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INFÉRENCE STATISTIQUE DANS LES ÉQUATIONS DIFFÉRENTIELLES STOCHASTIQUES

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Contents

Introduction

Stochastic differential equations (SDE) occupy a central place in the modeling of continuous time phenomena involving the hazard. On the other hand, The SDE are generalization of ordinary differential equations where we have in addition a random term. The fields of application of these SDE are vast and varied, they apply in the elaboration of phenomena of diffusion in physics, and in the modeling of localization of a given species in population dynamics, the applications of SDE also touch the field of ecology, signal processing, theory control and financial mathematics. However, in 1900, L. Bachelier in his thesis "Theory of speculation" to present a Gaussian model which fitted quite well with the data of the Paris Stock Exchange. This model had the form $X(t) = \alpha t + \sigma w(t)$; $t \in [0; T]$, $w(t)$ is a standard Brownian motion. It follows from Itô's formula, that the geometric Brownian motion is a solution of this SDE, $dX(t) = \mu X(t)dt + \sigma X(t)dw(t)$, its dynamics is the basis of the Black-Sholes model (1973) which deals with the evaluation and coverage of a European-type option on an action dividend.

We consider a diffusion process $(X(t))_{t\geq 0}$ defined by a SDE of the form :

$$dX(t) = a\,(X,t)\,dt + b\,(X,t)\,dw(t),\ X(0) = X_0,$$

where $w(t)$ is a standard Brownian motion, the functions a, b depend on certain parameters and X_0 is the initial state of this SDE. Fundamental questions about the existence and uniqueness of a solution for such equation exist in a similar way to ordinary equations, but there are two criteria for proving existence and uniqueness on a given interval I,

i. A Lipschitz condition that expresses the fact that for any couple (x, y) and all $t \in I$, there is a constant K such that

$$|a(x,t) - a(y,t)| + |b(x,t) - b(y,t)| \leq K|x - y|$$

ii. A condition of growth that expresses the fact that variations in $a(x,t)$ and $b(x,t)$ are not too fast, that is, there is a constant K' such as in the time interval I we have the condition

$$|a(x,t)|^2 + |b(x,t)|^2 \leq K'^2(1 + x^2)$$

For more details on the problem of existence and uniqueness, we can refer to Arnold [3], Lipcer and Shiryayev [49], Le breton and Musiela [44]. In the above equation, if the functions a, b are linear regarding to the first component, the SDE is called bilinear and behaves like a nonlinear stochastic differential equation. It is this direction that we have chosen to study SDE for certain reasons, among which the ability of these stochastic models in continuous time to model certain phenomena more appropriate than time series in discrete-time, such as the well known $ARMA$

models being an effective and easy tool which is used for describing many equidistant data. However, in many practical situations, the data generating the process is often observed irregularly spaced. This phenomenon occurs for example in physics, economics and so on. Therefore, the use of continuous time (which can be interpreted as a solution of some SDE) is inevitable. Moreover, researchers have often assumed that these models are linear and can be Gaussian, which continues to attract increasing interest from researchers (see for example Brockwell [15] and references therein). However, recent studies have shown that assumptions of linearity and / or gaussianity are very unrealistic. Thus, various nonlinear models have been widely proposed in economics, sociology, ... to describe these characteristics. Indeed, a $GARCH$ process in continuous time ($COGARCH$) was recently introduced and studied by Kluppelberg et al. [42] and by Brockwell et al. [14]. A general class of non-linear continuous time and threshold ($CTAR$) and $ARMA$ threshold autoregressive processes are constructed and briefly discussed by Brockwell [15]. A bilinear model in continuous time ($COBL$) was introduced by Mohler [53] in control theory and popularized in time series analysis by Le breton and Musiela [45], Subba Rao and Terdik [62] and by Iglói and Terdik [38]. Finally, another reason that motivates us to study this subject and that we remarked that there are not many researchers in Algeria working on this axis of research.

The bilinear stochastic model

In this thesis, we focused on the probabilistic and statistical study of the so called bilinear stochastic process COBL(1,1), so, we consider the process $(X(t))_{t\geq 0}$ which is given by the following bilinear stochastic differential equation

$$dX(t) = (\alpha(t)X(t) + \mu(t))\,dt + (\gamma(t)X(t) + \beta(t))\,dw(t), X(0) = X_0,$$

in which $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are measurable deterministic functions to the conditions $\forall t \geq 0$, $\alpha(t) \neq 0$, $\gamma(t)\mu(t) \neq \alpha(t)\beta(t)$ and the following conditions: $\forall T > 0$, $\int_0^T |\alpha(t)|\,dt < \infty$ and $\int_0^T |\mu(t)|\,dt < \infty$, $\int_0^T |\gamma(t)|^2\,dt < \infty$, $\int_0^T |\beta(t)|^2 \ dt < \infty$.The initial state $X(0)$ is a random variable, defined on $(\Omega, \mathcal{A}, P)$, independent of w such that $E\left\{X(0)\right\} = m(0)$ and $Var\left\{X(0)\right\} = K(0)$. This Equation is called continuous−time bilinear ($COBL\,(1,1)$) (resp. linear) SDE whenever $\gamma(t) \neq 0$ (resp. $\gamma(t) = 0$) for all $t > 0$, in other words, when the solution is not Gaussian or it is.

The Itô approach

The existence and uniqueness of the Itô solution process $(X(t))_{t\geq 0}$ of bilinear SDE in time domain is ensured by the general results on stochastic differential equations and under the above assumptions (see Le breton and Musiela (1984)), this solution is given by

$$X(t) = \varphi(t)\left\{X(0) + \int_0^t \varphi^{-1}(s)\left(\mu(s) - \gamma\left(s\right)\beta\left(s\right)\right)ds + \int_0^t \varphi^{-1}(s)\beta\left(s\right)d\epsilon(s)\right\}$$

where $\varphi(t) = \exp\left\{\int_0^t \left(\alpha(s) - \frac{1}{2}\gamma^2\left(s\right)\right)ds + \int_0^t \gamma\left(s\right)d\epsilon\left(s\right)\right\}$, $t \geq 0$, which reduce to that given by Iglói and Terdik (1999) in constant coefficients case and provide a solution for non-stationary

Gaussian Ornstein-Uhlenbeck process corresponding to the case when $\gamma(s) = 0$ for all s. In this case we obtain

$$X(t) = \psi(t)\left\{X(0) + \int_0^t \varphi^{-1}(s)\mu(s)ds + \int_0^t \varphi^{-1}(s)\beta(s)\,d\epsilon(s)\right\},$$

where $(\psi(t))_{t\geq 0}$ is the mean function of $(\varphi(t))_{t\geq 0}$ i.e., $\psi(t) = \exp\left\{\int_0^t \alpha(s)ds\right\}$, $t \geq 0$

