n_e n_f \\ &= e^2 e_i^e e^{fi} n_e n_f \\ &= E_i^e E^{fi} n_e n_f \end{aligned}$$

where we have used the definition of the normal (4.3), the definition of the densitized triad (3.7) and the relation:

$$g_{a[b} g_{d]c} = \frac{1}{2} q q^{ef} \epsilon_{ace} \epsilon_{bdf}$$

To construct an operator version of (4.2), one has to quantize it canonically using the quantization momenta rule (3.2b):

$$\hat{A}(S) = 8\pi G \hbar \gamma \int_S \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \sqrt{-n_a n_b \frac{\delta^2}{\delta A_i^a \delta A^{bi}}} \tag{4.4}$$

This expression poses a problem because there are two functional derivatives, upon acting on a state will form $6d$-Dirac delta function. four of them can be treated with the surface $2d$-integration as well as the $2d$-integral in the cylindrical functional for each functional derivative in (4.4), and the other will remain in the form of $\delta^4(0)$. This is an ill-defined quantity (infinities) which hinders the process of quantizing. Moreover the presence of the square root further complicates the situation. One way to solve this problem is by what is known as *regularization*; it can be easily dealt with this problem if we regularize the expression for the area in the following way: we consider a partition of the surface S in N 2-dimensional small cells $\{S_I, I = 1, \dots, N\}$,

$$S = \bigcup_{I=1}^{N} S_I \tag{4.5}$$

and write the integral as the limit of a Riemann sum,

$$A(S) = \lim_{N\to\infty} A_N(S), \tag{4.6}$$

where the Riemann sum can be expressed as

$$A_N(S) = \sum_{I=1}^{N} \sqrt{E_i(S_I)E^i(S_I)}. \tag{4.7}$$

Here N is the number of cells, and $E_i(S_I)$ is the flux of E_i through the I-th cell.

Checking the limit of a Riemann sum. In the limit of infinitesimal cells we have that

$$E_i(S_I) := \int_{S_I} \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 n_a E_i^a \approx E_i^a\big|_I n_a\big|_I S_I$$

In that limit the definition of the area

$$\begin{aligned} A(S) &= \int_S \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \sqrt{E_i^a E^{bi} n_a n_b} \\ &= \lim_{N\to\infty} \sum_{I=1}^{N} S_I \sqrt{E_i^a\big|_I E^{bi}\big|_I n_a\big|_I n_b\big|_I} \\ &= \lim_{N\to\infty} \sum_{I=1}^{N} \sqrt{(E_i^a\big|_I n_a\big|_I S_I)(E^{bi}\big|_I n_b\big|_I S_I)} \\ &= \lim_{N\to\infty} \sum_{I=1}^{N} \sqrt{E_i(S_I)E^i(S_I)} \end{aligned}$$

Accordingly, we define the area operator as

$$\hat{A}(S) = \lim_{N\to\infty} \hat{A}_N(S), \tag{4.8}$$

where in $A_N(S)$ we simply replace the classical flux $E_i(S_I)$ by the operator $\hat{E}_i(S_I)$. This operator now acts on a generic spin network state Ψ_Γ, where the graph Γ is generic and can intersect S many times. We already know that $\hat{E}_i(S_I)\hat{E}^i(S_I)$ gives zero if S_I is not intersected by any link of the graph. Therefore once the partition is sufficiently fine so that each surface S_I is punctured once and only once, taking a further refinement has no consequences. Therefore, the limit amounts to simply sum the contributions of the finite number of punctures p of S caused by the links of Γ. That is,

$$\hat{A}(S)\,\Psi_\Gamma = \lim_{N\to\infty}\sum_{I=1}^{N}\sqrt{\hat{E}_i(S_I)\hat{E}^i(S_I)}\,\Psi_\Gamma = \sum_{p\in S\cap\Gamma} 8\pi G\hbar|\gamma|\sqrt{j_p(j_p+1)}\,\Psi_\Gamma. \qquad (4.9)$$

There are three key remarks to make to this formula:

- The spectrum of the area operator is completely known and the area can only take up discrete values, with minimal excitation being proportional to the squared Planck length $L_P^2 = \frac{G\hbar}{c^3} \approx 10^{-66}cm^2$. Then the natural interpretation one obtain is that spacetime itself is discrete, much like matter field in the quantum scales.

- The spin network states are eigenstates of the area operator.

- In the classical theory, the value of the Immirzi parameter in the Holst action did not affect the physical solutions. Instead in the quantum theory, (4.9) suggests that the eigenvalue of the area depends on the value of the Immirzi parameter.

4.2 The volume operator

The next geometric operator we will provide in loop quantum gravity is the volume operator, it measures the quanta of volume. Given a region $R \subset \Sigma$, we can classically define its volume as:

$$V(R) := \int_R \mathrm{d}^3\boldsymbol{x}\sqrt{\left|\frac{1}{3!}\epsilon_{abc}\epsilon^{ijk}E_i^aE_j^bE_k^c\right|} \qquad (4.10)$$

where the quantity in absolute value can be recognized as the determinant of the densitised triad, $\det(E_i^a)$. To construct an operator version of (4.10), one has to quantize it canonically using the quantization momenta rule (3.2b):

$$\hat{V}(R) := (8\pi G\gamma\hbar)^{3/2} \int_R \mathrm{d}^3\boldsymbol{x} \sqrt{\left|\frac{1}{3!}\epsilon_{abc}\epsilon^{ijk}\frac{\delta^3}{\delta A_a^i \delta A_b^j \delta A_c^k}\right|} \tag{4.11}$$

This expression poses a problem because there are three functional derivatives, upon acting on a state will form $9d$-Dirac delta function. Six of them can be treated with the volume $3d$-integration as well the $1d$-integral in the cylindrical functional for each functional derivative in (4.11), and the other will remain in the form of $\delta^3(0)$. This is an ill-defined quantity (infinities) which hinders the process of quantizing. One way to solve this problem is by what is known as *regularization*; it is based on a given *fluxization* of the links with small surfaces around the region R. Two distinct mathematically fluxizations of the volume measure have been proposed in the literatur; one is due to *Rovelli and Smolin* V_{RS}, and the other to *Ashtekar and Lewandowski* V_{AL}. Both of them act non-trivially only at the nodes of a spin network state. Let us begin reviewing the construction by V_{RS}.

4.2.1 Rovelli and Smolin volume operator

The fluxization due to Rovelli-Smolin [20] is reached as follows:

- Choosing the partition in which the nodes of Γ can fall only in the interior of cells and each cubic cell C_I contains at most one node.

- In the case that the cell contains no node, then we assume that it contains at most one link.

- we consider a partition of the surfaces ∂C_I in cells S_I^α in which the links of Γ can intersect a surface S_I^α only in its interior and each cell S_I^α is punctured at most by one link.

As we did for the area, the volume integral (4.10) can be approximated to a limit of a Riemann sum,

$$V_{RS}(R) = \lim_{\varepsilon \to 0} \sum_I \mathrm{vol}(C_I) \tag{4.12}$$

where $R = \bigcup_I C_I$. Now, the only remaining step to be taken is to provide a fluxization of the small ε-cubic cells C_I. To do so, consider the following three-surfaces integral,

$$W_I = \frac{1}{48} \int_{\partial C_I} \mathrm{d}^2\boldsymbol{\sigma}_1 \int_{\partial C_I} \mathrm{d}^2\boldsymbol{\sigma}_2 \int_{\partial C_I} \mathrm{d}^2\boldsymbol{\sigma}_3 \left| \epsilon^{ijk} E_i^a(\boldsymbol{\sigma}_1) n_a(\boldsymbol{\sigma}_1) E_j^b(\boldsymbol{\sigma}_2) n_b(\boldsymbol{\sigma}_2) E_k^c(\boldsymbol{\sigma}_3) n_c(\boldsymbol{\sigma}_3) \right| \tag{4.13}$$

For each boundary surface ∂C_I of the small cubic cell C_I, the last quantitiy can be approximated to a sum of small surfaces $\partial C_I = \bigcup_\alpha S_I^\alpha$, one has then:

$$\begin{aligned} W_I &\approx \frac{1}{48} \sum_{\alpha\beta\gamma} S_I^\alpha S_I^\beta S_I^\gamma \left| \epsilon^{ijk} E_i^a|_I n_a|_I E_j^b|_I n_b|_I E_k^c|_I n_c|_I \right| \\ &= \frac{1}{48} \sum_{\alpha\beta\gamma} \left| \epsilon^{ijk} (E_i^a|_I n_a|_I S_I^\alpha)(E_j^b|_I n_b|_I S_I^\beta)(E_k^c|_I n_c|_I S_I^\gamma) \right| \\ &= \frac{1}{48} \sum_{\alpha\beta\gamma} \left| \epsilon^{ijk} E_i(S_I^\alpha) E_j(S_I^\beta) E_k(S_I^\gamma) \right| \end{aligned} \tag{4.14}$$

where we have labeled the approximate value of any field φ at the small cell C_I by the notation $\varphi|_I$. If we send the size of the cell ε to 0, we will get:

$$V_{RS}(R) = \lim_{\varepsilon \to 0} \sum_I \sqrt{W_I} \tag{4.15}$$

Therefore, this fluxization allows us to rewrite (4.10) in terms of fluxes as follows:

$$V_{RS}(R) = \lim_{\varepsilon \mapsto 0} \sum_I \sqrt{\frac{1}{48} \sum_{\alpha,\beta,\gamma} \left| \epsilon^{ijk} E_i(S_I^\alpha) E_j(S_I^\beta) E_k(S_I^\gamma) \right|} \tag{4.16}$$

Finally, quantizing (4.16) gives the Rovelli-Smolin volume operator:

$$\hat{V}_{RS}(R) = \lim_{\varepsilon \mapsto 0} \sum_I \sqrt{\frac{1}{48} \sum_{\alpha,\beta,\gamma} \left| \epsilon^{ijk} \hat{E}_i(S_I^\alpha) \hat{E}_j(S_I^\beta) \hat{E}_k(S_I^\gamma) \right|} \tag{4.17}$$

We can directly use (3.46) to write the contributions of the finite number of nodes inside a graph Γ and the one point puncture of S_n^α caused by the links of Γ to the volume operator. That is,

$$\hat{V}_{RS}(R)\Psi_\Gamma = \sum_{n \in R \cap \Gamma} \sqrt{\frac{(8\pi G\hbar\gamma)^3}{48} \sum_{\alpha,\beta,\gamma} \left| \epsilon^{ijk} J_i^{(j_\alpha)} J_j^{(j_\beta)} J_k^{(j_\gamma)} \right|} \Psi_\Gamma \tag{4.18}$$

This is a well-defined volume operator, whose action spectrum is again discrete, with minimal excitation proportional to $(8\pi G\hbar\gamma)^{3/2}$. Moreover, the spin network basis are not eigenstates of the volume operator (4.17).

Checking the limit of the Riemann sum (4.15). In the limit of infinitesimal cubic cells:

$$W_I \approx \frac{1}{48}\int_{\partial C_I} \mathrm{d}^2\boldsymbol{\sigma}_1 \int_{\partial C_I} \mathrm{d}^2\boldsymbol{\sigma}_2 \int_{\partial C_I} \mathrm{d}^2\boldsymbol{\sigma}_3 \big|\underbrace{\epsilon^{ijk}E^a_i|_I E^b_j|_I E^c_k|_I}_{\det(E^a_i)|_I\epsilon^{abc}} n_a(\boldsymbol{\sigma}_1)n_b(\boldsymbol{\sigma}_2)n_c(\boldsymbol{\sigma}_3)\big|$$
$$= \det(E^a_i)|_I \frac{1}{48}\int_{\partial C_I} \mathrm{d}^2\boldsymbol{\sigma}_1 \int_{\partial C_I} \mathrm{d}^2\boldsymbol{\sigma}_2 \int_{\partial C_I} \mathrm{d}^2\boldsymbol{\sigma}_3 \big|\epsilon^{abc} n_a(\boldsymbol{\sigma}_1)n_b(\boldsymbol{\sigma}_2)n_c(\boldsymbol{\sigma}_3)\big|$$

If we divide the boundary surface cell ∂C_I into its 6 square surfaces $\mathcal{S}^\alpha, \alpha = \mp 1, \mp 2, \mp 3$, where the unit normal vector $\boldsymbol{n}$ is then its is obvious to see the orthonormality relations $\boldsymbol{n}^\alpha \cdot \boldsymbol{n}^\beta = \delta^{\alpha\beta} - \delta^{\alpha,-\beta}$. then we get:

$$\begin{aligned} W_I &\approx \frac{\det(E^a_i)|_I}{48}\varepsilon^6 \underbrace{\sum_{\alpha,\beta,\gamma} |\boldsymbol{n}^\alpha|_I \cdot (\boldsymbol{n}^\beta|_I \times \boldsymbol{n}^\gamma|_I)|}_{48} \\ &= \det(E^a_i)|_I \varepsilon^6 \\ &= (\mathrm{vol}(C_I))^2 \end{aligned} \tag{4.19}$$

The last sum is equal to 48, there are 48 terms in the sum, each term equal to 1 due to the presence of the absolute value. The 48 terms come from the nature of $\alpha, \beta, \gamma = \pm 1, \pm 2, \pm 3$ and the presence of ϵ^{abc}; they must be different in each term, this leads to $3! = 6$ of different permutations. And for each permutation there are 8 different terms due to the presence of double parallel square surfaces ($\pm$) for the cubic cell.

4.2.2 Ashtekar and Lewandowski volume operator

The fluxization due to Ashtekar-Lewandowski [19] is reached as follows:

- Choosing the partition in which the nodes of Γ can fall only in the interior of cells and each cubic cell C_I contains at most one node.
- In the case that the cell contains no node, then we assume that it contains at most one link.
- Consider three surfaces $S^a_I, a = 1, 2, 3$ to be any surfaces orthogonal to each other inside the cube, in which every single node of the graph coincides with the intersection point of the three surfaces inside the cube.

This fluxization allows us to rewrite (4.10) in terms of fluxes as follows:

$$V_{AL}(R) = \lim_{\varepsilon \mapsto 0} \sum_I \sqrt{\left| \frac{1}{3!} \epsilon^{ijk} \epsilon_{abc} E_i(S_I^a) E_j(S_I^b) E_k(S_I^c) \right|} \tag{4.20}$$

Finally, quantizing (4.20) gives the Ashtekar-Lewandowski volume operator:

$$\hat{V}_{AL}(R) = \lim_{\varepsilon \mapsto 0} \sum_I \sqrt{\left| \frac{1}{3!} \epsilon^{ijk} \epsilon_{abc} \hat{E}_i(S_I^a) \hat{E}_j(S_I^b) \hat{E}_k(S_I^c) \right|} \tag{4.21}$$

We can directly use (3.46) (the case where the surface intersects with the end points of the link) to write the contributions of the finite number of nodes inside a graph Γ and the punctures p_a of S_n^a caused by the links of Γ to the volume operator. That is,

$$\hat{V}_{AL}(R)\Psi_\Gamma = \sum_{n \in R \cap \Gamma} \sqrt{\left| \frac{(8\pi G\hbar\gamma)^3}{48} \epsilon^{ijk} \sum_{e_n, e'_n, e''_n} \kappa(e_n, e'_n, e''_n) J_i^{(j_{e_n})} J_j^{(j_{e'_n})} J_k^{(j_{e''_n})} \right|} \Psi_\Gamma \tag{4.22}$$

The sum is over all possible triplet links e_n, e'_n, e''_n passing through the node n and $\kappa(e_n, e'_n, e''_n)$ is the orientation function depends on the sign functions (3.47):

$$\kappa(e_n, e'_n, e''_n) = \epsilon_{abc} \kappa_{S_n^a}(e_n) \kappa_{S_n^b}(e'_n) \kappa_{S_n^c}(e''_n) \tag{4.23}$$

This is a well-defined volume operator, whose action spectrum is again discrete, with minimal excitation proportional to $(8\pi G\hbar\gamma)^{3/2}$. Moreover, the spin network basis are not eigenstates of the volume operator (4.21).

4.2.3 Discussion of the volume operators

Let us now study the action of the volume operators (4.17) and (4.21):

- The presence of the epsilon tensor ϵ^{ijk} in both volume operators (4.17) and (4.21) requires all three fluxes to be different, this means that the volume does not act on links. We thus obtain the important result that *the volume operator acts only on nodes of the graph* (it acts on the intertwiners).
- The action of (4.17) and (4.21) on a single 3-valent node is zero due to the presence of ϵ^{ijk} as well as the closure relation (3.66).
- Non-trivial contributions to the volume comes from nodes of valency 4 or higher.

- Both volume operators (4.17) and (4.21) act only on nodes of the graph; their spectra are discrete with minimal excitations proportional to the Planck length cube L_P^3.

In loop quantum gravity, these results together with the discreteness of the area operator show that the *quantum space-geometry is discrete at the Plack scale.* Each spin network for a given graph Γ describes a *quantum geometry*, where each face dual to a link e_l has an area proportional to the spin j_l, and each region around a node n has a volume determined by the intertwiner i_n as well as the spins of the link sharing the node, one has the following two key results:

$$\text{link} \longleftrightarrow \text{irr-representation} \longleftrightarrow \text{face}$$

$$\text{node} \;\longleftrightarrow\; \text{intertwiner} \;\longleftrightarrow\; \text{region}$$

An important question is whether these spin network state for a given graph Γ can be understood as some approximate description of smooth $3d$-geometries [63, 64]. In what follows, we would like to establish a connection between the picture of the quantum degrees of freedom captured by an intertwiner space $\mathcal{H}_F$ of F-valency and the polyhedral description.

4.3 Polyhedra Interpretation of Intertwiner State

This section is based on the papers [26, 27, 65].

Now we would like to have a classical picture of the quantum degrees of freedom captured by the gauge $SU(2)$-invariant kinematical space $\mathcal{H}^0_\Gamma$. Since the intertwiner spaces (3.69) are the building blocks of $\mathcal{H}^0_\Gamma$, then we will keep our attention to an intertwiner space of a single node. From the above discussion and the closure relation (3.66) of the irr-representaion vectors at each node, a best visualizing picture of an intertwiner of valency $(V_n = F)$ is an F-faces convex Euclidean polyhedron. In the next, we will study this correspondence in more details.

4.3.1 Convex Euclidean polyhedron

A *convex Euclidean polyhedron* is a convex region consists of set of points Pol in 3d Euclidean space $\mathbb{R}^3$, which is:

$$Pol := \{\vec{x} \in \mathbb{R}^3 |\ \vec{n}_f \cdot \vec{x} \leq h_f, f = 1, \dots, F\} \tag{4.24}$$

it can be seen as the intersection of $F-$ suitably half-spaces, where F is the face number of the polyhedron and $\{h_f\}$ are the distances between the origin point and the planes $\{S_f\}$ with a unit normal $\{\vec{n}_f\}$. These surfaces is defined by:

$$S_f := \{\vec{x} \in \mathbb{R}^3 |\ \vec{n}_f \cdot \vec{x} = h_f\} \tag{4.25}$$

There is an important relation must be satisfied in terms of the areas $\{A_f\}$ and the unit normals $\{\vec{n}_f\}$, called *the closure condition*:

$$\sum_{f=1}^{F} A_f \vec{n}_f = \vec{0} \tag{4.26}$$

A convex Euclidean polyhedron with areas and normals that satisies the closure condition always exists and it is unique, up to rotations and translations. This result is proved by Hermann Minkowski [66].

4.3.2 The space of polyhedra shapes (Kapovich-Millson phase space)

The space of all convex Euclidean polyhedral shapes with faces of fixed areas $\{A_f\}, f = 1, \dots, F$ satisfying the closure relation (4.26) is the

$$\mathcal{S}_F := \{(\vec{n}_1, \dots, \vec{n}_F) \in (S^2)^F |\ \sum_{l=1}^{F} A_f \vec{n}_f = \vec{0}\}/_{SO(3)} \tag{4.27}$$
$$\dim(\mathcal{S}_F) = 2F - 3 - 3 = 2(F-3)$$

and is known as the *Kapovich-Millson phase space* [67]. One can canonically coordinatized the Kapovich-Millson phase space by $F-3$ invariant[1] pairs (μ_r, θ_r):

$$\mu_r := |\vec{\mu}_r| = \left| \sum_{f=1}^{r+1} A_f \vec{n}_f \right| \tag{4.28a}$$

$$\theta_r := \arctan\left[\frac{|(\vec{\mu}_{r-1} \times \vec{\mu}_r) \times (\vec{\mu}_r \times \vec{\mu}_{r+1})|}{(\vec{\mu}_{r-1} \times \vec{\mu}_r) \cdot (\vec{\mu}_r \times \vec{\mu}_{r+1})} \right] \tag{4.28b}$$

[1] A best coordinatization must be invariant under the action of $3d$-rotation $SO(3)$.

with a symplectic structure on this spas as:

$$\{\mu_r, \theta_{r'}\} = \delta_{rr'} \tag{4.29}$$

This naturally arises from the well-known Lie-Poisson bracket by considering the interpretation of each area vector $\vec{A}_l = A_l \vec{n}_l$ as a classical angular momentum (see for more details [68]),

$$\begin{aligned}\{f, g\}_{LP} &:= \sum_{f=1}^{F} \vec{A}_f \cdot \left(\frac{\partial f}{\partial \vec{A}_f} \times \frac{\partial g}{\partial \vec{A}_f} \right) \\ &= \sum_{f=1}^{F} A^i_f \epsilon_i{}^{jk} \frac{\partial f}{\partial A^j_f} \frac{\partial g}{\partial A^k_f}\end{aligned} \tag{4.30}$$

using this Lie-Poisson bracket, one can show the canonical coordinates of $\mathcal{S}_F$ satisfy (4.29). The fact that the space of polyhedra shapes with faces of fixed areas form a phase space will be important in the next where we discuss the relation between quantum polyhedra and intertwiner states.

Lie-Poisson bracket on angular momentum space. The space of angular momentum is constructed by $3d$-vector $\vec{J}$ of fixed norm $\vec{J}^2 = C^2$. It can be seen as a 2-sphere with a radius $C > 0$, (e.g. a dynamical system with spherically symmetric potential). We consider the symplectic structure to be determined by the Lie-Poisson bracket as:

$$\{J^j, J^j\}_{LP} = \epsilon^{ij}{}_k J^k \tag{4.31}$$

Then, this Lie-Poisson bracket induces a symplectic structure: a closed, nondegenerate, differential 2-form for the angular momentum space:

$$w_{LP} = \epsilon_{ijk} J^i dJ^j \wedge dJ^k \tag{4.32}$$

where if we refer $\{X_i\}$ to be the basis tangent vector in the angular momentum space, in which $\{dJ^i\}$ is its dual $(dJ^i(X_j) = \delta^i_j)$, then (4.32) is well defined,

$$w_{LP}(X_i, X_j) = \{J^j, J^j\}_{LP} \tag{4.33}$$

Since the Lie-Poisson bracket between two functions f and g is determined by the symplectic structure acting on their associated Hamiltonian vector field X_f and X_g respectively, one has the following

$$\{f, g\}_{LP} = w_{LP}(X_f, X_g) \tag{4.34}$$

In order to determine (4.34), we need to determine the Hamiltonian vector field X_f and X_g in terms of the structure constant in (4.31) and the angular momentum J^i. To do so, we use the relation between the variation df and $X_f = X_f^i X_i$:

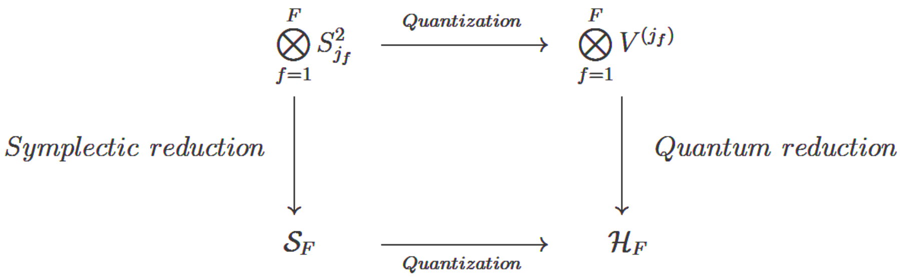
$$
\begin{aligned}
& w_{LP}(X_f, \cdot) = df \\
\Leftrightarrow & w_{LP}(X_f^i X_i, \cdot) = \frac{\partial f}{\partial J^i} dJ^i \\
\Leftrightarrow & X_f^i w_{LP}(X_i, X_j) = \frac{\partial f}{\partial J^i} dJ^i(X_j) \\
\Leftrightarrow & X_f^i \epsilon_{ijk} J^k = \frac{\partial f}{\partial J^i} \delta^i_j \\
\Leftrightarrow & X_f^i \underbrace{\epsilon^{jmn} \epsilon_{ijk}}_{2\delta_k^{[m} \delta_i^{n]}} J^k = \epsilon^{jmn} \frac{\partial f}{\partial J^j} \\
\Leftrightarrow & 2X_f^{[n} J^{m]} = \epsilon^{jmn} \frac{\partial f}{\partial J^j} \\
\Leftrightarrow & 2X_f^{[n} J^{m]} J_m = \epsilon^{jmn} \frac{\partial f}{\partial J^j} J_m \\
\Leftrightarrow & X_f^n \underbrace{J^m J_m}_{C^2} = \epsilon^{jmn} \frac{\partial f}{\partial J^j} J_m \\
\Leftrightarrow & X_f^i = \frac{1}{C^2} \epsilon^{ij}{}_k \frac{\partial f}{\partial J^j} J^k
\end{aligned}
$$

where in the eighth step, we use the fact that all Hamiltonian vector field is restricted to be tangent to the 2-sphere of radius $|\vec{J}|$, then $X_f^i J_i = 0$. Now, substituting this result in formula (4.34)

$$
\begin{aligned}
\{f, g\}_{LP} &= w_{LP}(X_f, X_g) \\
&= w_{LP}(X_f^i X_i, X_g^j X_j) \\
&= X_f^i X_g^j w_{LP}(X_i, X_j) \\
&= X_f^i X_g^j \epsilon_{ijk} J^k \\
&= \left(\frac{1}{C^2} \epsilon^{iq}{}_l \frac{\partial f}{\partial J^q} J^l \right) \left(\frac{1}{C^2} \epsilon^{jm}{}_n \frac{\partial g}{\partial J^m} J^n \right) \epsilon_{ijk} J^k \\
&= \frac{1}{C^4} \epsilon^{iq}{}_l \epsilon^{jm}{}_n \epsilon_{ijk} J^l J^n J^k \left(\frac{\partial f}{\partial J^q} \right) \left(\frac{\partial g}{\partial J^m} \right) \\
&= \frac{1}{C^4} \epsilon^{iq}{}_l \underbrace{\epsilon^{jm}{}_n \epsilon_{ijk}}_{(\delta_k^m \delta_{ni} - \delta_i^m \delta_{nk})} J^l J^n J^k \left(\frac{\partial f}{\partial J^q} \right) \left(\frac{\partial g}{\partial J^m} \right) \\
&= \frac{1}{C^4} (\underbrace{\epsilon^q{}_{ln} J^l J^n}_{0} J^m - \epsilon^{mq}{}_l J^l \underbrace{J_k J^k}_{C^2}) \left(\frac{\partial f}{\partial J^q} \right) \left(\frac{\partial g}{\partial J^m} \right) \\
&= \frac{1}{C^2} \epsilon_i{}^{jk} J^i \left(\frac{\partial f}{\partial J^j} \right) \left(\frac{\partial g}{\partial J^k} \right) \\
&= \frac{1}{C^2} \vec{J} \cdot \left(\frac{\partial f}{\partial \vec{J}} \times \frac{\partial g}{\partial \vec{J}} \right)
\end{aligned}
$$

4.3.3 Relation to loop quantum gravity

Let us consider the space of convex Euclidean shapes $\mathcal{S}_F$ with faces of fixed areas $\{A_f\}$. In view of the above discussion, the Lie-Poisson bracket structure for each face f is defined on the 2-sphere $S^2_{j_f}$ embedding in the area $3d$-space $\{\vec{A}_f\}$ in which we consider $A_f \sim j_f$, which is the area spectum corresponds to a surface that is intersected with only one link of a given graph. As is well known, the quantization of the the 2-sphere is the irreducible representation space $V^{(j)}$. Thanks to Guillemin-Sternberg's theorem [69] which shows the commutativity of the following diagram,



The quantization commutes with the reduction:

- One can reducing first the unconstrained phase space $\bigotimes_{f=1}^{F} S^2_{j_f}$ by summing area vectors to zero and up to rotations invariant, to end up with the Kapovich-Millson phase space $\mathcal{S}_F$, then canonically quantizing it by promoting the canonical variables (4.28) to be well-defined operators acting on an appropriate Hilbert space, which is the intertwiner space $\mathcal{H}_F$.

- One can quantize first the unconstrained phase space $\bigotimes_{f=1}^{F} S^2_{j_f}$ and then reducing it at the quantum level by solving Gauss constraints to end up with the intertwiner space $\mathcal{H}_F$.

The commutativity between quantization and reduction leads to an equivalence between the quantum polyedra space and the intertwiner space. As a consequence of this, intertwiners are the quantization of the Kapovich-Millson phase space; it can be visualized as the state of a quantum polyhedron, and spin network state as a collection of quantum polyhedra associated to each node. (See Fig. 4.1).

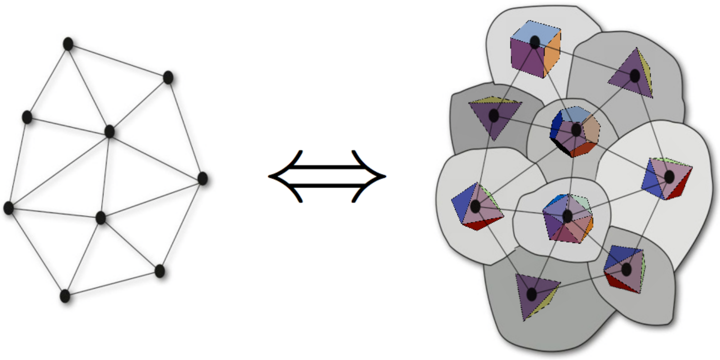


Figure 4.1: Spin network states as a collection of quantum polyhedra associated to each node. It is taken from [65].

The equivalence between convex Euclidean polyhedron and the intertwiner state can be understood as following:

The closure condition: $\sum_{f=1}^{F} \vec{A}_f = \vec{0} \longrightarrow \sum_{f=1}^{F} J_i^{(j_f)} = 0$

The area vector: $\vec{A}_f = A_f \vec{n}_f \longrightarrow \hat{E}_i(S^f) = 8\pi\gamma L_P^2 J_i^{(j_f)}$

The face area: $A_f = \sqrt{\vec{A}_f \cdot \vec{A}_f} \longrightarrow \hat{A}_f = \sqrt{\hat{E}_i(S^f)\hat{E}^i(S^f)} = 8\pi\gamma L_P^2 \sqrt{j_f(j_f+1)}$

4.3.4 Fuzzy Geometry

This correspondence allows us to interpret each atom of space on a node as quantum Euclidean polyhedra states and not a fixed one. Indeed, it offers infinite possible Euclidean polyhedra shapes for the same intertwiner state. In fact, after restricting the space of shapes of fixed areas to an arbitrary spectrum of the volume operator we will obtain $(2F-7)$ dimensions surface of relevant shapes. It is called *the volume spectrum orbit* and defined by:

$$\mathcal{O}rbit_V := \{(\mu_r, \theta_r) \in \mathcal{S}_F |\; V(A_f; \mu_r, \theta_r) = V\} \subset \mathcal{S}_F \tag{4.35}$$

More precisely, Consider a polyhedron with F faces, E edges and V vertices. The Euler formula $F - E + V = 2$ must satisfy. For the dominant class of polyhedra with all vertices 3-valent[2], one has $2E = 3V$, $E = 3(F-2)$ and $V = 2(F-2)$.

[2]Dominant class shapes: set of polyhedra with all vertices 3-valent, this condition maximizes both number of edges $E_{max} = 3(F-2)$ and vertices $V_{max} = 2(F-2)$. Subdominant classes are special kinds with some zero-length edges and then fewer vertices.

Thus, the geometry of a classical polyhedron with F faces is determined by $3(F-2)$ parameters, for instance, the $E = 3(F-2)$ lengths of its edges. But the corresponding quantum numbers that determine the quantum polyhedron states are not $3(F-2)$; they are only $F+1$: F for areas and one for volume. This fuzziness of quantum geometry is caused by the following two reasons:

1. The non-commutativity of the electric fluxes components (3.50) is an intrinsic feature of the kinematical quantum-geometry. It is different from the classical geometry where all the geometrical quantities are commute.

2. The fact that we consider a single graph Hilbert space which captures only finite degree of freedom number of the theory.

For instance if we take the case of 4-valent intertwiner with irr-representations $\{j_f, f = 1,2,3,4\}$ to see the fuzziness of the quantum geometry. We have 2-d space of tetrahedon shapes of fixed areas $\{A_f = 8\pi\gamma L_P^2\sqrt{j_f(j_f+1)}, f = 1,2,3,4\}$. restricting the volume variable to the spectrum of the volume operator through the 4-valent state, we obtain 1-d space of relevant shapes. Thus, we cannot identify a 4-valent intertwiner with a fixed tetrahedron due to the non-commutativity of the geometry.

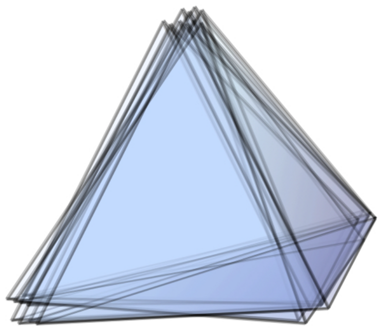


Figure 4.2: The fuzziness of quantum geometry. For instance, a quantum tetrahedron state. It is taken from [65]

Chapter 5

A Curvature and edge length Operators in LQG

This chapter is based on the the work in our publication [28]. Authors: N. Mebarki, O. Nemoul.

Article title: A curvature operator for a regular tetrahedron shape in LQG.

Journal reference: IJGMMP, Vol. 16, No. 06, 1950095 (2019).

A geometrical applications of loop quantum gravity is an important arena for more understanding the interpretation of intertwiner states as well as obtaining a nice semi-classical limit to the smooth picture of spacetime. A scalar curvature is one of the most important geometrical quantity that can allow us to know which kind of space at a given point. In what follows, we will try to introduce a notion of scalar curvature operator associated to a fixed polyhedron shape in Loop Quantum Gravity as well as the edge length operator. Then we will give a direct application of these new operators on the 4-valent intertwiner state. A suggested introduction to the curvature operator in terms of the length operator and the dihedral angles was provided by using 3d- Regge calculus [70]. Moreover, there are three proposals for length operator discussed in refs. [71, 72, 73].

5.1 Motivation for a new scalar curvature measure

We would like to introduce new expression of scalar curvature measure in terms of the well-known geometrical quantities (4.9,4.22). The 3d- Ricci scalar curvature in a given point of the hyper-surface S_t embedded in a smooth Riemannian manifold $\mathcal{M}$ is technically determined by the measure of volume and boundary area of a small neighborhood region around this point. It is obvious to observe that, doing these measurements separately does not give enough geometrical informations of the dynamical space in that region. Rather, it is mandatory to do this at the same time in order to get the complete information. To be more explicit, let us consider the simplest case of the 2-sphere $S^2_{r(t)}$ of radius $r(t)$ in 2+1 dimension (See Fig. 5.1).

$$\begin{aligned} S^2_{r(t)} &:= \{(x,y,z) \in \mathbb{R}^3 |\; x^2+y^2+z^2 = r^2(t)\} \\ &= \{(\theta,\varphi) \in \mathbb{R}^2 |\; 0<\theta<\pi, 0 \leqslant \varphi < 2\pi\} \cup \{N_p, S_p\} \end{aligned} \tag{5.1}$$

The spatial invariant interval of the 2-sphere $S^2_{t_0}$ at a given time t_0 is:

$$ds^2 = r(t)^2 \left(d\theta^2 + \sin^2(\theta) d\varphi^2\right) \tag{5.2}$$

where $r(t_0) = r_0$. The 2d-Ricci scalar curvature $R(t_0)$ is a constant in the 2-sphere $S^2_{t_0}$ and it can be written in terms of the curvature radius r_0 by the relation:

$$R(t_0) = \frac{2}{r_0^2} \tag{5.3}$$

Now, we want to measure the 2d-Ricci scalar curvature at a time t_0, this means we need to determine the radius r_0. To do so, we fix in $S^2_{r_0}$ a geodesic disc $\mathcal{D}_a(p)$ of a radius a centering at a point $p \in S^2_{r_0}$:

$$\mathcal{D}_a(m) = \left\{ p \in S^2_{r_0} \;\middle|\; l_{mp} \leq a \right\} \subset S^2_{r_0} \tag{5.4}$$

where l_{mp} is the geodesic length in the 2-sphere $S^2_{r_0}$ between the points m and p. The area $A(r_0,a)$ of the disc and its boundary curve length $L(r_0,a)$ are:

$$A(r_0,a) = 2\pi r_0^2 \left(1 - cos\left(\tfrac{a}{r_0}\right)\right) \tag{5.5a}$$

$$L(r_0,a) = 2\pi r_0 \; sin\left(\tfrac{a}{r_0}\right) \tag{5.5b}$$

Given the pair (r_0, a), one can determine the area of a disc and its boundary curve length (A, L). It is easy to invert these two functions to obtain:

$$R(A, L) = \frac{2(4\pi A - L^2)}{A^2} \tag{5.6a}$$

$$a(A, L) = \frac{A}{\sqrt{4\pi A - L^2}} \arctan\left(\frac{L\sqrt{4\pi A - L^2}}{2\pi A - L^2}\right) \tag{5.6b}$$

Accordingly to these result, the simultaneous measurement of the area and the boundary curve length (A, L) can allows us to estimate the value of the 2d-Ricci scalar curvature and the disc radius (R, a). In 2+1 dimension and for the 2-sphere shape, these two relations give us another way to measure the main important geometrical quantity which is the value of the 2d-Ricci scalar curvature $R(t_0) = R(A, L)$ as a function of the area measure A of a disc and its boundary curve length L. Remarkably, this technique does not depend on the choice of the region we chosen; one can use any shape of a region insted of the disc (5.4) and get the same 2d-scalar curvature. Our job now is to generalize this technique for arbitrary 3-dimensional topological spaces. To get such a generalization, we try to find a relation between the 3d- Ricci scalar curvature with the measurement of volume and boundary area of an arbitrary small region. It was done by using small geodesic ball [66], and for any arbitrary regular tetrahedron in a constant curvature spaces [29]. The curvature can be determined by inverting the resulting functions in all cases.

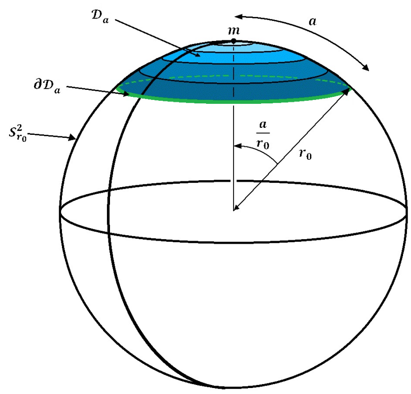


Figure 5.1: The geodesic disc $\mathcal{D}_a$ (blue) and its boundary circle $\partial\mathcal{D}_a$ (green) in the 2-sphere $S^2_{r_0}$. It is taken from [28].

5.2 Strategy for defining a new scalar curvature operator in LQG

In what follows, we will focus on introducing the notion of scalar curvature and edge length operators by considering the generalization of the correspondence between intertwiner state and quantum Euclidean polyhedra to a quantum geodesic polyhedra (it is not necessary Euclidean). The strategy is consisted of the following items:

1. We will interpret the intertwiner state by a fixed polyhedron shape (even if it doesn't belong to the volume orbit (4.35) of Euclidean polyhedra shapes).
2. We will try to find out what kind of a curved space one must have in order that this polyhedron grain be nicely consistent with the area and volume spectra of LQG:
   - Identifying the volume and areas operators of LQG with those of the corresponding polyhedron in an arbitrary curved space.
   - Inverting the resulting set of functions to end up to the classical formula of scalar curvature and edge lengths related to a fixed polyhedron.
   - Quantizing the resulting formula to obtain the quantum operators for the 3d- scalar curvature and the edge lengths.

It is worth to mention that the classical consistency of the 3d- Ricci scalar curvature measure as a function of the volume and boundary area measures is also well-defined at the quantum level since the commutativity between their associated geometrical operators is guaranteed in LQG[1] (there is no ordering problem of non-commutative operators). Unfortunately, we cannot exactly calculate the volume and boundary face area of a polyhedron in a general curved space, even if we make a perturbative series expansion around the Euclidean measure for a small polyhedron as it was mentioned for the small geodesic ball cases [74], we don't have any guidance to

[1] In LQG, the volume and area operators are commute.

estimate the uncertainty of this expansion. This problem occurred due the arbitrary degree of freedom of the considered general curved space. The solution is trivial; one can just relax the degree of freedom to spaces with a constant scalar curvature. In fact, a spin network state of a fixed graph induces naturally a discrete locally valued function of the 3d- Ricci scalar curvature. The reason is that all quantum geometric operators are not sensitive to all points inside the quantum atom of space; only nodes and links represent the quanta of space and its boundary surface respectively. Thus, each quantum atom of space corresponds to a constant 3d- Ricci scalar curvature value, i.e. all points inside the quantum atom of space share the same geometrical property. In the following, we will make our calculation concerning the volume and boundary area of a polyhedron in a constant curvature Riemannian manifolds. We remind that the Riemannian manifolds of a constant curvature can be classified into the Euclidean ($\mathbb{E}^3, R = 0$), spherical ($S^3_r, R > 0$) and hyperbolic ($H^3_r, R < 0$) geometries (other spaces that have a constant scalar curvature are isometric to the one of these three classes by the Killing-Hopf theorem [75, 76]). As a byproduct, the full expression of volume and boundary face area of a regular tetrahedron in the 3-sphere S^3_r and the 3-hyperbolic H^3_r has been derived explicitly in terms of the 3d- Ricci scalar curvature and the edge length in ref. [29]. In the monochromatic 4-valent node example, we will be interested to study the possibility of finding a correspondence with a regular geodesic tetrahedron. Applying the 3d- Ricci scalar curvature operator related to a regular tetrahedron region on the intertwiner state for constructing a space of a constant curvature where one can have the regular tetrahedron correspondence for any irreducible representation j.

5.3 Application: quantum tetrahedra

The quantum tetrahedra [77, 78, 79, 80, 81] is one of the most important topics proposed in LQG. Our task now is to determine new curvature operator related to a regular quantum tetrahedron acting on a monochromatic 4-valent intertwiner (links with the same irr-representation j) by using the approach similar to the one mentioned previously. Before we do that, let us discuss the main ingredients of the quantum tetrehadron state.