The frequency approach

Let $\Im = \Im(w) := \sigma(w(t), t \geq t_0)$ (resp $\Im_t := \sigma(w(s), t_0 \leq s \leq t)$) be the σ -algebra generated by $(w(t))_{t\geq 0}$ (resp. generated by $w(s)$ up to time t) and let $\mathbb{L}_2(\Im) = \mathbb{L}_2(\mathbb{R}, \Im, P)$ (resp. $\mathbb{L}_2(\Im_t)$) be the real Hilbert space of nonlinear $\mathbb{L}_2-$functional of $(w(t))_{t\geq 0}$ and let $\mathbb{L}_r(F)$ be the real Hilbert space of complex valued functions $f_t\left(\lambda_{(r)}\right)$ defined on $\mathbb{R}^r$ such that $f_t(-\lambda_{(r)}) = \overline{f_t(\lambda_{(r)})}$ with a inner product $\langle f_t, g_t\rangle_F = r!\int_{\mathbb{R}^r} Sym\left\{f_t(\lambda_{(r)})\right\}\overline{Sym\left\{g_s(\lambda_{(r)})\right\}}dF(\lambda_{(r)})$ where $\lambda_{(r)} = (\lambda_1, ..., \lambda_r) \in \mathbb{R}^r$, $Sym\left\{f_t(\lambda_{(r)})\right\} = \frac{1}{r!}\sum_{\pi\in\mathcal{P}} f\left(\lambda_{\pi(1),}, ..., \lambda_{\pi(r)}\right) \in \mathbb{L}_r(F)$ with $\mathcal{P}$ denotes the group of all permutations of the set $\{1, ..., r\}$ and $dF(\lambda_{(r)}) = \prod_{i=1}^r dF(\lambda_i)$. It is well known that for any regular second-order process $(X(t))_{t\geq t_0}$ (i.e., $X(t)$ is $\Im_t-$measurable not necessary stationary, belonging to $\mathbb{L}_2(\Im)$) admits the so-called Wiener-Itô spectral representation (see Major [50] and Dobrushin [23] for further discussions), i.e.,

$$X(t) = f_t(0) + \sum_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r} f_t(\underline{\lambda}_{(r)})e^{it\Sigma\underline{\lambda}_{(r)}}dZ\left(\underline{\lambda}_{(r)}\right),$$

where $\underline{\lambda}_{(r)} = (\lambda_1, ..., \lambda_r)$, $\Sigma\underline{\lambda}_{(r)} = \sum_{i=1}^r \lambda_i$ and $dZ\left(\underline{\lambda}_{(r)}\right) = \prod_{j=1}^r dZ(\lambda_i)$. This representation is unique up to the permutation of the arguments of the evolutionary transfer functions $f_t(\underline{\lambda}_{(r)})$, $r \geq 2$ and $f_t(\underline{\lambda}_{(r)}) \in \mathbb{L}_r(F)$ for all $t \geq t_0$, with $dF(\lambda_{(r)}) = \frac{1}{(2\pi)^r}\prod_{i=1}^r d\lambda_{(r)}$ and such that

$$\sum_{r\geq 0}\frac{1}{r!}\int_{\mathbb{R}^r}\left|f_t(\underline{\lambda}_{(r)})\right|^2 dF(\underline{\lambda}_{(r)}) < \infty \text{ for all } t \geq t_0.$$

Analytical tools

Our study needs to know some useful analytical tools to establish the proof of the results which are obtained in this study.

Itô's Formula

Itô's Formula is an important tool in stochastic calculus where its simplest form is given for any twice differentiable scalar function $f(t, x)$ of two real variables t and x, one has

$$df(t, X(t)) = \left(\frac{\partial f}{\partial t} + a(X(t), t)\frac{\partial f}{\partial x} + \frac{b^2(X(t), t)}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + b(X(t), t)\frac{\partial f}{\partial x}dw(t).$$

Diagram Formula

The diagram formula for spectral representation which play an important role in some subsequent proofs and that state that for all f and g defined on $\mathbb{R}$ and on $\mathbb{R}^r$ respectively such that $(f, g) \in \mathbb{L}_1(F) \times \mathbb{L}_r(F)$, if f is symmetric then

$$\int_{\mathbb{R}} f(\lambda) dZ(\lambda) \int_{\mathbb{R}^n} g\left(\lambda_{(n)}\right) dZ(\lambda_{(n)}) = \int_{\mathbb{R}^{n+1}} g\left(\lambda_{(n)}\right) f\left(\lambda_{n+1}\right) dZ(\lambda_{(n+1)})$$
$$+ \sum_{k=1}^{n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} g\left(\lambda_{(n)}\right) \overline{f\left(\lambda_k\right)} dF\left(\lambda_k\right) dZ(\lambda_{(n\backslash k)})$$

where $dZ(\lambda_{(n\backslash k)}) = dZ\left(\lambda_1\right) ... dZ\left(\lambda_{k-1}\right) .dZ\left(\lambda_{k+1}\right) ... dZ\left(\lambda_n\right)$.

Orthogonal property

As a property of the above spectral representation is that for any $f_t(\lambda_{(n)})$ and $f_s(\lambda_{(m)})$, we have

$$E\left\{\int_{\mathbb{R}^n} f_t(\lambda_{(n)}) dZ(\lambda_{(n)}) \int_{\mathbb{R}^m} f_s(\lambda_{(m)}) dZ(\lambda_{(m)})\right\} = \delta_n^m n! \int_{\mathbb{R}^n} \widetilde{f}_t(\lambda_{(n)}) \overline{\widetilde{f}_s(\lambda_{(n)})} dF(\lambda_{(n)}),$$

where δ_n^m is the delta function.

Results

In this thesis, we are studying a class of continuous-time bilinear processes ($COBL(1,1)$) generated by some stochastic differential equations where we have investigate some probabilistic properties and statistical inference.

In first part, we use Itô approach for studying the $\mathbb{L}_2$ structure of the $COBL(1,1)$ process and its powers for any order with time varying coefficients. Furthermore, we prove that these results can be obtained by using the transfer functions approach, moreover, by the spectral representation of the process, we give also conditions for the stability of moments, in particular the moments of the quadratic process provide us to checking the presence of the so called Taylor property for$COBL(1,1)$ process.

In a second part of this thesis, we use the results of the first part and we propose some methods of estimation for involving unknown parameters, so, we starting by the moments method (MM) to estimate the parameters by two methods, taking into consideration the relation that exists between the moments of the process and its quadratic version and those associate with the incremented processes where we have showed that the resulting estimators are strongly consistent and asymptotically normal under certain conditions. Using the linear representation of $COBL(1,1)$ process, we are able to propose three other methods, one is in frequency domain and the rest are in time domain and we prove the asymptotic properties of the proposed estimators. Simulation studies are presented in order to illustrate the performances of the different estimators, furthermore, this methods are used to model some real data such as the exchanges rate of the Algerian Dinar against the US-dollar and against the single European currency and the electricity consumption sampled each 15mn in Algeria.

Organization of the thesis

This thesis is composed in two main parts, It is organized as follow.

Probabilistic study

Chapter 1

We are interested in this chapter on the one hand, on the necessary and sufficient conditions for the existence of regular solution of a bilinear SDE with variable coefficients, this solution is given in the frequency domain by its evolutionary transfer functions, which makes it possible express explicitly the characteristics of the second order of the process generalizing thus of Iglòi and Terdik [38] where they studied this model with constant coefficients. On the other hand we used the frequency approach to study the probabilistic properties of the model already studied in the time domain by Le breton and Musiela [45].

Chapter 2

In this chapter, we study a bilinear SDE with variable coefficients driven by a fractional Brownian motion and under certain conditions imposed on the coefficients of the bilinear term, the explicit solution is given in the frequency domain, Moreover, the second-order structure for this class of processes is analyzed where we give explicitly the expectation, covariance, and spectral density using the Itô approach. A consequence of this study shows that this process has a long memory, or it is also said that the process has the property "long-rang dependent".

Chapter 3

We continue in this chapter the study of some probabilistic properties of a continuous-time bilinear process defined as a nonlinear SDE which attracts the attention of researchers in recent years. These properties are related to the strict and weak stationarity of the process and its quadratic version, thus deducing the autocorrelations of these two processes in order to analyze the presence of a Taylor effect and its relation with the property of the lepokurtosis of the corresponding process. Recall that Goncalves, Martins and Mendes-Lopes [29] analyzed this property for non-negative discrete-time linear models.