5.3.1 Area and volume operators in $\mathcal{H}_4$

The intertwiner space $\mathcal{H}_4$ has been constructed in subsection 3.5.3. The remaining steps are to introduce the appropriate area and volume operators such that the convex Euclidean tetrahedron has been inspired from it. It was done in ref. [77], we have then: The area operator corresponds to fth face of the quantum tetrahedron acts trivially on $\mathcal{H}_4$ (precisely acts trivially on the fth link of the intertwiner state (3.113)):

$$\hat{A}_f|k\rangle = \sqrt{\hat{\vec{E}}_f \cdot \hat{\vec{E}}_f}\ |k\rangle = 8\pi\gamma L_P^2 \sqrt{j_f(j_f+1)}\ |k\rangle \tag{5.7}$$

The volume operator corresponds to quantum tetrahedron acts non-trivially on $\mathcal{H}_4$ (precisely acts non-trivially on the node of the intertwiner state (3.113)):

$$\hat{V}|k\rangle = \frac{\sqrt{2}}{3}\sqrt{|\hat{\vec{E}}_1 \cdot (\hat{\vec{E}}_2 \times \hat{\vec{E}}_3)|}\ |k\rangle \tag{5.8}$$

due to the closure relation,

$$\left(\hat{\vec{E}}_1 + \hat{\vec{E}}_2 + \hat{\vec{E}}_3 + \hat{\vec{E}}_4\right)|k\rangle = 0 \tag{5.9}$$

a moment of reflection allow us to see how can this operator coincides with the Rovelli-Smolin operator (4.17) and the Ashtekar-Lewandowski operator (4.21). It is worth to mention that the presence of the square root in the volume operator (5.8) make a difficulty in computation of the spectrum. To overcome this problem, it is useful to introduce the volume square operator $\hat{Q}$. Thus, one has to diagonalize the volume matrix element by diagonalizing the matrix $[Q_{k'k}]$ of elements:

$$Q_{k'k} = \langle k'|\hat{Q}|k\rangle \tag{5.10}$$

One can show that (see for detail [27, 82, 83]) the matrix $[Q_{k'k}]$ is a $d \times d$ Hermitian matrix and it can be written as:

$$[Q_{k'k}] = (8\pi\gamma)^3 L_P^6 \begin{bmatrix} 0 & ia_1 & \cdots & \cdots & 0 \\ -ia_1 & 0 & ia_2 & & \vdots \\ \vdots & -ia_2 & \ddots & & \vdots \\ \vdots & & & 0 & ia_d \\ 0 & \cdots & \cdots & -ia_d & 0 \end{bmatrix} \tag{5.11}$$

with the real parameter a_k defined by:

$$a_k = iQ_{k+1,k} = \frac{\sqrt{(j_1+j_2+k+2)(j_1+j_2-k)(j_1-j_2+k+1)(j_2-j_1+k+1)}}{2\sqrt{2k+3}} \frac{\sqrt{(j_3+j_4+k+2)(j_3+j_4-k)(j_3-j_4+k+1)(j_4-j_3+k+1)}}{2\sqrt{2k+1}} \tag{5.12}$$

Since $\hat{Q}$ is a $d \times d$ Hermitian matrix, one has the following properties:

- Its spectrum is non-degenerate (it contains d distinct real eigenvalues).
- Its non-vanishing eigenvalues come always in pairs $\pm q$, then the volume operator is twice degenerate.
- A vanishing eigenvalue is present only if the dimension d of the intertwiner space $\mathcal{H}_4$ is odd.

As a conclusion, one can introduce the eigenstates of the volume operator $|\pm q\rangle$, labeled by the eigenvalues $\pm q$ of $\hat{Q}$, which is a linear combination of the intertwiner states:

$$|\pm q\rangle = \sum_{k=1}^{d} C^k_{(\pm q)} |k\rangle \tag{5.13}$$

The volume spectrum of the mixed intertwiner state $|\pm q\rangle$ is:

$$\hat{V}|\pm q\rangle = \sqrt{|q|}|\pm q\rangle \tag{5.14}$$

These are common eigenstates of both volume and area operators, then one has also:

$$\hat{A}_f|\pm q\rangle = 8\pi\gamma L_P^2 \sqrt{j_f(j_f+1)}\,|\pm q\rangle \tag{5.15}$$

5.3.2 Example: a monochromatic 4-valent node in $\mathcal{H}_4^j$

The gauge-invariant Hilbert space $\mathcal{H}_4^j$ of a monochromatic 4-valent node (the four links with the same spin j), the intertwiner basis is $\{|k\rangle, k = 0, \ldots, 2j\}$ with a dimension $2j+1$, we also label the common eigenstates of area and volume by $|\pm q\rangle_j$. We have then the following: the area operator gives the same quntity for any f-face,

$$\hat{A}| \pm q\rangle_j = 8\pi\gamma L_P^2 \sqrt{j(j+1)}\ | \pm q\rangle_j \tag{5.16}$$

and the volume operator gives,

$$\hat{V}| \pm q\rangle_j = \sqrt{|q|}| \pm q\rangle_j \tag{5.17}$$

where the spectrum $\pm q$ are the eigenvalues of the matrix element (5.11) for $d = 2j+1$, one can rewrite it in terms of components as:

$$Q_{n'n} = -ia_n\delta_{n',n+1} + ia_{n'}\delta_{n'+1,n} \quad , \quad n, n' = 1, \ldots, 2j+1 \tag{5.18}$$

where a_n is given by substituting $j_1 = j_2 = j_3 = j_4 = j$ in Eq. (5.12):

$$a_n = \frac{1}{4}\frac{(n^2 - (2j+1)^2)n^2}{\sqrt{4n^2 - 1}} \tag{5.19}$$

In order to determine the volume spectrum for arbitrary irreducible representation j, one has to compute the eigenvalues of $\hat{Q}$ of elements (5.18). This can be done by using numerically method and the result is exhibited in Fig. 5.2.

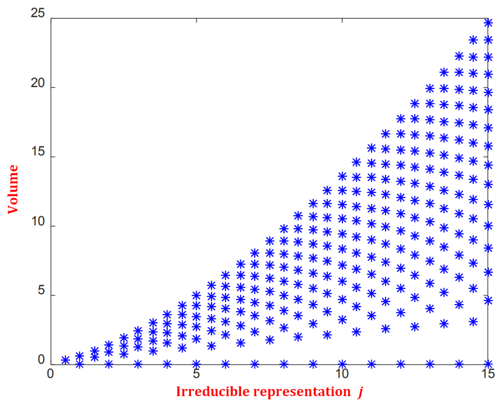


Figure 5.2: The volume spectra of a monochromatic 4-valent node for irreducible representations $\{\frac{1}{2}, 1, \frac{3}{2}, \ldots, 15\}$. We have taken the unit where $8\pi\gamma L_P^2 = 1$. It is taken from [84].

5.3.3 The space of equilateral tetrahedron shape $\mathcal{S}_4^A$

The space of all convex, Euclidean, equilateral, tetrahedron shapes S_4^A with fixed area norms $A_1 = A_2 = A_3 = A_4 = A$ satisfying the closure relation:

$$\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = \vec{0} \tag{5.20}$$

is defined by,

$$\mathcal{S}_4^A = \{(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4) \in (S^2)^4 | \sum_{f=1}^{4} \vec{n}_f = \vec{0}\}/_{SO(3)} \tag{5.21}$$

$$\dim(\mathcal{S}_4^A) = 2 \times 4 - 3 - 3 = 2 \tag{5.22}$$

From the definition (4.28), the canonical coordinates (p, q) in that case are:

$$\mu = |\vec{A}_1 + \vec{A}_2| \tag{5.23a}$$

$$\theta = \arctan\left[\frac{|(\vec{A}_1 \times \vec{A}_2) \times (\vec{A}_3 \times \vec{A}_4)|}{(\vec{A}_1 \times \vec{A}_2) \cdot (\vec{A}_3 \times \vec{A}_4)}\right] \tag{5.23b}$$

It is obvious that

$$0 \leq \mu \leq 2A \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \tag{5.24}$$

All geometrical informations of an Euclidean equilateral tetrahedron with fiexed faces area A can be constructed from its representation point $(\mu, \theta) \in \mathcal{S}_4^A$, such as the volume measure, one can easily show [27]:

$$V(A; \mu, \theta) = \frac{\mu^{\frac{3}{2}}}{3\sqrt{2}} \sqrt{|\sin(\theta)| \left(\frac{4A^2}{\mu^2} - 1\right)} \tag{5.25}$$

Notice that the volume function has a maximal value as it is shown in Fig. 5.3. In fact, one has to solve the equations:

$$\frac{\partial V(A; \mu, \theta)}{\partial \mu}\Big|_{(\mu_0, \theta_0)} = 0 \qquad \frac{\partial V(A; \mu, \theta)}{\partial \theta}\Big|_{(\mu_0, \theta_0)} = 0 \tag{5.26}$$

It is easily to check that,

$$\mu_0 = \frac{2\sqrt{3}}{3} A \qquad \theta_0 = \pm\, \frac{\pi}{2} \tag{5.27}$$

where

$$V_{max} = V(A;\mu_0,\theta_0) = 2^{3/2}\ 3^{-7/4}\ A^{3/2} \tag{5.28}$$

which is the expected Euclidean regular tetrahedron[2].

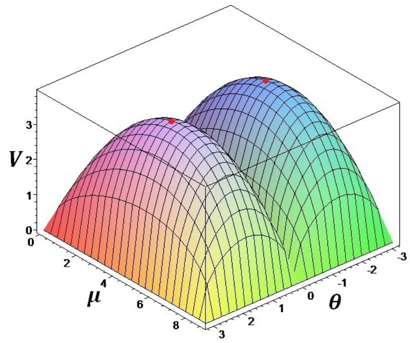


Figure 5.3: The volume function in the the Kapovich-Millson phase space $\mathcal{S}_4^A$ by taking the unit area $A = 1$. The two red points in the top of the volume surface corrsponds to two symmetrical regular tetrahedron shapes. It is taken from [28].

5.3.4 The relation between $\mathcal{H}_4^j$ and $\mathcal{S}_4^A$

Each volume spectrum $\sqrt{|q|}$ of the intertwiner space $\mathcal{H}_4^j$ corresponds to an orbit in the Kapovich-Millson phase space $\mathcal{S}_4^A$ with $A = 8\pi\gamma L_P^2\sqrt{j(j+1)}$:

$$\mathcal{O}rbit_q = \{(\mu,\theta) \in \mathcal{S}_4^A |\ V(A;\mu,\theta) = \sqrt{|q|}\} \tag{5.29}$$

These volume orbits are the possible Euclidean equilateral tetrahedron shapes corresponds to the intertwiner state $|\pm q\rangle_j$ (See an example of $j = 4$ in Fig. 5.4). The regular tetrahedron is the only state that has the maximum volume value. Therefore, the only intertwiner state corresponds to a unique equilateral tetrahedron shape is the one that has a volume eigenvalue equal to the maximum volume of the phase space $\mathcal{S}_4^A$ which is:

$$V_{max} = 2^{3/2}\ 3^{-7/4}(8\pi\gamma L_P^2)^{3/2}\Big(j(j+1)\Big)^{3/4} \tag{5.30}$$

[2]The regular, convex, Euclidean tetrahedron is a tetrahedron whose 6-edges are equal in length. Its volume is $V = \frac{\sqrt{2}}{12}a^3$ and its face area $A = \frac{\sqrt{3}}{4}a^2$ in which a is the edges length. The regular tetrahedron is special case of the equilateral tetrahedron corresponds to maximum volume value with fixed face area.

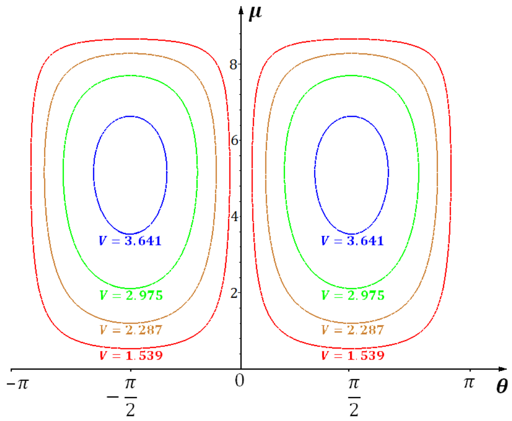


Figure 5.4: The Kapovich-Millson phase space $\mathcal{S}_4^A$. The colored orbits are quantized levels of the volume operator in the monochromatic 4-valent eigenstate with $j = 4$. We have taken the unit where $8\pi\gamma L_P^2 = 1$. It is taken from [28].

and it corresponds to the regular tetrahedron. In LQG, there is no quantum regular tetrahedron corresponding to a monochromatic 4-valent node state, since all quantum volume spectra are below the volume of a regular tetrahedron with a face area $A = 8\pi\gamma L_P^2\sqrt{j(j+1)}$ (See Fig. 5.5). Nonetheless, the existence of a such regular tetrahedron correspondence is guaranteed if we look for them in the space of equilateral tetrahedra shapes in a constant curvature space R [30]. In what follows, we will try to find which constant of scalar curvature in which the correspondence with a quantum regular tetrahedron existe.

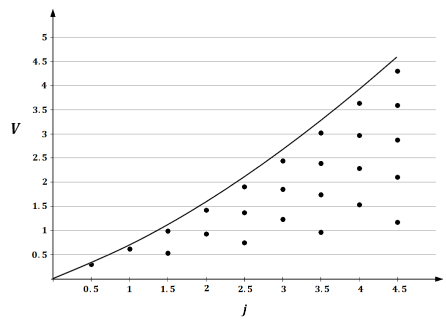


Figure 5.5: Comparison of the regular Euclidean tetrahedron volume (dark line) with the LQG volume spectra (dots) for the monochromatic 4-valent node state with different links color j. We have taken the unit where $8\pi\gamma L_P^2 = 1$. It is taken from [28].

5.4 A curvature and edge length operators for a regular quantum tetrahedron

Now, let us look for the 3d- Ricci scalar curvature value in which one can represent the monochromatic 4-valent quanta of space as a regular tetrahedron in a constant curvature space. In reference [29], the volume and the boundary face area of a regular spherical and hyperbolic tetrahedron given as explicit functions of the edge length a and the curvature radius $r = \sqrt{\frac{6}{|R|}}$ are shown to have the following expressions:

$$A\,(r,a) = \epsilon^2 r^2 \left[3 \arccos \left(\frac{\cos(\frac{a}{\epsilon r})}{\cos(\frac{a}{\epsilon r}) + 1} \right) - \pi \right] \tag{5.31a}$$

$$V(r,a) = 12\epsilon^3\, r^3 \int_0^{\tan(\frac{a}{2\epsilon r})} dt\, \frac{t \arctan(t)}{(3 - t^2)\sqrt{2 - t^2}} \tag{5.31b}$$

where

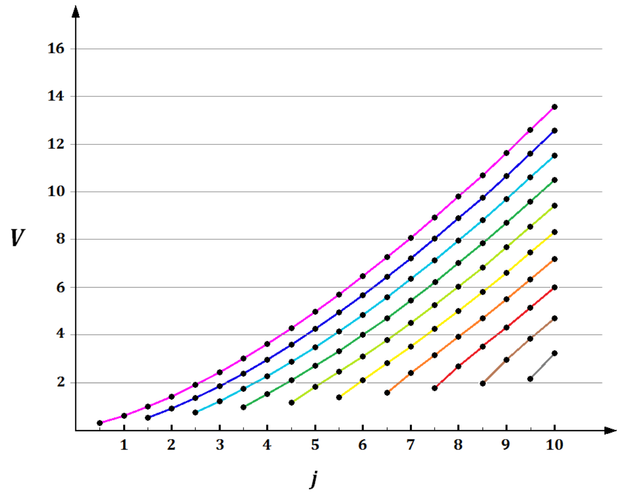
$$\epsilon = \begin{cases} 1, & \text{3d-sphere } S_r^3 \\ \sqrt{-1} \equiv i, & \text{3d-hyperbolic } H_r^3 \end{cases} \tag{5.32}$$

The Euclidean case is well-defined in the limit $r \to \infty$ $(R \to 0)$:

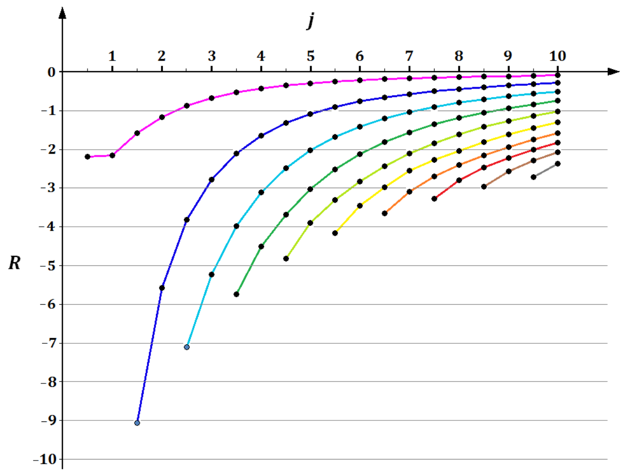
$$\lim_{r\to\infty} A\,(r,a) = \frac{\sqrt{3}}{4} a^2 \tag{5.33a}$$

$$\lim_{r\to\infty} V\,(r,a) = \frac{\sqrt{2}}{12} a^3 \tag{5.33b}$$

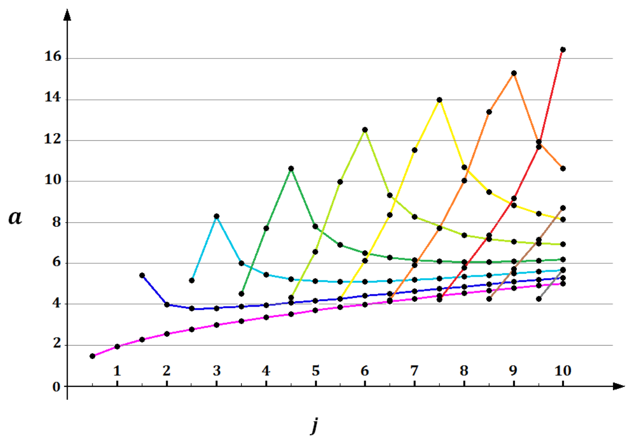
which is the face area and volume of an Euclidean regular tetrahedron with an edge length a. A direct application of the resulted formulas (5.31a,5.31b) in LQG is to determine a 3d- scalar curvature of the quantum tetrahedron state such that the monochromatic 4-valent intertwiner has an interpretation of a regular tetrahedron state of a constant curvature space (not Euclidean). For each area and volume spectra of the operators (5.16,5.17), inverting analytically these systems of functions is not so simple instead, we can deal with it numerically and construct the 3d- Ricci scalar curvature and the edge length spectra (See Figs. 5.6). In Figs. 5.6a, 5.6b and 5.6c, each curve with the same color corresponds to volume, scalar curvature and edge length spectra of the same states.



(a) The volume spectra



(b) The scalar curvature spectra



(c) The edge length spectra

Figure 5.6: Colored lines of different spectra levels for volume 5.6a, scalar curvature 5.6b and edge length 5.6c of a monochromatic 4-valent intertwiner for irreducible representations $\{\frac{1}{2}, 1, \frac{3}{2}, \ldots, 10\}$. We have taken the unit where $8\pi\gamma L_P^2 = 1$. We have used *Maple* to compute the spectra and draw these graphs. It is taken from [28].

From the the above figures 5.6, it is worth to shed light on the main following conclusions:

1. The existence of a regular tetrahedron consistent with LQG data (volume and area spectra) is guaranteed in the negative curvature regime, and then one can represent the monochromatic 4-valent state by a regular hyperbolic tetrahedron.

2. In general speaking, the 4-valent monochromatic state that has a biggest volume represented by a regular tetrahedron in negative constant curvature space is the closest to the Euclidean space with the smallest edge length and vice versa.

3. The lowest level value of the edges length (violet curve in Fig. 5.6c) are approximately the edges length of the Euclidean regular tetrahedron:

$$a_{min} \approx \left(\frac{4A}{\sqrt{3}}\right)^{\frac{1}{2}} = \left(\frac{32\pi\gamma}{\sqrt{3}}\right)^{\frac{1}{2}} L_P \,(j(j+1))^{\frac{1}{4}} \tag{5.34}$$

4. For a generic spin value $j \sim 1$, we find that the regular tetrahedron solutions of negative scalar curvature spectra are in the huge negative range:

$$R \approx -\frac{1}{\gamma L_P^2} \approx -10^{70}/\gamma \; m^{-2} \tag{5.35}$$

5. In the semi-classical limit $j \gg 1$, the monochromatic 4-valent will be more closer to be identified with the Euclidean regular tetrahedron, because all scalar curvature spectra (See Fig. 5.6b) tend to zero as well as the edge length spectra (See Fig. 5.6c) tend asymptotically to the edge length of a regular Euclidean tetrahedron given in (5.34). Accordingly, we are able to have a good approximation of the volume and boundary face area functions (5.31a,5.31b) around the zero constant curvature in the case of $j \gg 1$. In fact, by expanding these two functions (5.31a,5.31b) with respect to the

variable $\frac{a}{r}$, we obtain:

$$A(r,a) = \frac{\sqrt{3}}{4}a^2\left[1+\frac{1}{8}\left(\frac{a}{\epsilon r}\right)^2 + \mathcal{O}\left(\left(\tfrac{a}{\epsilon r}\right)^4\right)\right] \tag{5.36a}$$

$$V(r,a) = \frac{\sqrt{2}}{12}a^3\left[1+\frac{23}{80}\left(\frac{a}{\epsilon r}\right)^2 + \mathcal{O}\left(\left(\tfrac{a}{\epsilon r}\right)^4\right)\right] \tag{5.36b}$$

As we have previously said, the analytic inversion of the two functions (5.31a,5.31b) is not analytically possible, instead of doing the exact inversion with respect to the exact variables (r, a), we will use the good approximation functions (5.36a,5.36b) with respect to the approximate variables $(\tilde{r}, \tilde{a})$ and write:

$$A(\tilde{R},\tilde{a}) = \frac{\sqrt{3}}{4}\tilde{a}^2\left[1+\frac{1}{48}\tilde{R}\tilde{a}^2\right] \tag{5.37a}$$

$$V(\tilde{R},\tilde{a}) = \frac{\sqrt{2}}{12}\tilde{a}^3\left[1+\frac{23}{480}\tilde{R}\tilde{a}^2\right] \tag{5.37b}$$

where we have used the expression of 3d-Ricci scalar curvature in terms of the curvature radius $\tilde{R} = \frac{6}{\epsilon^2\tilde{r}^2}$. Inverting the two functions (5.37a,5.37b) for the two variables $(\tilde{R}, \tilde{a})$, we obtain approximated formulas of the scalar curvature as well as the edge length:

$$\tilde{R}(A,V) = 3\frac{\sqrt{3}}{2A}\tilde{x}\left[1+\frac{1}{8}\tilde{x}\right] \tag{5.38a}$$

$$\tilde{a}(A,V) = \left[\frac{4\sqrt{3}}{3}\frac{A}{1+\frac{1}{8}\tilde{x}}\right]^{1/2} \tag{5.38b}$$

with

$$\tilde{x}(A,V) = \frac{4\sqrt{3}\ A}{F(A,V)} - 8 \tag{5.39}$$

where

$$F(A,V) = \left[\frac{1}{78}G(A,V) + \frac{23\sqrt{3}A}{G(A,V)}\right]^2 \tag{5.40}$$

and

$$G(A,V) = \left[-205335\sqrt{3}V + 117\sqrt{-1265368\sqrt{3}A^3 + 9240075V^2}\right]^{1/3} \tag{5.41}$$

Now, one has to quantize the 3d-Ricci scalar curvature and edge length functions given in (5.38a,5.38b) by quantizing the area and volume operators to obtain quantum operators (5.16,5.17) that act on the intertwiner state $|\pm q\rangle_j$ of monochromatic 4-valent node with a j-color:

$$\tilde{R}(A,V) \longrightarrow \hat{\tilde{R}}(\hat{\tilde{A}},\hat{\tilde{V}})|\pm q\rangle_j = \tilde{R}\left(8\pi\gamma L_P^2\sqrt{j(j+1)},\sqrt{|q|}\right)|\pm q\rangle_j \tag{5.42a}$$

$$\tilde{a}(A,V) \longrightarrow \hat{\tilde{a}}(\hat{\tilde{A}},\hat{\tilde{V}})|\pm q\rangle_j = \tilde{a}\left(8\pi\gamma L_P^2\sqrt{j(j+1)},\sqrt{|q|}\right)|\pm q\rangle_j \tag{5.42b}$$

As the color j increases, the accuracy of these two operators (5.42a,5.42b) will be very high and their behavior spectra for $j \gg 1$ in the semi-classical limit is well known and it gives the Euclidean solution (See Table 5.1).

$$\hat{\tilde{R}}(\hat{\tilde{A}},\hat{\tilde{V}})|\pm q\rangle_{j\gg 1} \approx 0 \tag{5.43a}$$

$$\hat{\tilde{a}}(\hat{\tilde{A}},\hat{\tilde{V}})|\pm q\rangle_{j\gg 1} \approx L_p j^{\frac{1}{2}}|\pm q\rangle_{j\gg 1} \tag{5.43b}$$

Table 5.1: Comparison of the approximated spectra of the two operators $(\hat{\tilde{R}},\hat{\tilde{a}})$ associated to a regular quantum tetrahedron with their exact value (R,a) for the highest volume level (violet curves in Fig. 5.6) of the monochromatic 4-valent intertwiner state for the irreducible representation $\{1,2,3,\ldots,10\}$. We have taken the unit where $8\pi\gamma L_P^2 = 1$.

j	A	V_{max}	R	$\tilde{R}$	$\delta R\%$	a	$\tilde{a}$	$\delta a\%$
1	1.414	0.620	-2.146	-1.418	34%	1.954	1.914	2.07%
2	2.449	1.425	-1.156	-0.782	32%	2.557	2.511	1.82%
3	3.464	2.444	-0.663	-0.478	28%	2.998	2.960	1.25%
4	4.472	3.641	-0,422	-0.320	24%	3.369	3.340	0.87%
5	5.477	4.990	-0.291	-0.229	21%	3.700	3.677	0.63%
6	6.481	6.476	-0.212	-0.172	19%	4.003	3.983	0.48%
7	7.483	8.086	-0.161	-0.134	17%	4.283	4.267	0.37%
8	8.485	9.812	-0.127	-0.107	15%	4.545	4.532	0.30%
9	9.487	11.646	-0.102	-0.088	14%	4.793	4.782	0.24%
10	10.488	13.583	-0.084	-0.073	13%	5.029	5.019	0.20%

Chapter 6

Conclusion

Loop Quantum gravity is a background-independent,nonperturbative quantum field theory for describing the quantum structure of spacetime at the Planck scale. It is based on canonically quantizing the Ashtekar-Barbero phase space variables of the Holst action and then performing a suitable change of variables to the well-known holonomy-flux variables. The starting kinematical Hilbert space has been shown to be the space of all cylindrical wave functional through holonomies defined by the $su(2)$ connection along a system (graph) of smooth oriented paths (links). The resulted algebra is very suitable for quantization since the invariant Haar measure of the compact $SU(2)$ lie group is already exists. The kinematical Hilbert space is constructed thanks to Peter–Weyl theorem by using the orthonormal basis of $SU(2)$ irreducible representation. After we solved the Gauss constraints at the quantum level, spin network arises as a the basis of $SU(2)$ gauge invariant Hilbert space represented by an intertwiners (signlet states) for each point of intersection paths (node). We have also constructed well-defined geometrical operators: the area and volume acting on links and nodes of smooth paths system respectively. The spectrum of the area and volume operators was completely known and quantized. We have obtained the notion of quantum geometry (Γ, j_l, i_n) described by the spin network states where the intertwiner i_n assosiated to the node n is the quantum number of the volume and the irreducible representation j_l assosiated to the link e_l is the quantum number of the area. A beautiful interpretation of the intertwiners in terms of the quantum Euclidean polyhedral has been discussed where we have shown that each F-valent intertwiner state corresponds to $2F-5$ dimensions surface of relevant shapes in the space of polyhedra shapes. As anticipated, the main physical implication of Loop Quantum Gravity is that spacetime is fundamentally discrete and the minimal quanta of space is in the scale of the Planck length.

In the last, we have found a new approach of measuring the 3d- Ricci scalar curvature value by measuring the volume of a region and its boundary area. We have applied this technique in LQG, we sought to determine other possibilities of the correspondence in the context of non-zero curvature quantum polyhedra shapes (geodesic polyhedra) by acting the new proposal curvature operator on the intertwiner state to find other polyhedra shapes possibilities in the non-zero curvature regime. As a byproduct, we have studied the possibility of finding the regular tetrahedron correspondence with the monochromatic 4-valent node in other constant curvature spaces. It is shown that all quantum regular tetrahedron states are in the negative scalar curvature regime; for $j \gg 1$ the scalar curvature spectrum will be very close to the Euclidean regime. We conclude that the simultaneous measure of the volume and the boundary area of the monochromatic 4-valent node state allow us to estimate the appropriate case of a constant curvature space in which this state can be interpreted as a regular tetrahedron.

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Volume and Boundary Face Area of a Regular Tetrahedron in a Constant Curvature Space

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An example of the volume and boundary face area of a curved polyhedron for the case of regular spherical and hyperbolic tetrahedron is discussed. An exact formula is explicitly derived as a function of the scalar curvature and the edge length. This work can be used in loop quantum gravity and Regge calculus in the context of a non-vanishing cosmological constant.

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I. INTRODUCTION

In geometry, the calculation of volume and boundary face area of a curved polyhedron (geodesic polyhedron[1]) is one of the most difficult problems. In the case of spherical and hyperbolic tetrahedra, a lot of efforts has been made by mathematicians for calculating the volume and boundary face area: the volume formula are discussed by N. Lobachevsky and L. Schlafli in refs [1] for an orthoscheme tetrahedron, by G. Martin in ref [2] for a regular hyperbolic tetrahedron and by several authors in refs [3–9] for an arbitrary hyperbolic and spherical tetrahedron. All these results are based on the Schlafli differential equation where a unit sectional curvature was taken and they are given by a combination of dilogarithmic or Lobachevsky functions in terms of the dihedral angles. In the present paper, the volume and boundary face area of a regular spherical and hyperbolic tetrahedron are explicitly recalculated in terms of the curvature radius $r = \sqrt{\frac{6}{|R|}}$ and the edge length a. We directly perform the integration over the area and volume elements to end up with simple formula for the boundary face area and volume of a regular tetrahedron in a space of a constant scalar curvature R. This can be done by using the projection map to the Cayley-Klein-Hilbert coordinates system (CKHcs) which maps a regular geodesic tetrahedron $T(a)$ of an edge length a in the manifold of a constant curvature R to a regular Euclidean tetrahedron $T(a_0)$ of an edge length a_0 in the CKHcs. Then, one can express the area and volume measure elements in terms of their Euclidean ones. A comparison between the regular Euclidean, spherical and hyperbolic tetrahedron is studied and their implications are discussed. In physics, a direct application of the volume and boundary face area of a regular tetrahedron is essentially in loop quantum gravity (LQG) and Regge calculus. In LQG, the Euclidean tetrahedron interpretation of a 4-valent intertwiner state was shown in ref [10]. The main important feature of the formula which we are looking for is to find another possible correspondence between the 4-valent intertwiner state with a constant curvature regular tetrahedra shapes; this can be achieved by inverting the resulted functions. Thus, one can obtain the scalar curvature measure for a regular tetrahedron shape which allows us to know what kind of space in which the 4-valent intertwiner state can be represented by a regular tetrahedron [11]. It is worth mentioning that the idea supporting this new correspondence in the context of LQG with a non-vanishing cosmological constant was initiated in refs [11–14]. In the context of Regge calculus, the use of a constant curvature triangulation of spacetime was suggested in ref [15–17] and it can be useful for constructing a quantum gravity version with a non-vanishing cosmological

[1] Geodesic polyhedron is the convex region enclosed by the intersection of geodesic surfaces. A geodesic surface is a surface with vanishing extrinsic curvature and the intersection of two such surfaces is necessarily a geodesic curve.

constant. The paper is organized as follows: In section II, the volume and boundary face area of a geodesic polyhedron in general curved space are discussed. In section III, we give general integration formula of the volume and area for constant curvature spaces. In section IV, an exact formula for regular spherical and hyperbolic tetrahedra is explicitly derived as a function of the curvature radius and the edge length. Finally, in section V we draw our conclusions.

II. VOLUME AND BOUNDARY FACE AREA OF A POLYHEDRON IN A GENERAL CURVED SPACE

For any n-dimensional Riemannian manifold M equipped with an arbitrary metric g and a coordinates chart $\{U \subset M, \vec{\tilde{x}}\}$, one has to find another coordinates chart system $\{U \subset M, \vec{x}\}$, such that the straight lines in the second are geodesics of the manifold M. In other words, it maps the geodesic curves of the manifold in the first coordinates system to the straight line in the second one. Such a coordinates system denoted by CKHcs (Cayley-Klein-Hilbert coordinates system)[2] is very useful to calculate the volume and boundary face area of a geodesic polyhedron (i.e. every geodesic polygons and polyhedrons in the manifold maps to Euclidean polygons and polyhedrons in the CKHcs respectively). Finding such coordinates system is not an easy task for general metric spaces because it depends on the geometry itself and one has to solve a differential equation to find the CKHcs. If we denote by φ the coordinates transformation between the first and the CKHcs:

$$x^A = \varphi^A(\vec{\tilde{x}}) \qquad A = \overline{1.n}\,, \tag{1}$$

one can define the CKHcs by coordinates transformation that satisfying the following differential equation (See Appendix A):

$$\tilde{\nabla}_V \tilde{\nabla}_V \varphi^A(\vec{\tilde{x}}) = 0\,, \tag{2}$$

where

$$\tilde{\nabla}_V V = 0\,, \tag{3}$$

Eq. (2) holds for any vector field V tangent to geodesic curves and $\tilde{\nabla}_V$ stands for the covariant directional derivative along the vector field V in the coordinates system $\{U, \vec{\tilde{x}}\}$. By knowing the metric in the first coordinates system, one can determine the corresponding Christoffel symbols $\widetilde{\Gamma}'s$ and then solve the differential equation (2) to get the ideal frame

[2] It is usually known as the Klein projection.

CKHcs for calculating the volume of a geodesic polyhedron Pol and its boundary face area ∂Pol_f in an arbitrary n-dimensional Riemannian space:

$$\int_{Pol\subset U\subset M} dV^{Riem} = \int_{x(Pol)\subset x(U)\subset\mathbb{R}^n} \sqrt{|det(g(x))\;|}\; dV^{Euc}\,, \tag{4}$$

$$\int_{\partial Pol_f\subset U\subset M} dA_f^{Riem} = \int_{x(\partial Pol_f)\subset x(U)\subset\mathbb{R}^n} \sqrt{|det(g(x)|_{\partial Pol_f})\;|}\; dA_f^{Euc}\,, \tag{5}$$

where dA_f^{Euc} and dV^{Euc} are the Euclidean face area and volume measures of a geodesic polyhedron respectively, $g(x)$ is the metric in the CKHcs, $g(x)|_{\partial Pol_f}$ is the induced metric in the geodesic surface ∂Pol_f.

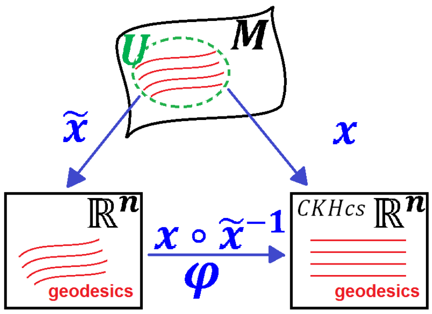

FIG. 1. The Cayley-Klein-Hilbert coordinates system (CKHcs).

III. VOLUME AND BOUNDARY FACE AREA OF A POLYHEDRON IN A 3D-CONSTANT CURVATURE SPACE

Let Σ be a 3-sphere or 3-hyperbolic metric space. The metric of the S_r^3 and H_r^3 can be combined in a unified expression and induced from the Euclidean Euc^4 and the Minkowski $Mink^4$ spaces respectively by using a compact form ϵ such that:

$$\epsilon = \begin{cases} 1 & for \quad S_r^3 \subset Euc^4 \\ i & for \quad H_r^3 \subset Mink^4 \end{cases}, \tag{6}$$

Let us consider the cartesian coordinates chart for the two spaces Euc^4 and $Mink^4$

$$\begin{aligned} X \;:\; M \quad &\longrightarrow \quad \mathbb{R}^3\times\epsilon\mathbb{R} \\ m \quad &\longmapsto \quad X^A(m) = \left(x^1, x^2, x^3, \epsilon x^4\right) \end{aligned}, \tag{7}$$

where

$$\epsilon\mathbb{R} = \begin{cases} \mathbb{R} & for \quad Euc^4 \\ i\mathbb{R} = Im\,(\mathbb{C}) & for \quad Mink^4 \end{cases}, \tag{8}$$

Basically, the metric of the Euc^4 and $Mink^4$ in this coordinates system is written as:

$$ds^2 = \delta_{AB} dX^A dX^B = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 + \epsilon^2 \left(dx^4\right)^2 , \tag{9}$$

In the spherical coordinates $\{\vec{x}\} = \{\rho, \psi, \theta, \varphi\}$ one has:

$$\begin{cases} \rho = \sqrt{\delta_{AB} X^A X^B} \\ \psi = \epsilon arctan\left(\frac{\sqrt{(X^1)^2+(X^2)^2+(X^3)^2}}{X^4}\right) \\ \theta = arctan\left(\frac{\sqrt{(X^1)^2+(X^2)^2}}{X^3}\right) \\ \varphi = arctan\left(\frac{X^2}{X^1}\right) \end{cases} \qquad \begin{cases} X^1 = \frac{\rho}{\epsilon} cos\,(\varphi)\; sin\,(\theta)\; sin(\epsilon\psi) \\ X^2 = \frac{\rho}{\epsilon} sin\,(\varphi)\; sin\,(\theta)\; sin(\epsilon\psi) \\ X^3 = \frac{\rho}{\epsilon} cos\,(\theta)\; sin(\epsilon\psi) \\ X^4 = \epsilon\; \rho cos\,(\epsilon\psi) \end{cases} , \tag{10}$$

$$ds^2 = \epsilon^2 d\rho^2 + \rho^2 \left[d\psi^2 + \epsilon^2 \sin^2(\epsilon\psi)\left(d\theta^2 + \sin^2(\theta) d\varphi^2\right)\right] , \tag{11}$$

Now, we define the 3d- metric spaces S_r^3 and H_r^3 as hyper-surfaces embedded in Euc^4 and $Mink^4$ respectively as:

$$X^2 = \delta_{AB} X^A X^B = (\epsilon r)^2 , \tag{12}$$

where r is a positive real number known as the radius of curvature. Geodesics can be obtained by the intersection of S_r^3 (or H_r^3) surface with two distinct 3d- hypersurfaces through the centre of the S_r^3 (or H_r^3):

$$\begin{cases} \delta_{AB} X^A X^B = (\epsilon r)^2 \\ a_A X^A = 0 \\ b_A X^A = 0 \end{cases} , \tag{13}$$

Where a_A and b_A are two non-collinear vectors of $\mathbb{R}^3 \times \epsilon\mathbb{R}$. After dividing Eq. (13) by $cos\,(\epsilon\psi)$, the geodesics satisfy:

$$\begin{cases} a_1 cos\,(\varphi)\; sin\,(\theta)\; tan\,(\epsilon\psi)\; + a_2 sin\,(\varphi)\; sin\,(\theta)\; tan\,(\epsilon\psi)\; + a_3 cos\,(\theta)\; tan\,(\epsilon\psi)\; + a_4 = 0 \\ b_1\; cos\,(\varphi)\; sin\,(\theta)\; tan\,(\epsilon\psi)\; + b_2 sin\,(\varphi)\; sin\,(\theta)\; tan\,(\epsilon\psi)\; + b_3 cos\,(\theta)\; tan\,(\epsilon\psi)\; + b_4 = 0 \end{cases} , \tag{14}$$

where $\psi \neq \frac{\pi}{2}$ is used in the case of the 3-sphere S_r^3. Therefore, we can get from the geodesic equations (14), the coordinates transformation to the CKHcs $\{\vec{x}\} = \{x, y, z\}$ that satisfying the differential equation condition (2) for both spherical and hyperbolic cases:

1. For the spherical case S_r^3 $(\epsilon = 1 \Rightarrow R = \frac{6}{r^2})$, the coordinates transformation to the CKHcs and its inverse read:

$$\begin{array}{ll} \varphi_{S_r^3} : \tilde{x}(U^{S_r^3} \subset S_r^3) \longrightarrow x(U^{S_r^3} \subset S_r^3) & \varphi_{S_r^3}^{-1} : x(U^{S_r^3} \subset S_r^3) \longrightarrow \tilde{x}(U^{S_r^3} \subset S_r^3) \\ \qquad (\psi, \theta, \varphi) \longmapsto (x, y, z) & \qquad (x, y, z) \longmapsto (\psi, \theta, \varphi) \end{array} , \tag{15}$$

and are defined by

$$\begin{cases} x = r\ cos\,(\varphi)\, sin\,(\theta)\, tan\,(\psi) \\ y = r\ sin\,(\varphi)\, sin\,(\theta)\, tan\,(\psi) \\ z = r\ cos\,(\theta)\, tan\,(\psi) \end{cases} \qquad \begin{cases} \psi = arctan\left(\frac{\sqrt{x^2+y^2+z^2}}{r}\right) \\ \theta = arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\ \varphi = arctan\left(\frac{y}{x}\right) \end{cases}, \quad (16)$$

Notice that $U^{S^3_r} \subset S^3_r$ is the top half 3-sphere divided by the hyper-surface of the equation $\psi = \frac{\pi}{2}$[3]:

$$\tilde{x}(U^{S^3_r}) = \{(\psi, \theta, \varphi)\ |.\ \psi \in [0, \frac{\pi}{2}], \theta \in [0, \pi], \varphi \in [0, 2\pi]\}\,, \quad (17)$$

2. For the hyperbolic case S^3_r $(\epsilon = i \Rightarrow R = \frac{-6}{r^2})$, the coordinates transformation to the CKHcs and its inverse read:

$$\begin{aligned} \varphi_{H^3_r} : \tilde{x}(U^{H^3_r} \subset H^3_r) &\longrightarrow [-r, r]^3 \\ (\psi, \theta, \varphi) &\longmapsto (x, y, z) \end{aligned} \qquad \begin{aligned} \varphi^{-1}_{H^3_r} : [-r, r]^3 &\longrightarrow \tilde{x}(U^{H^3_r} \subset H^3_r) \\ (x, y, z) &\longmapsto (\psi, \theta, \varphi) \end{aligned}, \quad (18)$$

and are defined by

$$\begin{cases} x = r\ cos\,(\varphi)\, sin\,(\theta)\, tanh\,(\psi) \\ y = r\ sin\,(\varphi)\, sin\,(\theta)\, tanh\,(\psi) \\ z = r\ cos\,(\theta)\, tanh\,(\psi) \end{cases} \qquad \begin{cases} \psi = arctanh\left(\frac{\sqrt{x^2+y^2+z^2}}{r}\right) \\ \theta\ = arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\ \varphi = arctan\left(\frac{y}{x}\right) \end{cases}, \quad (19)$$

Notice that, in order to get an isomorphism between the two coordinates systems, we have to take the cubic interval $[-r, r]^3$ since $\tanh(\psi)$ is bounded by the interval $[-1, 1]$. Moreover, we have also considered the region $U^{H^3_r} \subset H^3_r$ as the top sheet of the 3d- spherical hyperboloid H^3_r.