Statistical study

Chapter 4

In this chapter, we propose an estimation method for the first order continuous-time bilinear process ($COBL$) based on the Euler-Maruyama discretization of the Itô solution associated with the SDE defining the process. More precisely, certain relations connecting the parameters and the theoretical moments of the process and its quadratic version have been given. These relationships allow us to construct two algorithms to estimate the parameters based on the method of moments (MM). Using the fact that under certain conditions, the incremented processes are strongly mixing with exponential rate. We also show that the obtained estimators are strongly consistent and asymptotically normal. This method can be applied to the model $COGARCH(1,1)$, and Ornstein-Uhlenbeck models (OU) and for other specifications. The properties of the finished samples are also taken into account in Monte Carlo experiments. Finally, these algorithms are used to model the exchange rate of the Algerian dinar against the US dollar and against the single European currency.

Chapter 5

This chapter is devoted to the study of some probabilistic and statistical properties in the frequency domain of continuous time bilinear processes driven by a standard Brownian motion. Thus, the $\mathbb{L}_2-$structure of the process is studied and its covariance function is given. These structures lead us to study the strong consistency and asymptotic normality of Whittle's estimator of the unknown parameters involved in the process. The properties of the finite sample are also considered through Monte Carlo experiments.

Chapter 6

In this chapter, we examine the properties of the moments in the frequency domain of the class of first-order continuous-time bilinear processes ($COBL(1,1)$) with coefficients that depend on time. So, we use the associated transfer functions to study the second-order structure of the process and its powers. In time-invariant case, an expression of the moments of any order is given and some properties of the moments for special cases are also presented. Based on these results we are able to examine the statistical properties such that we develop an estimation method of the process via the so-called generalized method of moments (GMM)illustrated by a Monte Carlo study and applied to modelling two foreign exchange rates Dinar against US-Dollar (USD/DZD) and against the single European currency Euro (EUR/DZD).

Chapter 7

This chapter studies in the time domain, a class of diffusion process generated by first order continuous-time bilinear stochastic process ($COBL(1;1)$) with time-dependent coefficients of which we used the Itô formula approach to examine the $\mathbb{L}_2-$ structure of the process and its power of order $k \geq 2$, in particular and in time-invariant case, an expression of the moments of any order is given and the linear representation of such a process is given as well as the properties of the moments of certain specifications. Based on these results, we are able to examine the statistical properties to develop an estimation method of the process via the Yule-Walker (YW) algorithm for the unknown parameters of the CAR representation. The method is illustrated by a Monte Carlo study and applied to the modeling of the Algerian electricity consumption sampled every 15 minutes. We refer to the introduction of each of these chapters for a more detailed presentation of their respective contents.

Part I

PROBABILISTIC STUDY

Chapter 1

A note on $\mathbb{L}_2$-structure of continuous-time bilinear processes with time-varying coefficients[1]

1. Ce chapitre est publié dans le journal : International Journal of Statistics and Probability.

Abstract

This chapter is concerned with the investigation of $\mathbb{L}_2$-structure issue of time-varying coefficients continuous-time bilinear processes ($COBL$) driven by a Brownian motion (BM). Such processes are very useful for modeling irregular spacing non linear and non Gaussian datasets and may be proposed to model for instance some financial returns representing high amplitude oscillations and thus make it a serious candidate for describe processes with time-varying degree of persistence and other complex systems. Our attention is focused however on the probabilistic structure of $COBL$ processes, so, we establish necessary and sufficient conditions for the existence of regular solutions in term of their transfer function. Explicite formulas for the mean and covariance functions are given. As a consequence, we observe that the second order structure is similar to a $CARMA$ processes with some uncorrelated noise. Therefore, it is necessary to look into higher-order cumulant in order to distinguish between $COBL$ and $CARMA$ processes.

1.1 Introduction

Discrete-time series such as the well-known $ARMA$ models, provide an effective and tractable tool to describe many datasets assumed to be equally spaced. However, in many practical situations, the data generating the process, are often observed irregularly spaced. This phenomenon happens for instance in physics, engineering problems, economy and so on. Therefore, the resort to continuous-time (which can be interpreted as a solution of some stochastic differential equations (SDE)) models is unavoidable. These models are often assumed to be linear and may be Gaussian which continue to gain a growing interest of researchers (see for instance Brockwell (2001) and the references therein). However, recent studies have been shown that the linearity and/or Gaussianity assumptions is very unrealistic. So, various nonlinear models have been widely proposed in economics, sociology and in industrial panel data in order to describe these features. Indeed, a continuous-time $GARCH$ ($COGARCH$) process, was recently introduced and studied by Kluppelberg et al. (2004) and by Brockwell et al. (2006). A general class of nonlinear continuous-time Autoregressive and threshold ($CTAR$) and threshold $ARMA$ processes are constructed and briefly discussed by Brockwell (2001). A continuous-time bilinear models ($COBL$) was introduced firstly by Mohler (1973) in control theory and has popularized in time series analysis by Lebreton and Musiela (1984), Subba Rao and Terdik (2003) and by Iglói and Terdik (1999).

The main purpose of this chapter is, on hand, to generalize the time-invariant $COBL$ model proposed by Iglói and Terdik (1999) to time-varying one and on other hand, to extend the results by Lebreton and Musiela (1984) in time domain to frequency domain. So, in the next section, we present a powerful frame for studying the nonlinear SDE in terms of their transfer function. This approach allows us to distinguish between linear and nonlinear and between regular and singular solutions. Section 3, describes the $COBL$ equation with respect to its evolutionary (time-dependent) transfer function, so in Section 4 we use this representation to give sufficient and necessary conditions ensuring the existence of second-order regular solutions. In section 5, an exact expression for the covariance function and the spectral density are given and it is shown that the second-order structure is the same as a $CARMA$ process with time-varying coefficients. This result, we then conduct to investigate the third-order cumulants and showing that the bispectrum is zero if the process is linear. In Section 6, we conclude and discuss possible extensions.