By using the compact form (6), one can unify the transformation between the two coordinates charts for both spherical and hyperbolic cases:

$$\begin{cases} x = \epsilon r\ cos\,(\varphi)\, sin\,(\theta)\, tan\left(\frac{\psi}{\epsilon}\right) \\ y = \epsilon r\ sin\,(\varphi)\, sin\,(\theta)\, tan\left(\frac{\psi}{\epsilon}\right) \\ z = \epsilon r\ cos\,(\theta)\, tan\left(\frac{\psi}{\epsilon}\right) \end{cases} \qquad \begin{cases} \psi = \epsilon\ arctan\left(\frac{\sqrt{x^2+y^2+z^2}}{\epsilon r}\right) \\ \theta = arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\ \varphi = arctan\left(\frac{y}{x}\right) \end{cases}, \quad (20)$$

The metric in the 3-sphere S^3_r and 3-hyperbolic H^3_r spaces is:

$$ds^2 = r^2\left[d\psi^2 + \epsilon^2 \sin^2(\epsilon\psi)\left(d\theta^2 + \sin^2(\theta) d\varphi^2\right)\right]\,, \quad (21)$$

Using the differential form chain rule, one can write:

$$d\psi = \frac{\epsilon^2\ r\ x}{(\epsilon^2 r^2 + |\vec{x}|^2)|\vec{x}|}dx + \frac{\epsilon^2\ r\ y}{(\epsilon^2 r^2 + |\vec{x}|^2)|\vec{x}|}dy + \frac{\epsilon^2\ r\ z}{(\epsilon^2 r^2 + |\vec{x}|^2)|\vec{x}|}dz, \quad (22)$$

[3]Knowing that the biggest possible spherical tetrahedron is the half of 3-sphere S^3_r.

$$d\theta = \frac{x\ z}{|\vec{x}|^2\sqrt{x^2+y^2}}dx + \frac{y\ z}{|\vec{x}|^2\sqrt{x^2+y^2}}dy - \frac{\sqrt{x^2+y^2}}{|\vec{x}|^2}dz, \tag{23}$$

$$d\varphi = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy, \tag{24}$$

Thus, the metric in the CKHcs becomes:

$$ds^2 = g_{AB}dx^A dx^B = -\left(\frac{\sum_{A=1}^{3} x^A dx^B}{\epsilon^2 r^2 + |\vec{x}|^2}\right)^2 + \frac{\sum_{A=1}^{3}\left(dx^A\right)^2}{\epsilon^2 r^2 + |\vec{x}|^2}\,, \tag{25}$$

The components of the metric elements read:

$$g_{AB} = \begin{pmatrix} \frac{\epsilon^2 r^2\ \left(\epsilon^2 r^2 + y^2 + z^2\right)}{\left(\epsilon^2 r^2 + x^2 + y^2 + z^2\right)^2} & \frac{-\ \epsilon^2\ r^2\ xy}{\left(\epsilon^2 r^2 + x^2 + y^2 + z^2\right)^2} & \frac{-\ \epsilon^2\ r^2\ xz}{\left(\epsilon^2 r^2 + x^2 + y^2 + z^2\right)^2} \\ \frac{-\ \epsilon^2\ r^2\ xy}{\left(\epsilon^2 r^2 + x^2 + y^2 + z^2\right)^2} & \frac{\epsilon^2 r^2\ \left(\epsilon^2 r^2 + x^2 + z^2\right)}{\left(\epsilon^2 r^2 + x^2 + y^2 + z^2\right)^2} & \frac{-\ \epsilon^2\ r^2\ yz}{\left(\epsilon^2 r^2 + x^2 + y^2 + z^2\right)^2} \\ \frac{-\ \epsilon^2\ r^2\ xz}{\left(\epsilon^2 r^2 + x^2 + y^2 + z^2\right)^2} & \frac{-\ \epsilon^2\ r^2\ yz}{\left(\epsilon^2 r^2 + x^2 + y^2 + z^2\right)^2} & \frac{\epsilon^2 r^2\ \left(\epsilon^2 r^2 + x^2 + y^2\right)}{\left(\epsilon^2 r^2 + x^2 + y^2 + z^2\right)^2} \end{pmatrix}, \tag{26}$$

and the Jacobian $J(\vec{x})$

$$J(\vec{x}) = \sqrt{|det\left(g(x)\right)|} = \frac{r^4}{\left(\epsilon^2 r^2 + |\vec{x}|^2\right)^2}\,, \tag{27}$$

Finally, we can determine the volume of a geodesic polyhedron Pol and its boundary face area ∂Pol_f:

1. For a spherical polyhedron ($R = \frac{6}{r^2}$)

$$\int_{\partial Pol_f \subset U^{S_r^3} \subset S_r^3} dA_f^{S_r^3} = \int_{x\left(\partial Pol_f\right)\subset \mathbb{R}^3} dA_f^{Euc}\sqrt{|det(g(x)|_{\partial Pol_f}^{S_r^3})\ |}\,, \tag{28}$$

$$\int_{Pol \subset U^{S_r^3} \subset S_r^3} dV^{S_r^3} = \int_{x(Pol)\subset \mathbb{R}^3} dV^{Euc}\frac{r^4}{\left(r^2 + |\vec{x}|^2\right)^2}\ , \tag{29}$$

2. For a hyperbolic polyhedron ($R = \frac{-6}{r^2}$)

$$\int_{\partial Pol_f \subset U^{H_r^3} \subset H_r^3} dA^{H^3} = \int_{x\left(\partial Pol_f\right)\subset \mathbb{R}^3} dA_f^{Euc}\sqrt{|det(g(x)|_{\partial Pol_f}^{H_r^3})\ |}\,, \tag{30}$$

$$\int_{Pol \subset U^{H_r^3} \subset H_r^3} dV^{H_r^3} = \int_{x(Pol)\subset \mathbb{R}^3} dV^{Euc}\frac{r^4}{\left(-r^2 + |\vec{x}|^2\right)^2}\ , \tag{31}$$

The induced Jacobian $\sqrt{|det(g(x)|_{\partial Pol_f}^{S_r^3})\ |}$ and $\sqrt{|det(g(x)|_{\partial Pol_f}^{H_r^3})\ |}$ for both spherical and hyperbolic respectively can be determined after restricting the metric in the boundary surface area ∂Pol_f.

IV. APPLICATION: REGULAR TETRAHEDRON IN A CONSTANT CURVATURE SPACE

Let $T(a)$ be a regular geodesic tetrahedron with an edge length a embedded in a constant curvature 3d- space Σ, and $\left\{\vec{A}_f\right\}_{f=\overline{1.4}}$ be normal areas vectors of $T(a)$. In what follows, we will calculate the volume of a geodesic regular tetrahedron $T(a)$ and its boundary face area $\partial T(a)_f$ in 3d- sphere S_r^3 and Hyperbolic H_r^3 manifolds:

$$A_f^{\Sigma}(r,a) = \int_{x\left(\partial T(a)_f\right)\subset\mathbb{R}^3} dA_f^{Euc}\sqrt{\left|det(g(x)|_{\partial T(a)_f})\right|}, \tag{32}$$

$$V^{\Sigma}(r,a) = \int_{x(T(a))\subset\mathbb{R}^3} dV^{Euc}\frac{r^4}{\left(\epsilon^2 r^2 + |\vec{x}|^2\right)^2}, \tag{33}$$

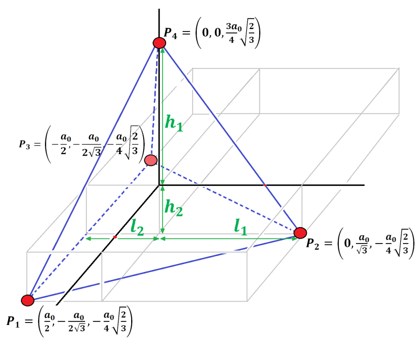

FIG. 2. A regular tetrahedron $T(a_0)$ in $\mathbb{R}^3$ (CKHcs).

The ignorance of how this new coordinates system CKHcs can map an Euclidean length to spherical and hyperbolic length measures, one has to be careful in choosing the location of the tetrahedron $T(a)$. From our choice in Fig. 2, it obvious to see that the image of a regular geodesic tetrahedron $T(a)$ of an edge length a in the manifold is an Euclidean regular tetrahedron $T(a_0)$ of a different edge length a_0 in the CKHcs:

$$x\left(T(a)\right) = T(a_0), \tag{34}$$

Our objective is to have an expression for the starting Euclidean length a_0 in terms of the geodesic length a. In order to determine how this coordinates system measure the length different from the original one, we have to consider two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ in the CKHcs where the corresponding geodesic line between them is parameterized by:

$$\begin{cases} y = \alpha x + \beta \\ z = \gamma x + \delta \end{cases}, \tag{35}$$

where

$$\alpha = \frac{y_2 - y_1}{x_2 - x_1} \qquad \beta = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}\,, \tag{36}$$

$$\gamma = \frac{z_2 - z_1}{x_2 - x_1} \qquad \delta = \frac{x_2 z_1 - x_1 z_2}{x_2 - x_1}\,, \tag{37}$$

The geodesic length between M_1 and M_2 is:

$$d\left(M_1 M_2\right) = \epsilon\ r\ arctan\left(\frac{\left(\alpha^2+\gamma^2+1\right)x+\alpha\beta+\gamma\delta}{\sqrt{\epsilon^2 r^2+\beta^2+\delta^2+\left(\alpha^2+\gamma^2\right)\epsilon^2 r^2+\alpha^2\delta^2+\gamma^2\beta^2-2\alpha\beta\gamma\delta}}\right)\Bigg|_{x_1}^{x_2}, \tag{38}$$

Since $d\left(M_1 M_2\right)$ depends strongly on the ending points, a special care has to be done in the location of the Euclidean regular tetrahedron in the CKHcs as it is shown in Fig. 2. One can check that:

$$a = 2\epsilon r\ arctan\left(\frac{1}{2}\frac{a_0}{\sqrt{\epsilon^2 r^2+\frac{a_0^2}{8}}}\right)\,, \tag{39}$$

In order to obtain a geodesic edge length a, one has to solve Eq. (39) for the unknown a_0 and get:

$$a_0 = \frac{2\ \epsilon\ r\ tan\left(\frac{a}{2\epsilon r}\right)}{\sqrt{\left(1-\frac{1}{2}tan^2\left(\frac{a}{2\epsilon r}\right)\right)}}\,, \tag{40}$$

1. For the spherical case S_r^3 $\left(\epsilon = 1 \Rightarrow R = \frac{6}{r^2}\right)$, one has:

$$a = 2r\ arctan\left(\frac{1}{2}\frac{a_0}{\sqrt{r^2+\frac{a_0^2}{8}}}\right)\,, \tag{41}$$

   In this case, one can check that the regular tetrahedron has a maximal edge a_{max} (for $a_0 \to \infty$) given by:

$$a_{max} = 2\ arctan\left(\sqrt{2}\right)\ r\,, \tag{42}$$

2. For the hyperbolic case S_r^3 $\left(\epsilon = i \Rightarrow R = \frac{-6}{r^2}\right)$, one has:

$$a = 2r\ arctanh\left(\frac{1}{2}\frac{a_0}{\sqrt{r^2-\frac{a_0^2}{8}}}\right)\,, \tag{43}$$

   Due to the compactness property (see Eq. (18)) of the coordinates chart, the initial value of the Euclidean length a_0 must be bounded $a_0 < \frac{2}{3}\sqrt{6}\ r$. However, a has no upper bound.

IV.1. Boundary area of a regular tetrahedron in S_r^3 and H_r^3

The faces area of a geodesic regular tetrahedron of an edge length a are all equal $\left(A_f^{\Sigma}(r,a) = A^{\Sigma}(r,a)\ ,\ \forall f = \overline{1.4}\right)$. In fact, the geodesic surface of the S_r^3 and H_r^3 are portions of the great 2-dimensional spheres S_r^2 and hyperbolic H_r^2 respectively. Accordingly, we expect to obtain the same area expression of the spherical and hyperbolic trigonometry. Due to the symmetric property of the constant curvature spaces, we restrict ourselves to geodesic triangle face $\partial T(a)_f \equiv P_1P_2P_3$ (See Fig. 2) in the geodesic surface $z = \frac{-a_0}{4}\sqrt{\frac{2}{3}}$ (with $dz = 0$). Then the induced Jacobian:

$$\sqrt{|det(g(x)|_{P_1P_2P_3})|} = \frac{\epsilon^2 r^2 \sqrt{\epsilon^2 r^2 + \frac{{a_0}^2}{24}}}{\left(\epsilon^2 r^2 + x^2 + y^2 + \frac{{a_0}^2}{24}\right)^{3/2}}, \tag{44}$$

The boundary face area is:

$$A^{\Sigma}(r,a) = \int_{P_1P_2P_3 \subset \mathbb{R}^3} dA_f^{Euc} \frac{\epsilon^2 r^2 \sqrt{\epsilon^2 r^2 + \frac{{a_0}^2}{24}}}{\left(\epsilon^2 r^2 + x^2 + y^2 + \frac{{a_0}^2}{24}\right)^{3/2}}, \tag{45}$$

with

$$dA_f^{Euc} = \frac{1}{2}\sum_{i,j,k=1}^{3} \epsilon_{ijk} A_f^i dx^j \wedge dx^k\,, \tag{46}$$

where A_f^i is the i^{th} component of the normal area vector $\vec{A}_f$. The integral in Eq. (45) is in general very hard to evaluate. To do so, one has to make a series expansion of the Jacobian $J(\vec{x})$ given in (27) with respect to the coordinates variables $\{\vec{x}\}$ and then easily perform the integration over one of the faces $P_1P_2P_3$, we get the following expression:

$$A^{\Sigma}(r,a) = \frac{\sqrt{3}}{4}a^2\{1 + \frac{1}{8}(\frac{a}{\epsilon r})^2 + \frac{1}{60}(\frac{a}{\epsilon r})^4 + \frac{583}{241920}(\frac{a}{\epsilon r})^6 + \frac{227}{604800}(\frac{a}{\epsilon r})^8 + \frac{23}{369600}(\frac{a}{\epsilon r})^{10} + \frac{1418693}{130767436800}(\frac{a}{\epsilon r})^{12} + \mathcal{O}((\frac{a}{\epsilon r})^{14})\}\,, \tag{47}$$

Using the symmetry of the triangle faces of a regular tetrahedron, the exact formula of the boundary face area reads:

$$A^{\Sigma}(r,a(a_0)) = 2\int_0^{\frac{a_0}{2}} dx \int_{\frac{-\sqrt{3}a_0}{6}}^{-\sqrt{3}x+\frac{\sqrt{3}a_0}{3}} dy\, \frac{\epsilon^2 r^2 \sqrt{\epsilon^2 r^2 + \frac{{a_0}^2}{24}}}{\left(\epsilon^2 r^2 + x^2 + y^2 + \frac{{a_0}^2}{24}\right)^{3/2}}, \tag{48}$$

Straightforward but tedious calculations (See Appendix B) give the following analytical expression of the boundary face area $A^{\Sigma}(r,a)$ of a regular spherical and hyperbolic tetrahedron with an edge length a in the curved space Σ of a constant curvature $R = \frac{6}{\epsilon^2 r^2}$:

$$A^{\Sigma}(r,a) = \epsilon^2 r^2 \left(3\arccos\left(\frac{\cos(\frac{a}{\epsilon r})}{\cos(\frac{a}{\epsilon r}) + 1}\right) - \pi\right)\,, \tag{49}$$

It is easy to check that the expansion of the resulted formula (49) in terms of the $\frac{a}{\epsilon r}$ variable is exactly the one in Eq. (47) and thus ensuring the correctness of the integration.

1. For the spherical case S^3_r $(\epsilon = 1 \Rightarrow R = \frac{6}{r^2})$, one has:

$$A^{S^3_r}(r,a) = r^2 \left(3 \arccos \left(\frac{\cos(\frac{a}{r})}{\cos(\frac{a}{r}) + 1} \right) - \pi \right) , \quad (50)$$

   As it is expected, it is the familiar expression of the regular spherical triangle embedded in the 2-sphere S^2_r where the dihedral angle is defined by $\Theta = \arccos\left(\frac{\cos(\frac{a}{r})}{\cos(\frac{a}{r})+1}\right)$ which is the cosine rule formula for spherical trigonometry. We can check that the boundary area $A^{S^3_r}$ for the maximal edge length a_{max} in Eq. (42) corresponds to an upper bound $A^{S^3_r}_{max} = \pi r^2$. The boundary area of a regular spherical tetrahedron is always greater than the boundary area of a regular Euclidean one.

2. For the hyperbolic case S^3_r $(\epsilon = i \Rightarrow R = \frac{-6}{r^2})$, one has:

$$A^{H^3_r}(r,a) = r^2 \left(\pi - 3 \arccos \left(\frac{\cosh(\frac{a}{r})}{\cosh(\frac{a}{r}) + 1} \right) \right) , \quad (51)$$

   As it is expected, it is the familiar expression of the regular hyperbolic triangle embedded in the 2-hyperbolic H^2_r where the dihedral angle is defined by $\Theta = \arccos\left(\frac{\cosh(\frac{a}{r})}{\cosh(\frac{a}{r})+1}\right)$ which is the cosine rule formula for hyperbolic trigonometry. Notice that in this case, there is no upper bound and for a given pair (r,a). The boundary area of a regular hyperbolic tetrahedron is always smaller than the boundary area of a regular Euclidean one.

3. For the Euclidean case Euc^3 $(R = 0)$, one has:

$$A^{Euc^3}(r,a) = \lim_{r\to\infty} A^{\Sigma}(r,a) \;= \frac{\sqrt{3}}{4}a^2 , \quad (52)$$

   The Euclidean limit is well-defined.

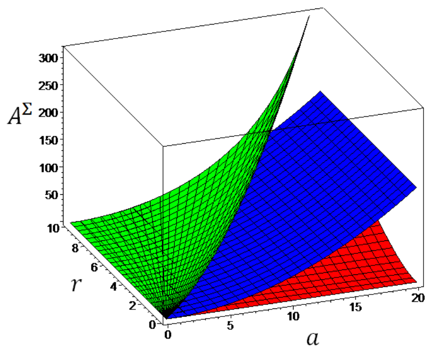

FIG. 3. : Function surface of the boundary face area for spherical (green), Euclidean (blue) and hyperbolic (red) regular tetrahedra.

IV.2. Volume of a regular tetrahedron in S_r^3 and H_r^3

The volume V^{Σ} of a regular spherical and hyperbolic tetrahedron is:

$$V^{\Sigma}\left(r,a\left(a_0\right)\right)=\int_{T(a_0)\subset\mathbb{R}^3} dV^{Euc}\frac{r^4}{\left(\epsilon^2r^2+|\vec{x}|^2\right)^2}\,, \tag{53}$$

Since the integration is very hard to deal with, it is better to make again a series expansion of the Jacobian $J(\vec{x})$ given in (27) in terms of the coordinates variables $\{\vec{x}\}$ and then easily perform the integration to end up with:

$$V^{\Sigma}(r,a)=\frac{\sqrt{2}}{12}a^3\{1+\frac{23}{80}(\frac{a}{\epsilon r})^2+\frac{3727}{53760}(\frac{a}{\epsilon r})^4+\frac{124627}{7741440}(\frac{a}{\epsilon r})^6 + \frac{20283401}{5449973760}(\frac{a}{\epsilon r})^8+\frac{14700653069}{17003918131200}(\frac{a}{\epsilon r})^{10}+\frac{1651049434189}{8161880702976000}(\frac{a}{\epsilon r})^{12}+\mathcal{O}(\frac{a}{r})^{14})\}\,, \tag{54}$$

Using the symmetry of the regular tetrahedron, the exact expression of the volume of a regular spherical and hyperbolic tetrahedron is:

$$V^{\Sigma}\left(r,a\left(a_0\right)\right)=2\int_{\frac{-\sqrt{6}a_0}{12}}^{\frac{\sqrt{6}a_0}{4}}dz\int_0^{\frac{\alpha(z)}{2}}dx\int_{\frac{-\sqrt{3}\alpha(z)}{6}}^{-\sqrt{3}x+\frac{\sqrt{3}\alpha(z)}{3}}dy\,\frac{r^4}{\left(\epsilon^2r^2+|\vec{x}|^2\right)^2}\,, \tag{55}$$

where

$$\alpha(z)=\frac{-\sqrt{6}}{2}z+\frac{3a_0}{4} \tag{56}$$

Which can be rewritten in the following integral form (See Appendix C) as:

$$V^{\Sigma}\left(r,a\right)=12\epsilon^3\,r^3\int_0^{tan\left(\frac{a}{2\epsilon r}\right)}dt\,\frac{t\;arctan\left(t\right)}{\left(3-t^2\right)\sqrt{2-t^2}}\,, \tag{57}$$

Notice that this integral has no analytic formula (we can carry the integration by using numerical methods) and can be expressed in terms of some special functions like the dilogarithm $Li_2(z)$, the Clausen of order 2 $Cl_2\left(\varphi\right)$ or the digamma $\Psi\left(x\right)$. It is easy to check that the expansion of the resulted formula (57) in terms of the $\frac{a}{\epsilon r}$ variable is exactly the one in Eq. (54) and thus ensuring the correctness of the integration.

1. For the spherical case S_r^3 $(\epsilon=1\Rightarrow R=\frac{6}{r^2})$, one has:

$$V^{S_r^3}\left(r,a\right)=12\;r^3\int_0^{tan\left(\frac{a}{2r}\right)}dt\,\frac{t\;arctan\left(t\right)}{\left(3-t^2\right)\sqrt{2-t^2}}\,, \tag{58}$$

The volume for a maximal edge length $V^{S_r^3}\left(r,a_{max}\right)$ (as it is expected) is half of the 3-dimensional cubic hyperarea of 3-sphere of radius r :

$$V^{S_r^3}\left(r,a_{max}\right)=\pi^2r^3=\frac{1}{2}Area\left(S_r^3\subset\mathbb{R}^4\right)\,, \tag{59}$$

Notice that for a given pair (r,a) the volume of a regular spherical tetrahedron is always greater than the regular Euclidean one.

2. For the hyperbolic case S_r^3 $(\epsilon = i \Rightarrow R = \frac{-6}{r^2})$, one has:

$$V^{H_r^3}(r,a) = 12\ r^3 \int_0^{tanh\left(\frac{a}{2r}\right)} dt\ \frac{t\ arctanh\,(t)}{(3+t^2)\sqrt{2+t^2}}\,, \tag{60}$$

has an upper bound :

$$\lim_{a\to\infty} V^{H_r^3}(r,a) \ = 1.0149416064096536250\ r^3\,, \tag{61}$$

$$= Im\left[Li_2\left(e^{i\frac{\pi}{3}}\right)\right] r^3 = \frac{\sqrt{6}}{3}\left(\Psi^1\left(\frac{1}{3}\right) - \frac{2}{3}\pi^2\right) r^3 \ = Cl_2\left(\frac{\pi}{3}\right) r^3\ , \tag{62}$$

Notice that for a given pair (r, a) the volume of a regular hyperbolic tetrahedron is always smaller than the regular Euclidean one.

3. For the Euclidean case Euc^3 $(R = 0)$, one has:

$$V^{Euc^3}(r,a) = \lim_{r\to\infty} V^{\Sigma}(r,a) \ = \frac{\sqrt{2}}{12} a^3\,, \tag{63}$$

The Euclidean limit is well-defined.

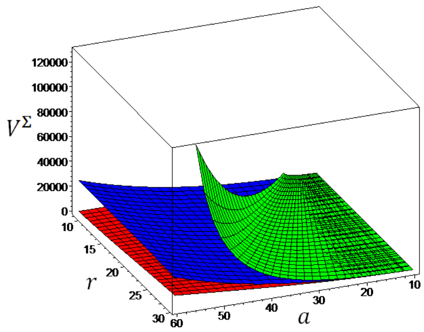

FIG. 4. Function surface of regular tetrahedron volume for spherical (green), Euclidean (blue) and hyperbolic (red) cases.