1.2 Wiener's chaos representation

Let $(\epsilon(t))_{t\geq 0}$ be a real Brownian motion defined on some filtered space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, P)$ with associated spectral representation $\epsilon(t) = \int_{\mathbb{R}} \dfrac{e^{it\lambda}-1}{i\lambda} dZ(\lambda)$, where $dZ(\lambda)$ is an orthogonal complex-valued stochastic measure on $\mathbb{R}$ with zero mean, $E\left\{|dZ(\lambda)|^2\right\} = dF(\lambda) = \dfrac{d\lambda}{2\pi}$ and uniquely determined by $Z([a,b[) = \dfrac{1}{2\pi}\int\limits_{\mathbb{R}} \dfrac{e^{-i\lambda a}-e^{-i\lambda b}}{i\lambda} d\epsilon(\lambda)$, for all $-\infty < a < b < +\infty$. So the process $(Z([0,t[))_{t>0}$ is also a Brownian motion. Consider the Hilbert space $\mathcal{H} = \mathbb{L}_2(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, F)$ of the complex squared integrable functions f satisfying $f(\lambda) = \overline{f(-\lambda)}$ for any $\lambda \in \mathbb{R}$ where $\overline{f(\lambda)}$ denotes the complex conjugate of $f(\lambda)$. For any $n \geq 1$, we associated three real Hilbert spaces based on $\mathcal{H}$. The first is $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ the n-fold tensor product of $\mathcal{H}$ endowed by the inner product $< f, f >= \int_{\mathbb{R}^n} f(\lambda_{(n)})\overline{f(\lambda_{(n)})}dF(\lambda_{(n)})$ where $\lambda_{(n)} = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n$, with $\lambda_{(0)} = 0$, $dF(\lambda_{(n)}) = \prod\limits_{i=1}^{n} dF(\lambda_i)$ and $f(\lambda_{(n)}) = \overline{f(-\lambda_{(n)})}$ such that $\|f\|^2 = \int_{\mathbb{R}^n} \left|f(\lambda_{(n)})\right|^2 dF(\lambda_{(n)}) < \infty$. The second one is $\widehat{\mathcal{H}}_n = \mathcal{H}^{\oplus n} \subset \mathcal{H}_n$ the $n-$fold symmetrized tensor product of $\mathcal{H}$ defined by $f \in \widehat{\mathcal{H}}_n$ if and only if f is invariant under permutation of their arguments i.e., $f(\lambda_{(n)}) = Sym\{f(\lambda_{(n)})\}$ where $Sym\left\{f(\lambda_{(n)})\right\} = \widetilde{f}(\lambda_{(n)}) = \frac{1}{n!}\sum\limits_{p\in\mathcal{P}_n} f(\lambda_{(p(n))})$, $\mathcal{P}_n$ denotes the group of all permutations of the set $\{1, 2, ..., n\}$, and we endowed $\widehat{\mathcal{H}}_n$ with the inner product $< f, f >_{\oplus}= n! < f, f >$ for $f, f \in \widehat{\mathcal{H}}_n$. The last space is called Fock-space over $\mathcal{H}$ denoted by $\Im(\mathcal{H})$ defined by $\Im(\mathcal{H}) = \bigoplus\limits_{r=0}^{\infty} \frac{1}{r!}\widehat{\mathcal{H}}_r$ with $\widehat{\mathcal{H}}_0 = \mathcal{H}_0 = \mathbb{R}$ ($\bigoplus$ denotes the direct orthogonal sum) whose elements $\underline{\mathbf{f}} = \left(f(\lambda_{(r)}), r \geq 0\right)$ fulfilled the condition

$$\|\underline{\mathbf{f}}\|^2 = \sum_{r\geq 0} \frac{1}{r!} \int_{\mathbb{R}^r} \left|f(\lambda_{(r)})\right|^2 dF(\lambda_{(r)}) < \infty. \tag{1.2.1}$$

Let $\Im = \Im(\epsilon) := \sigma(\epsilon(t), t \geq 0)$ (resp $\Im_{\leq t}(\epsilon) := \sigma(\epsilon(s), s \leq t)$) be the σ-algebra generated by $(\epsilon(t))_{t\geq 0}$ (resp. generated by $\epsilon(s)$ up to time t) and let $\mathbb{L}_2(\Im) = \mathbb{L}_2(\mathbb{R}, \Im, P)$ be the real Hilbert space of nonlinear $\mathbb{L}_2-$functional of $(\epsilon(t))_{t\geq 0}$. It is well known (see Major (1981) and Bibi (2006)) that $\mathbb{L}_2(\Im)$ is isometrically isomorphic to Fock-space $\Im(\mathcal{H})$, so, for any random process $(X(t))_{t\in\mathbb{R}}$ (not necessary stationary) of $\mathbb{L}_2(\Im)$ admits the so-called Wiener-Itô orthogonal representation

$$X(t) = f_t(0) + \sum_{r\geq 1} \frac{1}{r!} \int_{\mathbb{R}^r} e^{it\underline{\lambda}_{(r)}} f_t(\lambda_{(r)}) dZ(\lambda_{(r)}) \tag{1.2.2}$$

where $\underline{\lambda}_{(r)} = \sum_{i=1}^{r} \lambda_i$ and the integrals are the multiple Wiener-Itô stochastic integrals with respect to the stochastic measure $dZ\left(\lambda\right)$, $f_t(0) = E\{X(t)\}$, $dZ(\lambda_{(r)}) = \prod_{i=1}^{r} dZ(\lambda_i)$ and $f(\lambda_{(r)}) \in \widehat{\mathcal{H}}_r$ are referred as the $r-th$ evolutionary transfer function of $(X(t))_{t\in\mathbb{R}}$, uniquely determined and fulfill the condition $\sum_{r\geq 0} \frac{1}{r!} \int_{\mathbb{R}^r} \left|f_t(\lambda_{(r)})\right|^2 dF(\lambda_{(r)}) < \infty$. As a property of the representation (1.2.2) is that for any $f_t \in \mathcal{H}_n$ and $f_s \in \mathcal{H}_m$, we have

$$E\left\{\int_{\mathbb{R}^n} f_t(\lambda_{(n)}) dZ(\lambda_{(n)}) \int_{\mathbb{R}^m} f_s(\lambda_{(m)}) dZ(\lambda_{(m)})\right\} = \delta_n^m n! \int_{\mathbb{R}^n} \widetilde{f}_t(\lambda_{(n)}) \overline{\widetilde{f}_s(\lambda_{(n)})} dF(\lambda_{(n)}) \tag{1.2.3}$$

where δ_n^m is the delta function. This means that the spaces $\widehat{\mathcal{H}}_n$ are orthogonal. Two interesting properties related to the multiple Wiener-Itô stochastic integrals which is important for future use are the diagram and Itô formulas summarized in the following lemma due to Dobrushin (1979).

Lemma 1.2.1. *Let $(\varphi_i)_{1\leq i\leq k}$ be an orthonormal system in $\mathcal{H}$, $(n_j)_{1\leq j\leq k}$ is a sequence of positive integers such that $n = n_1 + ... + n_k$ and let h_j be the $j-th$ Hermite polynomial with highest coefficient 1, i.e., $h_j\left(x\right) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{-\frac{x^2}{2}}$. Then,*

1. *The Itô's formula states that*

$$\begin{aligned}
\prod_{j=1}^{k} h_{n_j}\left(\int_{\mathbb{R}} \varphi_j(\lambda) dZ(\lambda)\right) &= \int_{\mathbb{R}^n} \prod_{j=1}^{k} \prod_{i=1}^{n_j} \varphi_j(\lambda_{\underline{n}_{(j-1)}+i}) dZ(\lambda_{(n)}) \\
&= \int_{\mathbb{R}^n} Sym\left\{\prod_{j=1}^{n} \varphi_j(\lambda_j)\right\} dZ(\lambda_{(n)})
\end{aligned}$$

2. *The diagram formulas state that for any $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_n$ we have*

$$\begin{aligned}
\int_{\mathbb{R}} f(\lambda) dZ(\lambda) \int_{\mathbb{R}^n} g\left(\lambda_{(n)}\right) dZ(\lambda_{(n)}) &= \int_{\mathbb{R}^{n+1}} g\left(\lambda_{(n)}\right) f\left(\lambda_{n+1}\right) dZ(\lambda_{(n+1)}) \\
&+ \sum_{k=1}^{n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} g\left(\lambda_{(n)}\right) \overline{f\left(\lambda_k\right)} dF\left(\lambda_k\right) dZ(\lambda_{(n\setminus k)})
\end{aligned}$$

where $dZ(\lambda_{(n\setminus k)}) = dZ\left(\lambda_1\right) ...dZ\left(\lambda_{k-1}\right) .dZ\left(\lambda_{k+1}\right) ...dZ\left(\lambda_n\right)$.

Remark 1.2.2. *For any $n \geq 0$, let $\mathcal{K}_n$ be the closed vector subspace of $\mathbb{L}_2(\Im)$ spanned by $\{h_n(X), X \in \Im : Var(X) = 1\}$ with $\mathcal{K}_0 = \{C^t\}$. Then, it can be shown that the subspaces $\mathcal{K}_n$ are mutually orthogonal and $\mathbb{L}_2(\Im) = \bigoplus_{n=0}^{\infty} \mathcal{K}_n$ (see Peccati and Taqqu (2011)). Noting that the condition $Var(X) = 1$ in the definition of the subspaces $(\mathcal{K}_n)_{n \geq 0}$ is necessary. Indeed, $h_2(X) = X^2 - 1$ is orthogonal to $h_0(X) = C^t$, if and only if $E\{X^2 - 1\} = 0$.*

Noting here that the representation (1.2.2) is referred as regular solution if $\Im_{\leq t}(X) \subset \Im_{\leq t}(\epsilon)$ i.e., the process $X(t)$ depends only on the past of the noise $\epsilon(t)$. So, a stochastic process $(X(t))_{t\in\mathbb{R}}$ having a representation (1.2.2) has a regular (or also causal) second-order solution if for all $t \in \mathbb{R}$ and every $r \in \mathbb{N}$, the evolutionary transfer functions $f_t(\lambda_{(r)})$ satisfies Szego's condition, i.e., $\forall t \in \mathbb{R}$: $\int_{\mathbb{R}^r} \frac{1}{1 + \left\|\lambda_{(r)}\right\|^2} \log\left(\left|f_t(\lambda_{(r)})\right|\right) dF(\lambda_{(r)}) > -\infty$. (see Ibragimov and Rozanov (1978) for further discussions).

In the following section, a class of non linear diffusion processes admitting the representation (1.2.2) will be investigated.

1.3 Evolutionary transfer functions of time-varying COBL processes

The representation (1.2.2) can be describe general nonlinear stochastic differential equation (SDE) with a great accuracy and can be enlarged to include the processes $(X(t))_{t\in\mathbb{R}}$ solving the following SDE

$$dX(t) = f\left(X^{(s)}(t), \epsilon^{(r)}(t), 0 \leq s \leq p, 0 \leq r \leq q\right) dt + d\epsilon(t) \tag{1.3.1}$$

in which the superscript $^{(k)}$ denotes the $k-$fold differentiation with respect to t and some measurable function f. The derivative $\epsilon^{(j)}(t)$, $j > 0$ do not exist in usual sense, hence it can be interpret in the Itô sense. The main objective here is to derive the evolutionary transfer functions system associated with some subclass of (1.3.1) and we establish necessary and sufficient conditions ensuring the existence of second$-$order regular solutions. More precisely, we shall restrict ourself to the diffusion processes $(X(t))_{t\in\mathbb{R}}$ generated by the following SDE,

$$dX(t) = (\alpha(t)X(t) + \mu(t))\, dt + (\gamma(t)X(t) + \beta(t))\, d\epsilon(t), t \geq 0, X(0) = X_0 \tag{1.3.2}$$

in which $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are measurable deterministic functions to the conditions $\forall t \geq 0$, $\alpha(t) \neq 0$, $\gamma(t)\mu(t) \neq \alpha(t)\beta(t)$ and the following conditions: $\forall T > 0$, $\int_0^T |\alpha(t)|\, dt < \infty$ and $\int_0^T |\mu(t)|\, dt < \infty$, $\int_0^T |\gamma(t)|^2\, dt < \infty$, $\int_0^T |\beta(t)|^2\; dt < \infty$.The initial state $X(0)$ is a random variable, defined on $(\Omega, \mathcal{A}, P)$, independent of ϵ such that $E\{X(0)\} = m(0)$ and $Var\{X(0)\} = K(0)$. Equation (1.3.2) is called continuous$-$time bilinear ($COBL\,(1,1)$) (resp. linear) SDE whenever $\gamma(t) \neq 0$ (resp. $\gamma(t) = 0$) for all $t > 0$, in other words, when the solution is not Gaussian or it is.

SDE given by (1.3.2) encompasses many commonly used models in the literature. Some specific examples among others are:

1. $COGARCH(1,1)$: This classes of processes is defined as a SDE by $dX\,(t) = \sigma\,(t)\,dB_1(t)$ with $d\sigma^2\,(t) = \theta\left(\gamma - \sigma^2\,(t)\right) dt + \rho\sigma^2\,(t)\,dB_2\,(t)$, $t > 0$ where B_1 and B_2 are independent

Brownian motions. So, the stochastic volatility equation may be regarded as particular case of (1.3.2) by assuming constant the functions $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ with $\beta(t)=0$ for all $t\geq 0$. (see Kluppelberg et al. (2004) and the reference therein).

2. $CAR(1)$: This classes of SDE may be obtained by assuming $\gamma(t)=0$ for all $t\geq 0$. (see Brockwell (2001) and the reference therein)

3. Gaussian Ornstein-Uhlenbeck (GOU) process: The GOU process is defined as $dX(t)=\gamma(\mu-X(t))\,dt+\beta d\epsilon(t), t\geq 0$. So it can be obtained from (1.3.2) by assuming constant the functions $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ with $\gamma(t)=0$ for all $t\geq 0$. (see Brockwell (2001) and the reference therein).

The solution of equation (1.3.2) may be obtained according to :

1.3.1 The time domain approach

The existence and uniqueness of the Itô solution process $(X(t))_{t\geq 0}$ of equation (1.3.2) in time domain is ensured by the general results on stochastic differential equations and under the above assumptions (see Lebreton and Musiela (1984))

$$X(t)=\varphi(t)\left\{X(0)+\int_0^t\varphi^{-1}(s)\left(\mu(s)-\gamma(s)\beta(s)\right)ds+\int_0^t\varphi^{-1}(s)\beta(s)\,d\epsilon(s)\right\}$$

where $\varphi(t)=\exp\left\{\int_0^t\left(\alpha(s)-\frac{1}{2}\gamma^2(s)\right)ds+\int_0^t\gamma(s)\,d\epsilon(s)\right\}$, $t\geq 0$, which reduce to that given by Iglói and Terdik (1999) in constant coefficients case and provide a solution for non-stationary Gaussian Ornstein-Uhlenbeck process corresponding to the case when $\gamma(s)=0$ for all s. In this case we obtain

$$X(t)=\psi(t)\left\{X(0)+\int_0^t\varphi^{-1}(s)\mu(s)ds+\int_0^t\varphi^{-1}(s)\beta(s)\,d\epsilon(s)\right\}\tag{1.3.3}$$

where $(\psi(t))_{t\geq 0}$ is the mean function of $(\varphi(t))_{t\geq 0}$ i.e., $\psi(t)=\exp\left\{\int_0^t\alpha(s)ds\right\}$, $t\geq 0$ and the stochastic integral on the right-hand side of (1.3.3) has an expected value zero and by (1.2.3) we obtain

$$E\left\{\int_0^t f(s)\,d\epsilon(s)\int_0^t g(s')d\epsilon(s')\right\}=\left\{\int_0^t f(s)\,g(s)d(s)\right\}$$

for any squared integrable functions f and g with respect to Lebesgue measure on $[0,t]$.