IV.3. The volume-area ratio function

We define the volume-area ratio function VRA^{Σ} for a regular geodesic tetrahedron as:

$$VRA^{\Sigma}(r,a) = \frac{V^{\Sigma}(r,a)}{(A^{\Sigma}(r,a))^{\frac{3}{2}}}\,, \tag{64}$$

It is obvious that the VRA^{Σ} for a regular Euclidean tetrahedron is a constant:

$$VRA^{Euc^3} = \lim_{r\to\infty} VRA(r,a) \ = \frac{\sqrt{2}}{12\left(\frac{\sqrt{3}}{4}\right)^{\frac{3}{2}}} = 0.4136\,, \tag{65}$$

Corollary IV.0.1 *according to the useful inequality*

$$VRA^{H_r^3}(r,a) \le VRA^{Euc^3}(r,a) \le VRA^{S_r^3}(r,a)\,, \tag{66}$$

the VRA^{Σ} *function allows us to know what kind of geometry inside the regular geodesic tetrahedron: (see Fig. 5)*

$$\begin{cases} VRA^{\Sigma}(r,a) > 0.4136 & S_r^3 \\ VRA^{\Sigma}(r,a) = 0.4136 & Euc^3 \\ VRA^{\Sigma}(r,a) < 0.4136 & H_r^3 \end{cases}, \tag{67}$$

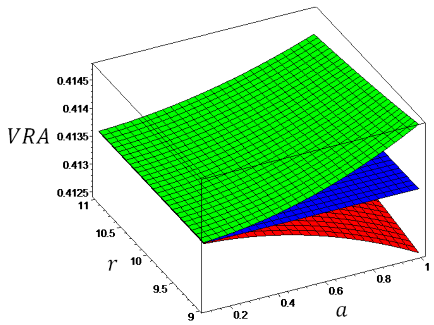

FIG. 5. The volume-area ratio function for spherical (green), Euclidean (blue) and hyperbolic (red) cases.

IV.4. The volume function in terms of scalar curvature and area

From the area formula (49), one can express the edge length a by:

$$a(A,R) = \left(\pi - \arccos\left(\frac{\sin(\frac{-\pi}{6} + \frac{A}{3\epsilon^2 r^2})}{\sin(\frac{-\pi}{6} + \frac{A}{3\epsilon^2 r^2}) + 1}\right)\right)\epsilon r\,, \tag{68}$$

substitute it in Eq. (57) to get a volume function in terms of the 3d- Ricci scalar curvature and boundary face area of a regular tetrahedron:

$$V^{\Sigma} = V^{\Sigma}\left(R, a\left(R, A\right)\right) = V^{\Sigma}\left(R, A\right)\,, \tag{69}$$

Corollary IV.0.2 *the volume of a regular geodesic tetrahedron for a fixed boundary area satisfies the following inequality*

$$For\ any\quad R_1, R_2 \in \mathbb{R}\quad if\quad R_1 < R_2\quad then\quad V^{\Sigma}(R_1, A) < V^{\Sigma}(R_2, A)\,, \tag{70}$$

this results from the fact that the function V^{Σ} *increases with respect to* R *for a fixed area norm* A *(see Fig. 6).*

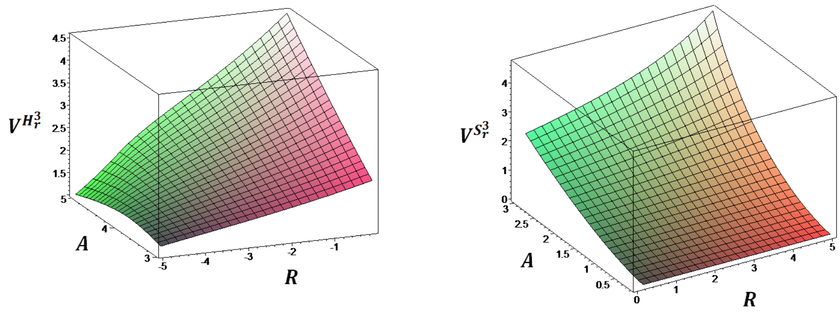

FIG. 6. The volume function in terms of scalar curvature R and area A for spherical (right) and hyperbolic (left) regular tetrahedron.

V. CONCLUSION

In this paper, we explicitly derived the boundary face area and volume of a regular spherical and hyperbolic tetrahedron in terms of the curvature radius (or the scalar curvature) and the edge length. We have directly performed the integration over the area and volume elements by using the Cayley-Klein-Hilbert coordinates system (CKHcs) to end up with simple formula given in Eqs. (49,57). A comparison between the Euclidean, spherical and hyperbolic cases is studied and their implications are discussed. It is shown that the volume function of a regular geodesic tetrahedron for a fixed boundary face area is a strictly increasing in the scalar curvature interval.

Appendix A: Proof of the relation (2)

The geodesics in the CKHcs $\{U \subset M, \vec{x}\}$ are straight lines, one has:

$$\ddot{x}^A = 0\,, \tag{A1}$$

The condition

$$\Gamma^A_{BC}(x)\dot{x}^B\dot{x}^C = 0\,, \tag{A2}$$

must be hold, which implies:

$$\Gamma^A_{BC}(x)\frac{\partial\varphi^B(\tilde{x})}{\partial\tilde{x}^I}\frac{\partial\varphi^C(\tilde{x})}{\partial\tilde{x}^J}\dot{\tilde{x}}^I\dot{\tilde{x}}^J = 0\,, \tag{A3}$$

Under the transformation (1), the Christoffel symbols transform as:

$$\Gamma^A_{BC}(x) = \frac{\partial\tilde{x}^J}{\partial x^B}\frac{\partial\tilde{x}^K}{\partial x^C}\frac{\partial\varphi^A}{\partial\tilde{x}^I}\tilde{\Gamma}^I_{JK}(\tilde{x}) - \frac{\partial\tilde{x}^J}{\partial x^B}\frac{\partial\tilde{x}^K}{\partial x^C}\frac{\partial^2\varphi^A}{\partial\tilde{x}^J\partial\tilde{x}^K}\,, \tag{A4}$$

By substituting it in Eq. (A3), one can obtain the transformation condition Eq. (2) to the ideal CKHcs frame.

Appendix B: Proof of the area formula

The boundary face area $(P_1P_2P_3)$ of a regular spherical and hyperbolic tetrahedron of an edge length a is given by an integral form in Eq. (48). For simplicity, we drop the triangle face $P_1P_2P_3$ to $\Pi(P_1P_2P_3)$ in the XY-plane (since the area of a fixed triangle is the same wherever its location inside the constant curvature manifold). In this case, the induced Jacobian can be written as:

$$\sqrt{|det(g(x)|_{\Pi(P_1P_2P_3)})|} = \frac{\epsilon^3 r^3}{(\epsilon^2 r^2 + x^2 + y^2)^{3/2}}, \tag{B1}$$

The boundary face area is given by:

$$A^{\Sigma}(r,a) = 2\int_0^{\frac{a_0}{2}} dx \int_{\frac{-\sqrt{3}a_0}{6}}^{-\sqrt{3}x+\frac{\sqrt{3}a_0}{3}} dy \; \frac{\epsilon^3 r^3}{(\epsilon^2 r^2 + x^2 + y^2)^{3/2}}, \tag{B2}$$

where one can check the starting Euclidean length a_0 in this case is given by:

$$a_0 = 2\epsilon r \frac{\tan(\frac{a}{2\epsilon r})}{\sqrt{1 - \frac{1}{3}\tan(\frac{a}{2\epsilon r})^2}}, \tag{B3}$$

Performing the Integral over y variable, one get:

$$\int_{\frac{-\sqrt{3}a_0}{6}}^{-\sqrt{3}x+\frac{\sqrt{3}a_0}{3}} dy \; \frac{\epsilon^3 r^3}{(\epsilon^2 r^2 + x^2 + y^2)^{3/2}} = \frac{\epsilon^3 r^3(-\sqrt{3}x + \frac{\sqrt{3}a_0}{3})}{(\epsilon^2 r^2 + x^2)\sqrt{\epsilon^2 r^2 + x^2 + (-\sqrt{3}x + \frac{\sqrt{3}a_0}{3})^2}} + \frac{\epsilon^3 r^3 \frac{\sqrt{3}a_0}{6}}{(\epsilon^2 r^2 + x^2)\sqrt{\epsilon^2 r^2 + x^2 + \frac{a_0^2}{12}}}, \tag{B4}$$

Let us preform the second integral over the x variable. By integrating each term separately, one has:

$$t_1(x) = \int_0^{\frac{a_0}{2}} dx \frac{\epsilon^3 r^3(-\sqrt{3}x + \frac{\sqrt{3}a_0}{3})}{(\epsilon^2 r^2 + x^2)\sqrt{\epsilon^2 r^2 + x^2 + (-\sqrt{3}x + \frac{\sqrt{3}a_0}{3})^2}} = \epsilon^2 r^2 \arctan(\frac{F(a_0, r; x)}{G(a_0, r; x)}), \tag{B5}$$

where

$$F(a_0, r; x) = -\frac{\sqrt{3}}{3}\sqrt{\epsilon^2 r^2 + 4x^2 - 2xa_0 + \frac{a_0^2}{3}}(\epsilon^2 r^2 + \frac{a_0^2}{9})(-\epsilon^2 r^2 + \frac{a_0 x}{3}) - a_0(\frac{5\epsilon^2 r^2}{9} + \frac{a_0^2}{27})(\epsilon^2 r^2 + x^2) + \frac{a_0^4 x}{81} + r^4 x + \frac{2a_0^2\epsilon^2 r^2 x}{9}, \tag{B6}$$

and

$$G(a_0, r; x) = \frac{\epsilon r\sqrt{3}}{3}\sqrt{\epsilon^2 r^2 + 4x^2 - 2xa_0 + \frac{a_0^2}{3}}(\epsilon^2 r^2 + \frac{a_0^2}{9})(x + \frac{a_0}{3}) + \frac{2a_0^2\epsilon r x^2}{27} + \frac{\epsilon^5 r^5}{3} - \frac{4a_0^2\epsilon^3 r^3}{27} - \frac{a_0^4\epsilon r}{81} + \frac{4\epsilon^3 r^3 x^2}{3}, \tag{B7}$$

$$t_2(x) = \int_0^{\frac{a_0}{2}} dx \frac{\epsilon^3 r^3 \frac{\sqrt{3}a_0}{6}}{(\epsilon^2 r^2 + x^2)\sqrt{\epsilon^2 r^2 + x^2 + \frac{a_0^2}{12}}} = \epsilon^2 r^2 \arctan\left(\frac{a_0 x}{\epsilon r \sqrt{12\epsilon^2 r^2 + 12x^2 + a_0^2}}\right), \tag{B8}$$

Adding the two terms together, we obtain:

$$A^{\Sigma}(r,a) = 2(t_1(x) + t_2(x))|_{x=0}^{x=a_0/2} =$$
$$2\epsilon^2 r^2 \arctan\left(\frac{9a_0^2 \epsilon r(3a_0^2 - \sqrt{3}a_0\sqrt{9\epsilon^2 r^2 + 3a_0^2} + 18\epsilon^2 r^2)}{3a_0^5 - 63a_0^3\epsilon^2 r^2 - 216 a_0 r^4 + \sqrt{3}\sqrt{9\epsilon^2 r^2 + 3a_0^2}(18a_0^2\epsilon^2 r^2 + 144 r^4 - a_0^4)}\right) \tag{B9}$$

When we replace a_0 given in Eq. (B3), we get the area function formula of Eq. (49).

Appendix C: Proof of the volume formula

The volume of a regular spherical and hyperbolic tetrahedron of an edge length a is given by an integral form in Eq. (55). Using the integration by shell method (taking the sum of parallel triangles of constant z). Performing the Integral over the y variable, one get:

$$\int_{\frac{-\sqrt{3}\alpha(z)}{6}}^{-\sqrt{3}x + \frac{\sqrt{3}\alpha(z)}{3}} dy \frac{r^4}{\left(\epsilon^2 r^2 + x^2 + y^2 + \frac{\alpha(z)^2}{24}\right)^2} =$$
$$\frac{32\sqrt{3}\, r^4\ (-3x + \alpha(z))}{\left(32x^2 - 16\alpha(z)x + 8\epsilon^2 r^2 + 3\alpha(z)^2\right)\left(24x^2 + 24\epsilon^2 r^2 + \alpha(z)^2\right)} + \frac{24\sqrt{6}\, r^4\ arctan\left(\frac{2\sqrt{2}(-3x+\alpha(z))}{\sqrt{24x^2+24\epsilon^2 r^2+\alpha(z)^2}}\right)}{\left(24x^2 + 24\epsilon^2 r^2 + \alpha(z)^2\right)}$$
$$+ \frac{48\sqrt{3}\, r^4 \alpha(z)}{\left(24x^2 + 24\epsilon^2 r^2 + 3\alpha(z)^2\right)\left(24x^2 + 24\epsilon^2 r^2 + \alpha(z)^2\right)} + \frac{24\sqrt{6}\, r^4\ arctan\left(\frac{\sqrt{2}\alpha(z)}{\sqrt{24x^2+24\epsilon^2 r^2+\alpha(z)^2}}\right)}{\left(24x^2 + 24\epsilon^2 r^2 + \alpha(z)^2\right)^{\frac{3}{2}}}, \tag{C1}$$

Now, let us focus on the second integral over the x variable. By integrating each term separately, one has:

$$T_1\ (x) = \int dx \frac{32\sqrt{3}\, r^4\ (-3x + \alpha(z))}{\left(32x^2 - 16\alpha(z)x + 8\epsilon^2 r^2 + 3\alpha(z)^2\right)\left(24x^2 + 24\epsilon^2 r^2 + \alpha(z)^2\right)} =$$
$$\frac{-6\sqrt{3}\, r^4\ ln\left(32x^2 - 16\alpha(z)x + 8\epsilon^2 r^2 + 3\alpha(z)^2\right)}{72\epsilon^2 r^2 + 11\alpha(z)^2} - \frac{8\sqrt{3}\, r^4 \alpha(z) arctan\left(\frac{8x - 2\alpha(z)}{\sqrt{16\epsilon^2 r^2 + 2\alpha(z)^2}}\right)}{\left(72\epsilon^2 r^2 + 11\alpha(z)^2\right)\sqrt{16\epsilon^2 r^2 + 2\alpha(z)^2}}$$
$$+ \frac{6\sqrt{3}\, r^4\ r^4\ ln\left(24x^2 + 24\epsilon^2 r^2 + \alpha(z)^2\right)}{72\epsilon^2 r^2 + 11\alpha(z)^2} + \frac{48\sqrt{3}\, r^4 \alpha(z) arctan\left(\frac{12x}{\sqrt{144\epsilon^2 r^2 + 6\alpha(z)^2}}\right)}{\left(72\epsilon^2 r^2 + 11\alpha(z)^2\right)\sqrt{144\epsilon^2 r^2 + 6\alpha(z)^2}}, \tag{C2}$$

$$T_2\ (x) = \int dx \frac{24\sqrt{6}\ r^4\ arctan\left(\frac{2\sqrt{2}(-3x+\alpha(z))}{\sqrt{24x^2+24\epsilon^2r^2+\alpha(z)^2}}\right)}{\left(24x^2+24\epsilon^2r^2+\alpha(z)^2\right)} =$$

$$\frac{48\sqrt{6}\ r^4\sqrt{8\epsilon^2r^2+\alpha(z)^2}\ \arctan\left(\frac{\sqrt{2}(4x-\alpha(z))}{\sqrt{8\epsilon^2r^2+\alpha(z)^2}}\right)}{\left(24\epsilon^2r^2+\alpha(z)^2\right)\left(72\epsilon^2r^2+11\alpha(z)^2\right)} - \frac{6\sqrt{3}\ r^4\ ln\left(24x^2+24\epsilon^2r^2+\alpha(z)^2\right)}{\left(72\epsilon^2r^2+11\alpha(z)^2\right)}$$

$$+\frac{6\sqrt{3}\ r^4\ ln\left(96x^2-48\alpha(z)x+24\epsilon^2r^2+9\alpha(z)^2\right)}{\left(72\epsilon^2r^2+11\alpha(z)^2\right)} - \frac{24\sqrt{2}\ arctan\left(\frac{\sqrt{24}\ x^2}{\sqrt{24\epsilon^2r^2+\alpha(z)^2}}\right)}{\left(72\epsilon^2r^2+11\alpha(z)^2\right)\sqrt{24\epsilon^2r^2+\alpha(z)^2}}$$

$$+\frac{24\sqrt{6}\ r^4\ x\ arctan\left(\frac{2\sqrt{2}(-3x+\alpha(z))}{\sqrt{24x^2+24\epsilon^2r^2+\alpha(z)^2}}\right)}{\left(24\epsilon^2r^2+\alpha(z)^2\right)\sqrt{24x^2+24\epsilon^2r^2+\alpha(z)^2}}, \tag{C3}$$

$$T_3\ (x) = \int dx\ \frac{48\sqrt{3}\ r^4\alpha(z)}{\left(24x^2+24\epsilon^2r^2+3\alpha(z)^2\right)\left(24x^2+24\epsilon^2r^2+\alpha(z)^2\right)} =$$

$$\frac{6\sqrt{2}\ r^4\ arctan\left(\frac{2\sqrt{6}\ x}{\sqrt{24\epsilon^2r^2+\alpha(z)^2}}\right)}{\alpha(z)\sqrt{24\epsilon^2r^2+\alpha(z)^2}} - \frac{2\sqrt{6}\ r^4\ arctan\left(\frac{2\sqrt{2}\ x}{\sqrt{8\epsilon^2r^2+\alpha(z)^2}}\right)}{\alpha(z)\sqrt{8\epsilon^2r^2+\alpha(z)^2}}, \tag{C4}$$

$$T_4\ (x) = \int dx\ \frac{24\sqrt{6}\ r^4\ arctan\left(\frac{\sqrt{2}\alpha(z)}{\sqrt{24x^2+24\epsilon^2r^2+\alpha(z)^2}}\right)}{\left(24x^2+24\epsilon^2r^2+\alpha(z)^2\right)^{\frac{3}{2}}} =$$

$$\frac{6\sqrt{6}\ r^4\sqrt{8\epsilon^2r^2+\alpha(z)^2}\ arctan\left(\frac{2\sqrt{2}\ x}{\sqrt{8\epsilon^2r^2+\alpha(z)^2}}\right)}{\left(24\epsilon^2r^2+\alpha(z)^2\right)} + \frac{24\sqrt{6}\ r^4x\ arctan\left(\frac{\sqrt{2}\alpha(z)}{\sqrt{24x^2+24\epsilon^2r^2+\alpha(z)^2}}\right)}{\left(24\epsilon^2r^2+\alpha(z)^2\right)\sqrt{24x^2+24\epsilon^2r^2+\alpha(z)^2}}$$

$$-\frac{6\sqrt{2}\ r^4\ arctan\left(\frac{2\sqrt{6}\ x}{\sqrt{24\epsilon^2r^2+\alpha(z)^2}}\right)}{\alpha(z)\sqrt{24\epsilon^2r^2+\alpha(z)^2}}, \tag{C5}$$

Adding all four terms together, we obtain:

$$2\left(T_1\ (x)+T_2\ (x)+T_3\ (x)+T_4\ (x)\right)\Big|_{x=0}^{x=\alpha(z)/2} = \frac{24\sqrt{6}\ r^4\ \alpha(z)\ arctan\left(\frac{\sqrt{2}\alpha(z)}{\sqrt{8\epsilon^2r^2+\alpha(z)^2}}\right)}{\left(24\epsilon^2r^2+\alpha(z)^2\right)\sqrt{8\epsilon^2r^2+\alpha(z)^2}}, \tag{C6}$$

Making the following change of variable in the third integral over z:

$$t = \frac{\sqrt{2}\alpha(z)}{\sqrt{8\epsilon r^2+\alpha(z)^2}}, \tag{C7}$$

When we replace a_0 given in Eq. (40), we get the volume function formula of Eq. (57).

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A curvature operator for a regular tetrahedron shape in LQG

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An alternative approach introducing a 3-dimensional (3D) Ricci scalar curvature quantum operator given in terms of volume and area as well as new edge length operators is proposed. An example of monochromatic 4-valent node intertwiner state (equilateral tetrahedra) is studied and the scalar curvature measure for a regular tetrahedron shape is constructed. It is shown that all regular tetrahedron states are in the negative scalar curvature regime and for the semi-classical limit the spectrum is very close to the Euclidean regime.

Keywords: Loop quantum gravity; curvature operator; quantum polyhedra.