1.3.2 The frequency domain approach

In frequency domain, we have

Theorem 1.3.1. *Assume that the process* $(X(t))_{t\geq 0}$ *generated by the SDE* (1.3.2) *has a regular second-order solution. Then the evolutionary symmetrized transfer functions* $f_t(\lambda_{(r)}), (t,r) \in \mathbb{R}\times\mathbb{N}$ *of this solution are given by symmetrization of the following differential equations of order* 1*,*

$$f_t^{(1)}(\lambda_{(r)}) = \begin{cases} \alpha(t)f_t(0)+\mu(t), & \text{if } r=0 \\ \left(\alpha(t)-i\underline{\lambda}_{(r)}\right) f_t(\lambda_{(r)}) + r\left(\gamma(t)f_t(\lambda_{(r-1)})+\delta_{\{r=1\}}\beta(t)\right), & \text{if } r\geq 1 \end{cases} \tag{1.3.4}$$

Proof. Suppose that the system (1.3.2) has a regular second-order solution of the form (1.2.2) with evolutionary symmetrized transfer functions $f_t(\lambda_{(r)}), (t,r) \in \mathbb{R}\times\mathbb{N}$, then by putting the solution (1.2.2) into (1.3.2) and using the diagram formula we obtain
$X(t)d\epsilon(t)$

$$= \left\{ f_t(0) + \sum_{r=1}^{\infty}\frac{1}{r!}\int_{\mathbb{R}^r} f_t(\lambda_{(r)})e^{it\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})\right\}\int_{\mathbb{R}} e^{it\lambda}dZ(\lambda)$$
$$= \int_{\mathbb{R}} f_t(0)e^{it\lambda}dZ(\lambda)$$
$$+\sum_{r=1}^{\infty}\frac{1}{r!}\left\{\int_{\mathbb{R}^{r+1}} f_t(\lambda_{(r)})e^{it\underline{\lambda}_{(r+1)}}dZ(\lambda_{(r+1)}) + r\int_{\mathbb{R}^{r-1}} e^{it\underline{\lambda}_{(r-1)}}\left(\int_{\mathbb{R}} f_t(\lambda_{(r)})dF(\lambda_r)\right)dZ(\lambda_{(r-1)})\right\},$$

since a regular solution does not depends on $\epsilon(s), s>t$, and depends on $\epsilon(t)$ linearly, then the last term in above expression is equal to 0, and hence, the recursion (1.3.4) follows by identifying the $r-th$ order evolutionary transfer functions. □

Remark 1.3.2. *The existence and uniqueness of the evolutionary symmetrized transfer functions* $f_t(\lambda_{(r)})$*,* $(t,r)\in\mathbb{R}\times\mathbb{N}$ *of this solution is ensured by general results on linear ordinary differential equations (see, Kelly (2010), ch. 1) so,*

$$f_t(\lambda_{(r)}) = \begin{cases} \varphi_t(0)\left(f_0(0)+\int_0^t \varphi_s^{-1}(0)\,\mu(s)ds\right) & \text{if } r=0 \\ \varphi_t\left(\underline{\lambda}_{(r)}\right)\left(f_0(\lambda_{(r)})+r\int_0^t \varphi_s^{-1}\left(\underline{\lambda}_{(r)}\right)\left(\gamma(s)f_s(\lambda_{(r-1)})+\delta_{\{r=1\}}\beta(s)\right)ds\right) & \text{if } r\geq 1 \end{cases} \tag{1.3.5}$$

where $\varphi_t\left(\underline{\lambda}_{(r)}\right) = \exp\left\{\int_0^t\left(\alpha(s)-i\underline{\lambda}_{(r)}\right)ds\right\}$.

Remark 1.3.3. *When* $\alpha(t)$*,* $\mu(t)$*,* $\gamma(t)$ *and* $\beta(t)$ *are constant, then the recursion* (1.3.4) *reduces to*

$$f(0) = -\frac{\mu}{\alpha}, \text{ and } f(\lambda_{(r)}) = r\left(i\underline{\lambda}_{(r)}-\alpha\right)^{-1}\left(\gamma f(\lambda_{(r-1)})+\delta_{\{r=1\}}\beta\right) \text{ if } r\geq 1$$

or equivalently $f(\lambda_{(r)}) = r!\gamma^{r-1}\left(\beta-\frac{\mu}{\alpha}\gamma\right)\prod_{j=1}^{r}\left(i\underline{\lambda}_{(j)}-\alpha\right)^{-1}$ *and the symmetrized version can be rewritten as* $Sym\left\{f(\lambda_{(r)})\right\} = (\mu\gamma-\alpha\beta)\,\gamma^{r-1}\int_0^{+\infty}\exp\{\alpha\lambda\}\prod_{j=1}^{r}\frac{1-\exp\{-i\lambda\lambda_j\}}{i\lambda_j}d\lambda.$

Remark 1.3.4. *Under the coditions specified in Remark 1.3.3 and when $\gamma = 0$, it is not difficult to see that $(X(t))_{t\geq 0}$ has a unique strictly stationary solution given by $X(t) = f(0) + \int_{\mathbb{R}} g(t-u)d\epsilon(u)$ where $\int_{\mathbb{R}} g(u)e^{-i\lambda u}du = \frac{\beta}{i\lambda - \alpha}$, $\lambda \in \mathbb{R}$.*

1.4 Condition for the existence of regular solutions

In Theorem 1.3.1 a first-order ordinary differential equation is derived for evolutionary transfer function of SDE (1.3.2). In order that these transfer function define a second-order regular solutions, we need to check that its belong to $\Im(\mathcal{H})$ and satisfies the condition (1.2.1). For this purpose, let $\mathbb{L}_\infty$ (resp. $\mathbb{L}_\infty \otimes \mathbb{L}_\infty$) be the Banach space of infinite dimensional vectors of essentially bounded complex functions on $\mathbb{C}$ (resp. on $\mathbb{C}^2$) and denote by $\underline{\mathbf{f}}(\lambda_{(r)}) = \left(f_t(\lambda_{(r)}), t \in \mathbb{R}\right) \in \mathbb{L}_\infty$ for any $\lambda_{(r)} \in \mathbb{R}^r$. Define the following bounded operators $\underline{\alpha} = diag\left\{\alpha(t)\right\}$, $\underline{\gamma} = diag\left\{\gamma(t)\right\}$, $\underline{\mu} = diag\left\{\mu(t)\right\}$ and $\underline{\beta} = diag\left\{\beta(t)\right\}$ where "$diag$" denotes the diagonal operator induced by a function in $\mathbb{L}_\infty$, i.e., $\left(\underline{\alpha} v\right)_t = \alpha(t)v(t)$ for all $v = (v(t), t \in \mathbb{R}) \in \mathbb{L}_\infty$. Moreover, let D (resp. I) be the differentiation (with respect to t) (resp. identity) operator on $\mathbb{L}_\infty$, i.e., $(Dv)_t = v^{(1)}(t), t \in \mathbb{R}$ (resp $(Iv)_t = v(t)$) for all $v \in \mathbb{L}_\infty$ with derivable components and set $P(z) = Iz - \Phi$ with $\Phi = \underline{\alpha} - D$ (see Dunford and Schwartz, 1963 for further details). With this notation, the functions (1.3.4) may be rewritten as $\underline{\mathbf{f}}^{(1)}(0) = \underline{\alpha}\underline{\mathbf{f}}(0) + \underline{\mu}\mathbf{1}$, $\underline{\mathbf{f}}^{(1)}\left(\lambda_{(r)}\right) = \left(\underline{\alpha} - i\underline{\lambda}_{(r)} I\right)\underline{\mathbf{f}}\left(\lambda_{(r)}\right) + r\left(\underline{\gamma}\underline{\mathbf{f}}\left(\lambda_{(r-1)}\right) + \delta_{\{r=1\}}\underline{\beta}\mathbf{1}\right)$, or equivalently

$$\underline{\mathbf{f}}(0) = P^{-1}(0)\underline{\mu}\mathbf{1},\ \underline{\mathbf{f}}(\lambda) = P^{-1}(i\lambda)\left(\underline{\gamma}\underline{\mathbf{f}}(0) + \underline{\beta}\mathbf{1}\right),\ \underline{\mathbf{f}}\left(\lambda_{(r)}\right) = rP^{-1}\left(i\underline{\lambda}_{(r)}\right)\underline{\gamma}\underline{\mathbf{f}}\left(\lambda_{(r-1)}\right),\ r \geq 2. \tag{1.4.1}$$

However, it is easily follows from (1.2.2) that the necessary and sufficient condition for the existence of second- order regular solution of SDE (1.3.2) is that the components of $\sum_{r\geq 0} \frac{1}{r!}\int_{\mathbb{R}^r} \widetilde{\underline{\mathbf{f}}}\left(\lambda_{(r)}\right) \otimes \overline{\widetilde{\underline{\mathbf{f}}}\left(\lambda_{(r)}\right)} dF\left(\lambda_{(r)}\right)$ be finite where $\widetilde{\underline{\mathbf{f}}}\left(\lambda_{(r)}\right) = Sym\left\{\underline{\mathbf{f}}\left(\lambda_{(r)}\right)\right\}$. Thus we have

Proposition 1.4.1. *The SDE (1.3.2) has a regular solution if and only if the following two conditions hold true.*

C1. *The spectrum of the operator Φ lie in $\overline{C} = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$.*

C2. *The spectrum of the operator $\left(I \otimes \Phi + \Phi \otimes I + \underline{\gamma}^{\otimes 2}\right)^{-1}\left(\underline{\gamma}^{\otimes 2} - \Phi \otimes I - I \otimes \Phi\right)$ lie in $\overline{C}$.*

Moreover, the solution process is unique, bounded up to the second order moments and its infinite dimensional evolutionary transfer functions $\underline{\mathbf{f}}\left(\lambda_{(r)}\right)$ satisfy the recursion (1.4.1) is such that $\left\{r!\underline{\mathbf{f}}_t\left(\lambda_{(r)}\right), r \in \mathbb{N}\right\} \in \Im(\mathcal{H})$.

To show the Proposition 1.4.1, we need the following lemmas

Lemma 1.4.2. *Under the condition **C1** of Proposition 1.4.1 the functions $\underline{\mathbf{f}}_t(\lambda_{(r)})$ and $\widetilde{\underline{\mathbf{f}}}_t(\lambda_{(r)})$ are well defined.*

Proof. First we define the tensor product of two bounded linear operators A and B on $\mathbb{L}_\infty$ by $\left((A \otimes B)(u \otimes v)\right)_{t,s} = (Au)_t \cdot (Av)_s$ for all $u, v \in \mathbb{L}_\infty$ and note that under **C1**, the operator $P(z)$ has an analytic inverse. For any $r \in \mathbb{N}$, let $\left\|\underline{\mathbf{f}}(r)\right\|$ be the norm of the vector $\underline{\mathbf{f}}\left(\lambda_{(r)}\right)$, then

$\|\underline{\mathbf{f}}(0)\| \leq \|P^{-1}\| \|\underline{\mu}\|$, $\|\underline{\mathbf{f}}(1)\| \leq \|P^{-1}\| \|\underline{\mathbf{f}}\|$ where $\underline{\mathbf{f}} = \underline{\gamma}\underline{\mathbf{f}}(0)+\underline{\beta}\mathbf{1}$ and $\|\underline{\mathbf{f}}(r)\| \leq r! \|P^{-1}\underline{\gamma}\|^{r-1} \|\underline{\mathbf{f}}\|$ for $r \geq 2$. This mean that $(\underline{\mathbf{f}}(\lambda_{(r)}), r \geq 0)$ is well defined and $\underline{\mathbf{f}}(\lambda_{(r)}) \in \mathbb{L}_\infty$ for all $r \in \mathbb{N}$. Since $\|Sym\underline{\mathbf{f}}(r)\| \leq \|\underline{\mathbf{f}}(r)\|$, then $\left(\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)}), r \geq 0\right)$ is also well defined. Now we prove that the coordinates $f_t(\lambda_{(r)})$ of the vector $\underline{\mathbf{f}}(\lambda_{(r)})$ belong to $\Im(\mathcal{H})$ for all $(t, n) \in \mathbb{R}\times\mathbb{N}$ or more generally, the integral $v_{t,s}(r) = \frac{1}{r!}\int_{\mathbb{R}^r} \widetilde{f}_t(\lambda_{(r)})\overline{\widetilde{f}_s(\lambda_{(r)})}dF(\lambda_{(r)})$ (which is the $r-th$ term of $Cov(X_t, X_s)$) exist and finite. Indeed, $v_{t,s}(r) \leq \frac{1}{r!}\int_{\mathbb{R}^r} \left|\widetilde{f}_t(\lambda_{(r)})\right| \left|\overline{\widetilde{f}_s(\lambda_{(r)})}\right| dF(\lambda_{(r)}) \leq \left(\sup_{\lambda_{(r)}\in\mathbb{R}^r} \|\underline{\mathbf{f}}(\lambda_{(r)})\|\right)^2$. Since $v_{t,t}(r) = \frac{1}{r!}\left\|\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)})\right\|^2$, then the functions $f_t(\lambda_{(r)}) \in \Im(\mathcal{H})$ for all $(t, r) \in \mathbb{R}\times\mathbb{N}$. $\square$

In the following lemma an explicit formula for the quantities $v_{t,s}(r)$ is given.

Lemma 1.4.3. *Let* $V(r) = (v_{t,s}(r), t, s \in \mathbb{R}) = \frac{1}{r!}\int_{\mathbb{R}^r} \widetilde{\underline{\mathbf{f}}}(\lambda_{(r)}) \otimes \overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)})}dF(\lambda_{(r)}$, *then under the condition* **C1** *of Proposition 1.4.1 we have*

$$V(r) = \frac{1}{r!}\int_{\mathbb{R}^r} \widetilde{\underline{\mathbf{f}}}(\lambda_{(r)}) \otimes \overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)})}dF(\lambda_{(r)} = G^{r-1}V(1),$$

where $G = -(\Phi\otimes I + I\otimes\Phi)^{-1}\underline{\gamma}^{\otimes 2}$ *and* $V(1) = -(\Phi\otimes I + I\otimes\Phi)^{-1}\underline{\mathbf{f}}^{\otimes 2}$.

Proof. First we note that $V(r) \in \mathbb{L}_\infty\otimes\mathbb{L}_\infty$ for all $r \geq 0$. On the other hand from (1.4.1) we obtain for $r = 1$,

$$V(1) = \int_{\mathbb{R}} \underline{\mathbf{f}}(\lambda)\otimes\overline{\underline{\mathbf{f}}(\lambda)}\frac{d\lambda}{2\pi} = \int_{\mathbb{R}} P^{-1}(i\lambda)\otimes P^{-1}(-i\lambda)\frac{d\lambda}{2\pi}\underline{\mathbf{f}}^{\otimes 2},$$

so by the residue theorem $V(1)$ reduces to $-(\Phi\otimes I + I\otimes\Phi)^{-1}\underline{\mathbf{f}}^{\otimes 2}$. For $r = 2$, we obtain