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1. Introduction

Loop Quantum Gravity (LQG) [1, 2] is a background-independent quantum field theory, it has been described as the best way to build a consistent quantum version of General Relativity. Canonically, it is based on the implementation of the Holst action [3] and the Ashtekar–Barbero variables (the configuration variable is the real $su(2)$ connection $A_a^i(x)$ and its canonical conjugate is the gravitational electric field $E_j^b(x)$ with a real Immirzi parameter γ [4, 5] by the Dirac quantization procedure [6]. In order to construct the starting kinematical Hilbert space, one has to use the well-known representation of the holonomy-flux algebra [7]: it is represented by the space of all cylindrical wave functionals through holonomies defined by the $su(2)$ connection along a system of smooth oriented paths and flux variables as the smeared electric field along the dual surface for each path. Due to the background-independent property of LQG, it was possible to use Wilson loops [8] which are the natural gauge invariant holonomy of the gauge connection

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as a basis for the gauge invariant Hilbert space [9]. Another useful basis state of the quantum geometry known as the Penrose's spin networks is frequently used [10]. Spin network arises as a generalization of Wilson loops necessary to deal with mutually intersecting loops "nodes" which are represented by a space of intertwiners at each node [11]. One can construct well-defined operators such as the area and volume acting on links and nodes, respectively, of smooth paths system [12]. The fuzziness and discreteness property of space [13–15] is predicted. A beautiful interpretation of the intertwiners in terms of quantum Euclidean polyhedra [16, 17] naturally arises. In this work, we construct a new geometrical information from LQG spin network based on the polyhedra interpretation of spin network states, which is the value of the 3D Ricci scalar curvature and the edge length as a function of volume and boundary areas operators. A suggested introduction to the curvature operator in terms of the length operator and the dihedral angles was provided by using 3D Regge calculus [18]. Moreover, there are three proposals for length operators discussed in [9, 20, 21]. The main idea of our work comes from the determination of the volume and the boundary area of a fixed region in a Riemannian manifold as a function of the 3d scalar curvature inside that region as well as its parameterization. One can invert these functions to get the explicit formula of the 3d scalar curvature in terms of volume and boundary area of a fixed region. Similar idea can be done using a geodesic polyhedron shape[a] [28]. One can use the new proposed scalar curvature operator related to a fixed polyhedron measure and try to determine the curvature in which the intertwiner state is represented. This geometrical approach can be considered as a natural arena for considering LQG including a cosmological constant. in the case $\Lambda \neq 0$, The $SU(2)$ gauge invariant is still representing the kinematical space of LQG (since the cosmological constant just appears in the Hamiltonian constraint). Thus, one can describe the intertwiner state by a curved chunk of a curved polyhedron and then the main feature of our proposed curvature operator is to determine in a straightforward manner which curvature value can an intertwiner state interpret as a fixed geodesic polyhedron. Moreover, a proposal to introduce a non-vanishing cosmological constant in LQG is to work with the q-deformed $\mathcal{U}_q(su(2))$ rather than the $su(2)$ itself [11, 22–25] and the use of curvature tetrahedron was suggested in [26]. In our approach, an example of such a monochromatic 4-valent node state was studied in detail and its associated Kapovich–Millson phase space (i.e. the space of all equilateral Euclidean tetrahedron shapes) was constructed. Moreover, we will show the absence of a regular Euclidean tetrahedron from the volume orbit of relevant shapes in that phase space, instead of this it is possible to find a regular tetrahedron correspondence in the context of a nonzero curvature tetrahedron. It is worth to mention that the

[a]Geodesic polyhedron is the convex region enclosed by the intersection of geodesic surfaces. A geodesic surface is a surface with vanishing extrinsic curvature and the intersection of two such surfaces is necessarily a geodesic curve.

phase space of curved tetrahedron shapes idea has been initiated in [27]. In our present paper, full expressions of volume and boundary face area of a regular tetrahedron in a constant curvature space (in terms of the scalar curvature and the edge length [28]) are explicitly derived then inverted to get the exact form of the 3D Ricci scalar curvature and the edge length. At the quantum level, we obtain two well-defined operators acting on the monochromatic 4-valent nodes state. Their spectra show that all quantum atoms of space can be represented by chunks of regular hyperbolic tetrahedron of a negative curvature $R \sim -(8\pi G h\gamma)^{-1}$. It also produces the Euclidean regular tetrahedron $R \sim 0$ in the semi-classical limit $j \gg 1$ (j is links color). The importance of this mathematical model was investigated by the authors of [32]. In what follows, we will work in a unit, where $8\pi G h\gamma = 1$. The paper is organized as follows: In Sec. 2, we give a motivation for a new scalar curvature measure. In Sec. 3, a strategy of defining new curvature operator in LQG is presented. In Sec. 4, a 3d Ricci scalar curvature and edge length operators are constructed for a regular tetrahedron state. Finally in Sec. 5, we draw our conclusions.

2. Motivation for a 2D Scalar Curvature

The 2D Ricci scalar curvature in some point of the 2D hypersurface Σ_t embedded in a smooth 3D Riemannian manifold M is technically determined by the measure of volume and boundary area of a neighborhood region around this point. Doing it separately does not give enough geometrical informations of the space. Rather, it is mandatory to do this at the same time in order to get the complete information. To be more explicit, let us consider the simplest case of the 2-sphere $S^2_{r_0}$ of radius $r(t)$ n (see Fig. 1). The spatial

$$ds^2|_{\Sigma_t} = r(t)(d\theta^2 + \sin^2(\varphi)). \tag{1}$$

At $t = t_0$, we want to measure the 2D Ricci scalar curvature R_{t_0} such that $r(t_0) = r_0$. This means we have to measure the radius r_0 (because $R_{t_0} = \frac{2}{r_0^2}$). To do so, we fix a region $\mathcal{D}_a(p)$ of a geodesic disc with a radius a centering at a point $p \in S^2_{r_0}$

$$\mathcal{D}_a(p) = \{q \in S^2_{r_0} \,|\, l_{pq} \le a\}, \tag{2}$$

where l_{pq} is the geodesic length of the $S^2_{r_0}$ space between the points p and q. The area of the disc $A(r_0, a)$ and its boundary curve length $L(r_0, a)$ are

$$A(r_0, a) = 2\pi r_0^2\left(1 - \cos\left(\frac{a}{r_0}\right)\right), \tag{3}$$

$$L(r_0, a) = 2\pi r_0 \sin\left(\frac{a}{r_0}\right). \tag{4}$$

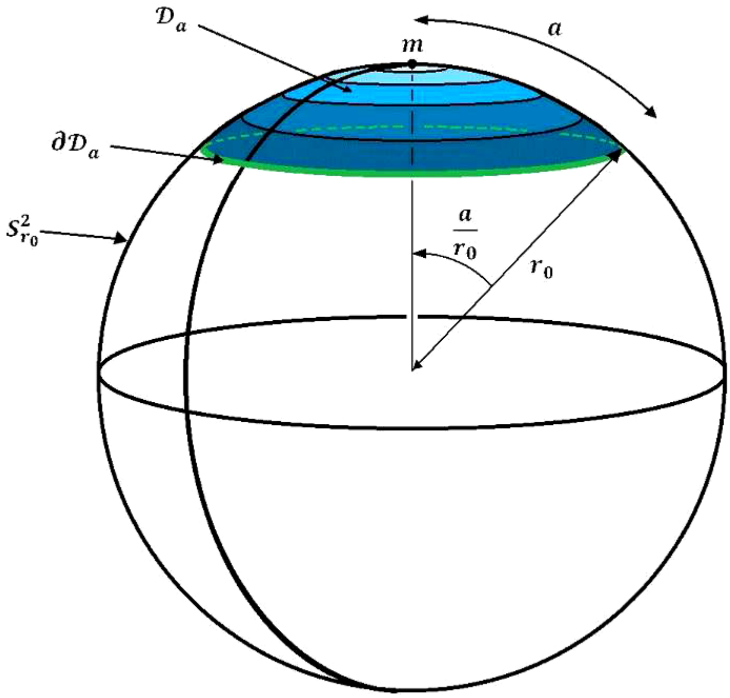


Fig. 1. (Color online) The geodesic disc $\mathcal{D}_a$ (blue) and its boundary circle $\partial\mathcal{D}_a$ (green) in the 2-sphere $S^2_{r_0}$.

Given the pair (r_0, a), one can determine the area of a disc and its boundary curve length (A, L). It is easy to invert these two functions to obtain

$$R_{t_0}(A, L) = \frac{2}{r_0} = \frac{2(4\pi A - L)}{A^2}. \tag{5}$$

$$a(A, L) = \frac{A}{\sqrt{4\pi - L^2}} \arctan\left(\frac{L\sqrt{4\pi A - L^2}}{2\pi A - L^2}\right). \tag{6}$$

Thus, The simultaneous measurement of the area and the boundary curve length of a geodesic disc can allow us to estimate the value of the 2D Ricci scalar curvature $(R_{t_0} = \frac{2}{r_0^2})$ and the disc radius a.

In $2+1$ dimension and for the 2-sphere case, these two relations give us another way to measure the main important geometrical quantity which is the value of the 2D Ricci scalar curvature $R_{t_0}(A, L)$ as a function of the area measure and its boundary curve length of a disc. Remarkably, this technique does not depend on the choice of the region; one can choose any shape of a region and get the same 2D scalar curvature. But how can we generalize this technique for arbitrary 3D topological spaces? To get such a generalization, we try to find a relationship between the 3D Ricci scalar curvature with the measurement of volume and boundary area of an arbitrary region. It was done by using small geodesic ball [29], and for any arbitrary regular tetrahedron in a constant curvature spaces [28] which is the relevant one. The curvature can be determined by inverting the resulting functions in all cases.

3. Strategy for Defining a New Curvature Operator in LQG

In LQG, the $SU(2)$ invariant Hilbert space at each F-valent node is the intertwiner space $\mathcal{H}_F \equiv \text{inv}(V^{(j_1)} \otimes \cdots V^{(j_F)})$. Since the geometry is a genuine quantum theory, the finer details of the geometrical picture remain unclear. A suggested solution of the quantum picture came in the proposal that vertices correspond to polyhedra in flat space. In fact, there is a close correspondence between the intertwiner space $\mathcal{H}_F$ and the quantization of the Kapovich–Millson phase space $\mathcal{S}_F$, i.e. the space of all Euclidean polyhedron shapes with fixed F-areas norms $\{A_f \sim j_f\}$, $f = 1, \ldots, F$. This correspondence allows us to interpret each atom of space on a node (volume eigenstate) as quantum Euclidean polyhedra states. It offers infinite possible Euclidean polyhedra shapes for the same intertwiner state. In fact, after restricting the space of shapes of fixed areas A_f to a spectrum of volume operator, we will obtain $2F-5$ dimensions hyper-surface of relevant shapes (since the $\mathcal{S}_F$ phase space has $2F-6$ dimensions). Now, it is legitimate to ask the following question:

- Can we find other polyhedra shape possibilities in the nonzero curvature regime? For instance, the absence of the regular Euclidean tetrahedron correspondence with the monochromatic 4-valent node intertwiners (except for the semiclassical limit, the state is really regular tetrahedron) means that there is no regular tetrahedron belonging to the volume orbits in the space of equilateral tetrahedra shapes; can we find this correspondence in the context of nonzero curvature spaces?

In what follows, we will interpret the intertwiner state by a fixed polyhedron shape (even if it doesn't belong to the volume orbit of Euclidean polyhedra shapes) and try to find out what kind of a curved space one must have in order that this polyhedron grain is nicely consistent with the area and volume spectra of LQG. The task now is to determine new curvature operator related to a fixed polyhedron shape by using the approach similar to the one mentioned previously consisting in identifying the volume and areas operators of LQG with those of the corresponding polyhedron in an arbitrary curved space and inverting the resulting set of functions to end up to the classical and quantum formula of scalar curvature related to a fixed polyhedron. It is worth to mention that the classical consistency of the 3D Ricci scalar curvature measure as a function of the volume and boundary area measures is also well-defined at the quantum level since the commutativity between their associated geometrical operators is guaranteed in LQG. Unfortunately, we cannot exactly calculate the volume and boundary face area of a polyhedron in a general curved space, even if we make a perturbative series expansion around the Euclidean measure for a small polyhedron as it was mentioned for the small geodesic ball cases [29], we don't have any guidance to estimate the uncertainty of this expansion. The first problem occurred due to the arbitrary degree of freedom of the considered general curved space. The solution is trivial; one can just relax the degree of freedom to spaces with a constant scalar curvature (one degree of freedom). In fact, a spin network state of a fixed graph (dual to a fixed discretization) induces naturally a discrete locally valued function of the 3D Ricci scalar curvature. The reason is that all quantum geometric operators are not sensitive to all points inside the quantum atom of space; only nodes and links represent the quanta of space and its boundary surface, respectively. Thus, each quantum atom of space corresponds to a constant 3D Ricci scalar curvature value, i.e. all points inside the quantum atom of space share the same geometrical property. In the following, we will make our calculation concerning the volume and boundary area of a polyhedron in a constant curvature Riemannian manifold. We remind that the Riemannian manifolds of a constant curvature can be classified into the Euclidean $(E^3, R = 0)$, spherical $(S^3, R > 0)$ and hyperbolic $(H^3, R < 0)$ geometries (other spaces that have a constant curvature are isometric to one of these three classes by the Killing–Hopf theorem [30, 31]). As a byproduct, the full expression of volume and boundary face area of a regular tetrahedron in the 3-sphere S^3 and the 3-hyperbolic H^3 has been derived explicitly in terms of the 3D Ricci scalar curvature and the edge length in [28]. In the monochromatic 4-valent node example, we will be interested to study the possibility of finding a correspondence with a regular geodesic tetrahedron. Applying the 3D Ricci scalar curvature operator related to a regular tetrahedron region on the intertwiner state for constructing a space of a constant curvature where one can have the regular tetrahedron correspondence for any irreducible representation j. Finally, since the semiclassical intertwiner state is identical with the regular tetrahedron, a crucial test of this 3D scalar curvature is to re-find the regular tetrahedron in the semiclassical limit as we really expect.

4. Application: A Monochromatic 4-Valent Node State

4.1. *Quantum equilateral Euclidean tetrahedron*

The corresponding system of a monochromatic 4-valent intertwiner node is an equilateral Euclidean tetrahedron (tetrahedron with faces of equal areas, see Fig. 2) and the main ingredients that comprise this system can be summarized as follows.

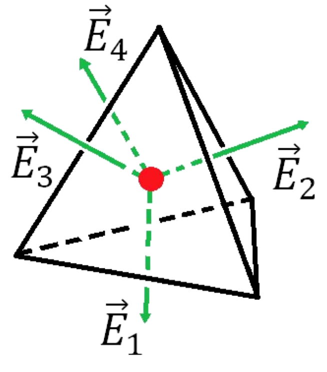


Fig. 2. Descriptions of the classical geometry of an equilateral Euclidean tetrahedron.

4.1.1. *The intertwiner space* $\mathcal{H}_4$

In LQG, the $SU(2)$ invariant Hilbert space of a monochromatic 4-valent node ($j_1 = j_2 = j_3 = j_4 = j$) is the intertwiner space $\mathcal{H}_4 \equiv \mathrm{inv}(V^{(j)} \otimes V^{(j)} \otimes V^{(j)} \otimes V^{(j)})$ with a dimension $3j+1$. There are two well-defined geometric operators acting on the gauge invariant intertwiner state $\{|i_K^{(j)}\rangle\}, K = 0, \ldots, 2j$:

The area operator acts trivially on the links as

$$\hat{A}_f|i_K^{(j)}\rangle = \sqrt{|\overrightarrow{\hat{E}}_f|^2}\,|i_K^{(j)}\rangle = \sqrt{j(j+1)}|i_K^{(j)}\rangle. \tag{7}$$

The volume operator acts nontrivially on the node [17]

$$\hat{V}|i_K^{(j)}\rangle = \frac{2}{\sqrt{3}}\sqrt{|\overrightarrow{\hat{E}}_1 \cdot (\overrightarrow{\hat{E}}_2 \times \overrightarrow{\hat{E}}_3)|}|i_K^{(j)}\rangle \equiv \frac{2}{\sqrt{3}}\sqrt{\hat{Q}}|i_K^{(j)}\rangle. \tag{8}$$

We have to diagonalize the volume matrix element by diagonalizing the matrix $[Q_{K'K}^{(j)}]$ of elements

$$Q_{K'K}^{(j)} = \langle i_{K'}^{j}|\hat{Q}|i_K^{(j)}\rangle \tag{9}$$

with

$$[Q_{K'K}^{(j)}] = \begin{pmatrix} 0 & ia_1 & \cdots & & 0 & \\ -ia_1 & 0 & & & & \\ \vdots & & \ddots & & \vdots & \\ & & & 0 & & ia_{2j+1} \\ 0 & & \cdots & & & \\ & & & -ia_{2j+1} & & 0 \end{pmatrix}, \tag{10}$$

where

$$a_n = \frac{n^2}{4}\frac{n^2 - (2j+1)^2}{\sqrt{4n^2-1}}, \quad n = 1, \ldots, 2j+1. \tag{11}$$

4.1.2. *The Kapovich–Millson phase space* $\mathcal{S}_4$

The space of all Euclidean equilateral tetrahedron [17] shapes with fixed area norms $A_1 = A_2 = A_3 = A_4 = A = \sqrt{2j+1}$, satisfying the closure relation:

$$\overrightarrow{A}_1 + \overrightarrow{A}_2 + \overrightarrow{A}_3 + \overrightarrow{A}_4 = \overrightarrow{0}. \tag{12}$$

The phase space canonical coordinates are

$$p = |\overrightarrow{A}_1 + \overrightarrow{A}_2| \quad q = \arccos\frac{(\overrightarrow{A}_1 \times \overrightarrow{A}_2)\cdot \overrightarrow{A}_3 \times \overrightarrow{A}_4)}{|\overrightarrow{A}_1 \times \overrightarrow{A}_2||\overrightarrow{A}_3 \times \overrightarrow{A}_4|}. \tag{13}$$

It is obvious that

$$0 \le p \le 2A \quad -\frac{\pi}{2} \le q \le \frac{\pi}{2}. \tag{14}$$

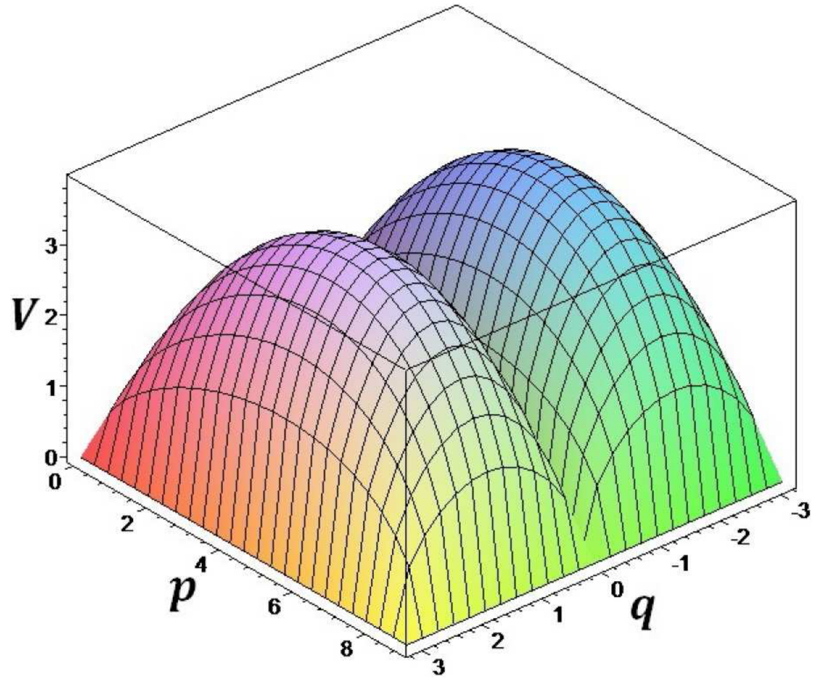


Fig. 3. The volume function in the Kapovich-Millson phase space $\mathcal{S}_4$.

All geometrical informations of a Euclidean equilateral tetrahedron can be constructed from its representation point $(p, q) \in \mathcal{S}_4$, such as the volume

$$V(A; p, q) = \frac{2}{\sqrt{3}}\sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|} = \frac{1}{3\sqrt{2}}\sqrt{|\sin(q)|\left(\frac{4A^2}{p^2} - 1\right)}. \tag{15}$$

Notice that the volume function has a maximal value as it is shown in Fig. 3. In fact, one has to solve the equations

$$\left.\frac{\partial V(A; p, q)}{\partial p}\right|_{(p_0,q_0)} = 0 \quad \left.\frac{\partial V(A; p, q)}{\partial q}\right|_{(p_0,q_0)}. \tag{16}$$

It is easy to check that

$$p_0 = \frac{2\sqrt{3}}{3}A \quad q_0 = \pm\frac{\pi}{2}, \tag{17}$$

where

$$V_{\max} = V(A; p_0, q_0) = 2^{3/2}3^{-7/4}A^{3/2} \tag{18}$$

which is the expected Euclidean regular tetrahedron.

4.1.3. *The correspondence* $\mathcal{H}_4 \leftrightarrow \mathcal{S}_4$

Each volume spectrum (8) of the intertwiner space $\mathcal{H}_4$ corresponds to an orbit in the Kapovich–Millson phase space $\mathcal{S}_4$. These volume orbits are the possible Euclidean

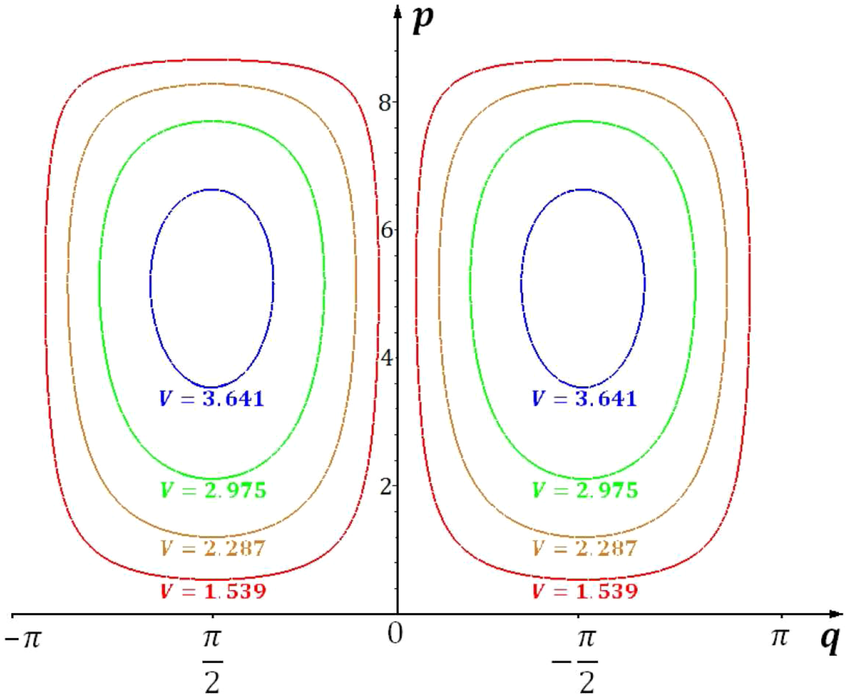


Fig. 4. The Kapovich–Millson phase space $\mathcal{S}_4$. The colored orbits are quantized levels of the volume operator in the monochromatic 4-valent eigenstate of $j = 4$.

equilateral tetrahedron shapes of the volume eigenstate with a fixed face area norm $A = \sqrt{j(j+1)}$ (see Fig. 4).

The regular tetrahedron is the only state that has the maximum volume value. Therefore, the only atom of space state that corresponds to a unique equilateral tetrahedron shape is the one that has a volume eigenvalue equal to the maximum volume of the phase space $\mathcal{S}_4$

$$V_{\max} = 2^{3/2}3^{-7/4}(j(j+1))^{3/4}. \tag{19}$$

and it corresponds to the regular tetrahedron. In LQG, there is no quantum regular tetrahedron corresponding to a monochromatic 4-valent node state, since all quantum volume spectra are below the volume of a regular tetrahedron with a face area $A = \sqrt{j(j+1)}$ (see Fig. 5). The existence of such a regular tetrahedron solution is guaranteed by the correspondence of the 4-valent node intertwiner space $\mathcal{H}_4$ with a constant curvature R tetrahedron shape.

4.2. *Ricci scalar curvature and edge length operators for regular tetrahedron state*

Now, let us look for the 3D Ricci scalar curvature value in which one can represent the monochromatic 4-valent quanta of space as a regular tetrahedron in a constant curvature space. In [28], the volume and the boundary face area of a regular spherical and hyperbolic tetrahedron given as explicit functions of the edge length a and

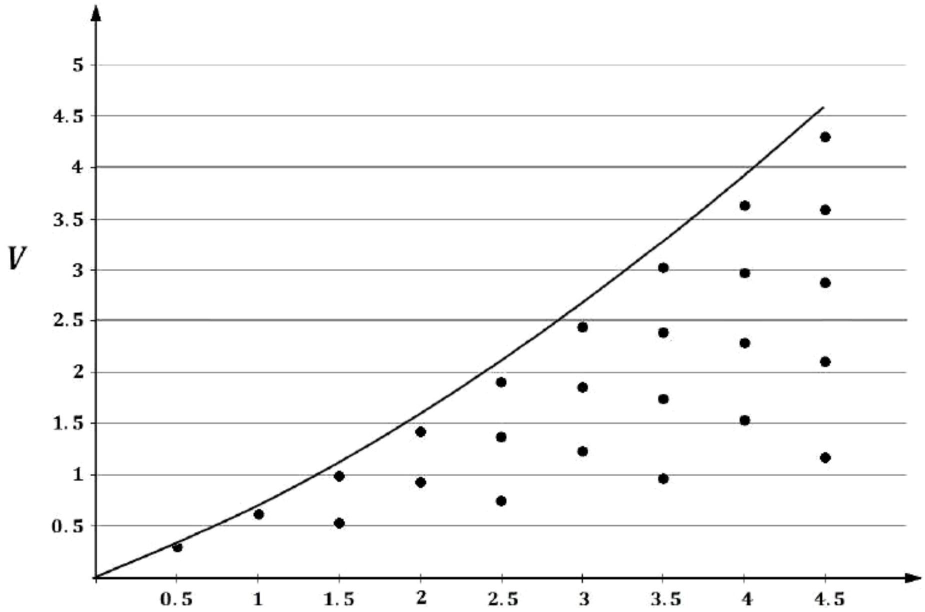


Fig. 5. Comparison of the regular Euclidean tetrahedron volume (dark line) with the LQG volume spectra (dots) for the monochromatic 4-valent node state with different links color j.

the radius $r = \sqrt{\frac{6}{|R|}}$ are shown to have the following expressions[b]:

$$A(r,a) = \epsilon^2 r^2 \left(3\arccos\left(\frac{\cos\left(\frac{a}{\epsilon r}\right)}{\cos\left(\frac{a}{\epsilon r}\right)+1}\right) - \pi\right) \tag{20}$$

$$V(r,a) = 12\epsilon^3 r^3 \int_0^{\tan\left(\frac{a}{2\epsilon r}\right)} dt \frac{t\arctan(t)}{(3-t^2)\sqrt{2-t^2}}, \tag{21}$$

where

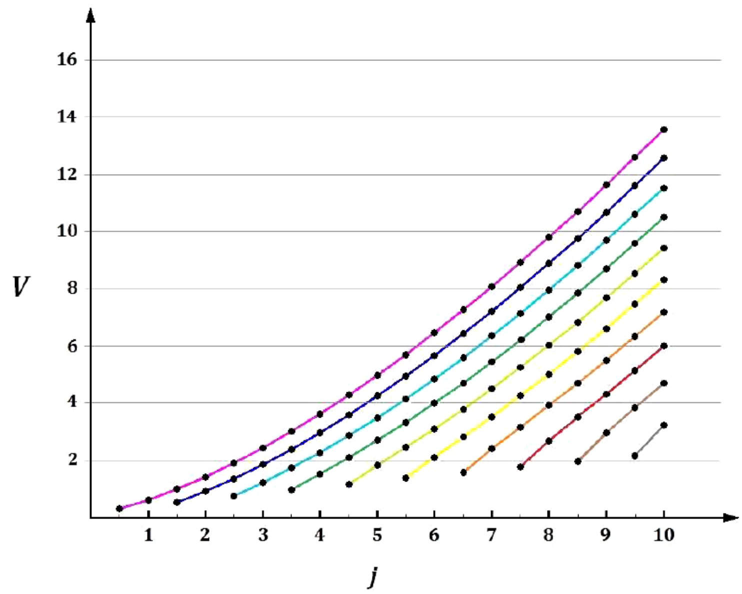
$$\epsilon = \begin{cases} 1 & \text{for } S_r^3, \\ i & \text{for } H_r^3. \end{cases} \tag{22}$$

The Euclidean case is well defined in the limit $r \to \infty$. A direct application of the resulted formulas (20), (21) in LQG is to find a 3D scalar curvature of the quantum atom of space such that the monochromatic 4-valent node has an interpretation of a regular tetrahedron in a constant curvature space. For each area and volume spectra of the operators (7), (8), inverting analytically these systems of functions is not so simple, instead we can deal with it numerically and construct the 3D Ricci scalar curvature and the edge length spectra. From the above Fig. 6, it is worth

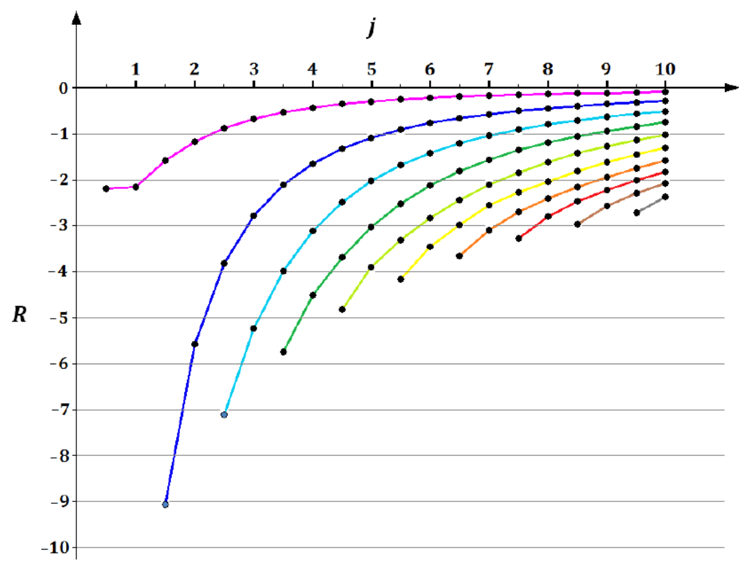
[b]Notice that the geodesic surfaces of the S_r^3 and H_r^3 are portions of the great 2-dimensional spheres S_r^2 and hyperbolic H_r^2, respectively. Indeed, the area expression (20) of a regular triangle is a combination of the area formula given by the dihedral angle Θ and the cosine rule $\cos(\Theta) = \frac{\cos(\frac{a}{\epsilon r})}{\cos(\frac{a}{\epsilon r})+1}$ in the context of spherical and hyperbolic trigonometry.

shedding light on the following main conclusions:

(1) The existence of a regular tetrahedron consistent with LQG data (volume and area spectra) is guaranteed in the negative curvature regime, and then one can represent the monochromatic 4-valent state by a regular hyperbolic tetrahedron.
(2) In general, the 4-valent monochromatic state that has a biggest volume represented by a regular tetrahedron in negative constant curvature space is

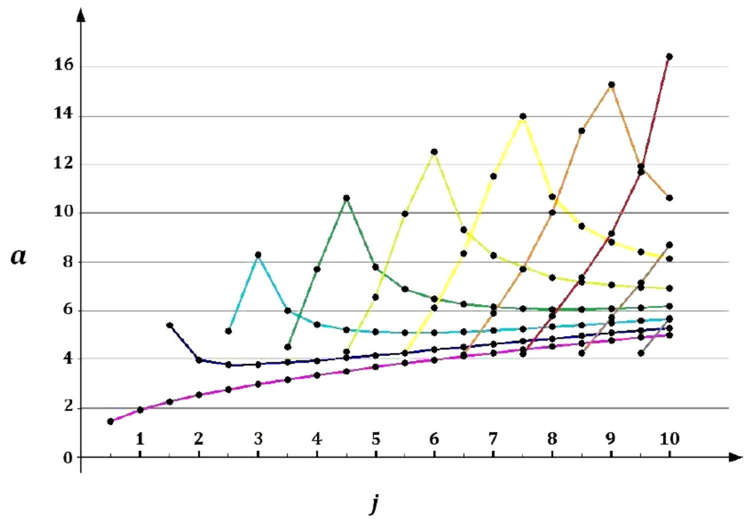


(a) The volume spectrum



(b) The curvature spectrum

Fig. 6. (Color online) Colored lines of different spectra levels for volume (a), scalar curvature (b) and edge length (c) of a monochromatic 4-valent intertwiner.



(c) The edge length spectrum

Fig. 6. (*Continued*)

the closest to the Euclidean space with the smallest edge length and vice versa.

(3) The lowest level value of the edges length (violet curve in Fig. 6(c)) is approximately the edges length of the Euclidean regular tetrahedron with a face area $A = \sqrt{j(j+1)}$

$$a_{\min} \approx \left(\frac{4A}{\sqrt{3}}\right)^{\frac{1}{2}} = \left(\frac{4\sqrt{j(j+1)}}{\sqrt{3}}\right)^{\frac{1}{2}}. \tag{23}$$

(4) For a generic spin value $j \sim 1$, we find that the regular tetrahedron solutions of negative scalar curvature spectra are in the range

$$R \sim -(8\pi G h \gamma)^{-1} \sim -10^{70}/\gamma\,\mathrm{m}^{-2}. \tag{24}$$

(5) In the semi-classical limit $j \gg 1$, the new operators worked perfectly as we had hoped, the monochromatic 4-valent will be more closer to be identified with the Euclidean regular tetrahedron, because all scalar curvature spectra vanish as well as the edge length spectra tend asymptotically to the edge length of a regular Euclidean tetrahedron given in (23) (see Figs. 6(b) and 6(c)). Accordingly, we are able to have a good approximation of the volume and boundary face area functions (20), (21) around the zero constant curvature in the case of $j \gg 1$. In fact, by expanding these two functions (20), (21) with respect to the variable $\frac{a}{r}$, we obtain:

$$A(r,a) = \frac{\sqrt{3}}{4}a^2\left[1 + \frac{1}{8}\left(\frac{a}{\epsilon r}\right)^2 + \mathcal{O}\left(\left(\frac{a}{\epsilon r}\right)^4\right)\right], \tag{25}$$

$$V(r,a) = \frac{\sqrt{2}}{12}a^3\left[1 + \frac{23}{80}\left(\frac{a}{\epsilon r}\right)^2 + \mathcal{O}\left(\left(\frac{a}{\epsilon r}\right)^4\right)\right]. \tag{26}$$

As we have previously said, the analytic inversion of the two functions (20), (21) is not analytically possible, instead of doing the exact inversion with respect to the exact variables (r, a), we will use the good approximation functions (25), (26) with respect to the approximate variables $(\tilde{r}, \tilde{a})$ and write

$$A(\tilde{r}, \tilde{a}) = \frac{\sqrt{3}}{4}\tilde{a}^2\left[1 + \frac{1}{8}\tilde{x}\right], \tag{27}$$

$$V(\tilde{r}, \tilde{a}) = \frac{\sqrt{2}}{12}\tilde{a}^3\left[1 + \frac{23}{80}\tilde{x}\right], \tag{28}$$

where

$$\tilde{x} = \left(\frac{\tilde{a}}{\epsilon\tilde{r}}\right)^2 = \frac{\tilde{R}\tilde{a}^2}{6}. \tag{29}$$

Inverting the two functions (27), (28) for the two variables $\tilde{R}$ and $\tilde{a}$, we obtain approximated formulas of the scalar curvature as well as the edge length:

$$\tilde{R}(A, V) = \frac{3\sqrt{3}}{2A}\tilde{x}\left(1 + \frac{\tilde{x}}{8}\right), \tag{30}$$

$$\tilde{a}(A, V) = \left(\frac{4\sqrt{3}}{3}\frac{A}{1 + \frac{\tilde{x}}{8}}\right)^{1/2}, \tag{31}$$

where

$$\tilde{x} = \frac{4\sqrt{3}A}{F(A, V)} - 8. \tag{32}$$

The function F is defined by

$$F(A, V) = \left[\frac{G(A, V)}{78} + \frac{23\sqrt{3}A}{G(A, V)}\right]^2 \tag{33}$$

and

$$G(A, V) = (-205335\sqrt{3}V + 117\sqrt{9240075V^2 - 1265368\sqrt{3}A^3})^{1/3}. \tag{34}$$

Now, one has to quantize the 3D Ricci scalar curvature and edge length functions given in (30), (31) by quantizing the area and volume operators to obtain quantum operators that act on the state of monochromatic 4-valent node quantum atom of space (the volume eigenstate):

$$\tilde{R}(A, V) \to \hat{\tilde{R}}(\hat{A}, \hat{V}), \tag{35}$$

$$\tilde{a}(A, V) \to \hat{\tilde{a}}(\hat{A}, \hat{V}). \tag{36}$$

As the color j increases, the accuracy of these two operators (35), (36) will be very high and their behavior spectra for $j \to \infty$ in the semi-classical limit is well known

Table 1. Comparison of the approximated spectra of the two operators $(\hat{\tilde{R}}, \hat{\tilde{a}})$ associated to a regular tetrahedron with their exact value (R, a) for the highest volume level (violet curve in Fig. 6(a)) of the monochromatic 4-valent node state for $j = 1, \ldots, 10$.

j	A	$V_{\max}$	R	$\tilde{R}$	δR (%)	a	$\tilde{a}$	δa (%)
1	1.414	0.620	−2.146	−1.418	34	1.954	1.914	2.07
2	2.449	1.425	−1.156	−0.782	32	2.557	2.511	1.82
3	3.464	2.444	−0.663	−0.478	28	2.998	2.960	1.25
4	4.472	3.641	−0,422	−0.320	24	3.369	3.340	0.87
5	5.477	4.990	−0.291	−0.229	21	3.700	3.677	0.63
6	6.481	6.476	−0.212	−0.172	19	4.003	3.983	0.48
7	7.483	8.086	−0.161	−0.134	17	4.283	4.267	0.37
8	8.485	9.812	−0.127	−0.107	15	4.545	4.532	0.30
9	9.487	11.646	−0.102	−0.088	14	4.793	4.782	0.24
10	10.488	13.583	−0.084	−0.073	13	5.029	5.019	0.20

and it gives the Euclidean solution (see Table 1)

$$\hat{\tilde{R}}(\hat{A}, \hat{V}) \underset{j\to\infty}{|i_K^{(j)}\rangle} = \tilde{R}(\sqrt{j(j+1)}, V_K) \underset{j\to\infty}{|i_K^{(j)}\rangle} \approx 0, \tag{37}$$

$$\hat{\tilde{a}}(\hat{A}, \hat{V}) \underset{j\to\infty}{|i_K^{(j)}\rangle} = \tilde{a}(\sqrt{j(j+1)}, V_K) \underset{j\to\infty}{|i_K^{(j)}\rangle} \approx \left(\frac{4j}{\sqrt{3}}\right)^{1/2} \underset{j\to\infty}{|i_K^{(j)}\rangle}. \tag{38}$$

5. Conclusion

We have found an alternative approach of measuring the 3D Ricci scalar curvature value by measuring the volume of a region and its boundary area. We have applied this technique in LQG and the main feature of our proposed curvature operator is to determine in a straightforward manner at which curvature value can an intertwiner state be interpreted as a geodesic polyhedron. As a byproduct, we have studied the possibility of finding the regular tetrahedron corresponding with the monochromatic 4-valent node in nonzero constant curvature spaces. It is shown that all regular tetrahedron states are in the negative scalar curvature regime; for $j \gg 1$, the scalar curvature spectrum will be very close to the Euclidean regime, as we have expected. We conclude that the simultaneous measure of the volume and the boundary area of the monochromatic 4-valent node state allows us to estimate the appropriate case of a constant curvature space in which this state can be interpreted as a regular tetrahedron.

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ملخص

الجاذبية الكوانتية الحلقية وتطبيقاتها الهندسية

الجاذبيـة الكوانتيـة الحلقيـة هي نظرية حقل كمومـي للفضاء، غير اضطرابية ومستقلـة عن الخلفيـة الزمكانيـة. تعتمـد هذه النظريـة علــى المعاملـة الكموميـة للنسبيـة العامـة من خلال منهجية ديراك للتكميـم. الفضاء الهلبرتــي الكينيماتيكــي لهذه النظرية يتألـف من التابعيات الموجيـة الأسطوانية عبر الهـولونوميات المعرفـة بـالروابط الجبرية (روابط جبر لــي) على طـول أنظمـة من المنحنيـات الموجهـة. الفضاء المعياري الثابت لهذه النظرية هو فضاء نـواة قيـد غوص على المستـوى الكمومـي. يتشكـل من الشبكـات المغزليـة (السبيـن) وهي عبارة عن مجموعة من المنحنيات الموجهة مرفوقة برقم مغزلي لكل منحنى ومتشابـك معيـاري ثابـت في كل عقـدة. تم الوصـول إلى مؤثر المساحة والحجم في نظريـة الجاذبية الكوانتية الحلقيـة. من خلال طيفهمـا الكمومي، تم التنبـؤ بالخاصيـة الغامضة والانفصاليـة للفضاء. لقد أظهرت لنا أن الزمكان له طبيعـة منفصلة على مقيـاس طـول بلانـك. و تـم توفيـر تفسيـر جميـل لذرات الفضـاء المكممـة باستعمـال متعـدد السطـوح الإقليديـي الكمومـي.

في هذه الأطروحـــة، نقتـرح منهـج بديـل لتقديم مؤثر ريشي للانحنـاء الثلاثي الأبعاد معطـى كدالة لمؤثرات الحجم والمساحـة، بالإضافـة إلى تقديم مؤثر الطـول. تمـت دراسـة كمثـال حالـة عقدة رباعيـة التكافـؤ مع أرقــام مغزليــة متساويــة (ربــاعي السطوح متســاوي المساحات) وتـم بنـاء مؤثـر الانحنـاء لشكـل رباعـي السطـوح متسـاوي الأضـلاع. لقد بيــن هذا أن جميع رباعيات السطوح متساوية الأضـلاع موجـودة في نظـام الانحنـاء السلبـي (الفضــاء الهيبـربـوليكـي) وبالـنسبـة للـنهايـة شبـه التقليديـة ، يكـون الطيـف قريبًـا جـدًا من رباعـي السطـوح متساوي الأضــلاع.

الكلمات المفتاحية: الجاذبيـة الكوانتيـة الحلقيـة، رباعي السطوح الكمومي، مؤثر الانحناء، مؤثر الطول.

Résumé

Gravitation quantique à boucles et ses applications géométriques

La Gravitation quantique à boucles (LQG) est une théorie des champs quantique de la géométrie elle-même, non perturbatif et indépendante du fond. Elle est basée sur l'implémentation quantique de la relativité générale (RG) à l'aide de programme de quantification de Dirac. L'espace cinématique de Hilbert est construit par des fonctionnelles d'onde cylindriques à travers des holonomies définies par la connexion $su(2)$ le long d'un système des trajectoires orientées. L'espace de Hilbert invariant par la jauge est l'espace noyau des contraintes de Gauss; il est construit par l'état de réseau de spin qui est un ensemble des courbes orientées, un nombre de spin à chaque courbe et un intertwiner invariant à chaque nœud. Les opérateurs d'aire et du volume dans LQG sont fournis. À partir de leur spectre quantique, la propriété de l'espace est mystérieuse et discontinu; il est démontré que l'espace-temps est fondamentalement discret à l'échelle de la longueur de Planck. Une belle interprétation de l'atome de l'espace en termes de polyèdre Euclidien quantique est fournie.

Dans cette thèse, une approche alternative introduisant un opérateur de courbure scalaire 3d-Ricci donnée en termes de volume et d'aire ainsi qu'un nouvel opérateur de longueur est proposée. Un exemple d'état d'intertwiner de nœud monochromatique à 4 valences (tétraèdres équilatéraux) est étudié et la mesure de courbure scalaire pour une forme de tétraèdre régulier est construite. Nous montrons que tous les états de tétraèdre réguliers sont dans le régime de courbure scalaire négatif et que pour la limite semi-classique le spectre est très proche du régime Euclidien.

Mots clés: Gravitation quantique à boucles, Tétraèdres quantiques, Opérateur de courbure, Opérateur de longueur.

Abstract

Loop Quantum Gravity is a non-perturbative, background-independent and quantum field theory of geometry itself. It is based on the quantum implementation of General relativity (GR) by using Dirac quantization program. The kinematical Hilbert space is constructed by cylindrical wave functionals through holonomies defined by the su(2) connection along a system of smooth oriented paths. The gauge invariant Hilbert space is the kernel space of the Gauss constraints; it is constructed by the spin network state which is a collection of oriented curves, a spin number at each curve and an invariant intertwiner at each node. The area and volume operators in LQG has been provided. From their quantum spectrum, the fuzziness and discreteness property of space is predicted; it is shown that spacetime is fundamentally discrete and at the scale of the Planck length. A beautiful interpretation of the space atom in terms of the quantum Euclidean polyhedral is provided.

In this thesis, an alternative approach introducing a 3d- Ricci scalar curvature operator given in terms of volume and boundary area as well as new edge length operator is proposed. An example of monochromatic 4-valent node intertwiner state (equilateral tetrahedra) is studied and the scalar curvature measure for a regular tetrahedron shape is constructed. We show that all regular tetrahedron states are in the negative scalar curvature regime and for the semi-classical limit the spectrum is very close to the Euclidean regime.

Key words: Loop Quantum Gravity, Quantum tetrahedra, Curvature operator, Edge length operator.