$$\begin{aligned}
V(2) &= \frac{1}{2!}\int_{\mathbb{R}^2} \widetilde{\underline{\mathbf{f}}}(\lambda_{(2)})\otimes\overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(2)})}\frac{d\lambda_1 d\lambda_2}{(2\pi)^2} \\
&= \frac{1}{2!}\int_{\mathbb{R}^2} P^{-1}(i\underline{\lambda}_{(2)})\underline{\gamma}\otimes P^{-1}(-i\underline{\lambda}_{(2)})\underline{\gamma}\{P^{-1}(i\lambda_1)\otimes P^{-1}(-i\lambda_1) + P^{-1}(i\lambda_1)\otimes P^{-1}(-i\lambda_2) \\
&+ P^{-1}(i\lambda_2)\otimes P^{-1}(-i\lambda_1) + P^{-1}(i\lambda_2)\otimes P^{-1}(-i\lambda_2)\}\frac{d\lambda_1 d\lambda_2}{(2\pi)^2}\underline{\mathbf{f}}^{\otimes 2} \\
&= \frac{1}{2!}\int_{\mathbb{R}^2} P^{-1}(i\underline{\lambda}_{(2)})\underline{\gamma}\otimes P^{-1}(-i\underline{\lambda}_{(2)})\underline{\gamma}\left\{P^{-1}(i\lambda_1)\otimes P^{-1}(-i\lambda_1) + P^{-1}(i\lambda_2)\otimes P^{-1}(-i\lambda_2)\right\}\frac{d\lambda_{(2)}}{(2\pi)^2}\underline{\mathbf{f}}^{\otimes 2} \\
&= GV(1).
\end{aligned}$$

Now, assume that the Lemma 1.4.3 is valid up to $r-1$. Then using the identity $\widetilde{\underline{\mathbf{f}}}(\lambda_{(r-1)}) = \frac{1}{r}\sum_{k=1}^{r}\widetilde{\underline{\mathbf{f}}}(\lambda_{(r\backslash k)})$, we obtain after some tedious computations

$$\frac{1}{r!}\int_{\mathbb{R}^r} \widetilde{\underline{\mathbf{f}}}(\lambda_{(r)}) \otimes \overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(r)})} \frac{d\lambda_{(r)}}{(2\pi)^r}$$
$$= \frac{1}{r!}\int_{\mathbb{R}^r} P^{-1}(i\underline{\lambda}_{(r)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(r)})\underline{\gamma} r\widetilde{\underline{\mathbf{f}}}(\lambda_{(r-1)}) \otimes r\overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(r-1)})} \frac{d\lambda_{(r)}}{(2\pi)^r}$$

$$= \frac{1}{r!}\sum_{k,l=1}^{r}\int_{\mathbb{R}^r} P^{-1}(i\underline{\lambda}_{(r)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(r)})\underline{\gamma}\widetilde{\underline{\mathbf{f}}}(\lambda_{(r\backslash l)}) \otimes \overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(r\backslash k)})} \frac{d\lambda_{(r)}}{(2\pi)^r}$$
$$= \frac{1}{r!}\sum_{k=1}^{r}\int_{\mathbb{R}^r} P^{-1}(i\underline{\lambda}_{(r)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(r)})\underline{\gamma}\widetilde{\underline{\mathbf{f}}}(\lambda_{(r\backslash k)}) \otimes \overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(r\backslash k)})} \frac{d\lambda_{(r)}}{(2\pi)^r}$$
$$+ \frac{1}{r!}\sum_{k\neq l}^{r}\int_{\mathbb{R}^r} P^{-1}(i\underline{\lambda}_{(r)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(r)})\underline{\gamma}\widetilde{\underline{\mathbf{f}}}(\lambda_{(r\backslash k)}) \otimes \overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(r\backslash l)})} \frac{d\lambda_{(r)}}{(2\pi)^r}$$
$$= \frac{r}{r!}\int_{\mathbb{R}^r} P^{-1}(i\underline{\lambda}_{(r)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(r)})\underline{\gamma}\widetilde{\underline{\mathbf{f}}}(\lambda_{(r-1)}) \otimes \overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(r-1)})} \frac{d\lambda_{(r)}}{(2\pi)^r}$$
$$+ \frac{r(r-1)}{r!}\int_{\mathbb{R}^r} P^{-1}(i\underline{\lambda}_{(r)})\underline{\gamma} \otimes P^{-1}(-i\underline{\lambda}_{(r)})\underline{\gamma}\widetilde{\underline{\mathbf{f}}}(\lambda_{(r-1)}) \otimes \overline{\widetilde{\underline{\mathbf{f}}}(\lambda_{(r\backslash r-1)})} \frac{d\lambda_{(r)}}{(2\pi)^r}.$$

Now, since the last term is zero (See Terdik [64]), the Lemma 1.4.3 follows. □

Proof. **of Proposition** 1.4.1. The proof follows essentially from the fact that if the spectrum of any operator A lies in $\overline{C} = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$, then the spectrum of $(I-A)^{-1}(I+A)$ lies in $\{|z| \in \mathbb{C} : z < 1\}$. Hence, for any $r \geq 1, V(r)$ converges to zero at an exponential rate as $r \to \infty$, so $\left(r!\underline{\mathbf{f}}_t\left(\lambda_{(r)}\right), r \geq 0\right) \in \Im(\mathcal{H})$ for all $t \in \mathbb{R}$ and their components constitute however a regular second-order solution for SDE (1.3.2). □

Remark 1.4.4. *The assumption **C1** ensure also that the CAR part has a regular solution, however, **C2** is the infinite dimensional generalization of the condition that we found in literature of differential equations systems (see Kelly (2010) chap. 6).*

Remark 1.4.5. *It is worth noting that the condition **C2** in Proposition 1.4.1, may be replaced by*

C0. *The spectrum of the operator*

$$\Psi = \int_{\mathbb{R}} \left(P(i\lambda) \otimes P(-i\lambda)\right)^{-1} \underline{\gamma}^{\otimes 2} dF(\lambda)$$

lies in $\overline{C} = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$.

Though the conditions **C1** and **C2** (or equivalently **C0**) of Proposition 1.4.1 could be used as a sufficient condition for the existence of regular solution of *SDE* (1.3.2), they are of little use in practice because they are based on the properties of infinite dimentional operators. Hence, some simple sufficient conditions can be given. Indeed, define $P_t(z) = z - \alpha(t)$ and $R_t(z) = \gamma(t)z$, it can be seen that $P(z)\mathbf{1} = (P_t(z))_{t\in\mathbb{R}}$ and $R(z)\mathbf{1} = (R_t(z))_{t\in\mathbb{R}}$. So,

Proposition 1.4.6. *A sufficient condition for the SDE* (1.3.2) *to have a second-order regular solution is that the following two conditions hold true*

C′1. $P_t(z) \neq 0$ *for all* $t \in \mathbb{R}$ *and* $z \in \mathbb{C}$.

C′2. $\Upsilon = \int\limits_{\mathbb{R}} \sup\limits_{t} |R_t(1)/P_t(i\lambda)|^2 \, d\lambda < 1$.

Proof. The condition **C′1** implie the analycity of $P^{-1}(z)$, so for all $u, v \in \mathbb{L}_\infty$, the equation $P^*(z)u = v$ has a solution. On the other hand, the condition **C′2** ensure the convergence of the series $V_\infty = \sum\limits_{r\geq 1} G^{r-1}$ (in the operator norm) or equivalently by remark 4.5, $\Psi(\mathbf{1}\otimes\mathbf{1}) \leq \Upsilon$. Indeed, for any $v \in \ell_1$ and $V \in \ell_\infty \otimes \ell_\infty$ let $\langle v\otimes\overline{v}, V\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} v_s\overline{v}_t V_{s,t}dtds$, $G(z) = P^{-1}(z)R(z)$ and set $k(z) = \sup\limits_{t} |R_t(1)/P_t(z)|^2$. Then we have

$$\begin{aligned}
\langle v\otimes\overline{v}, (G(z)\otimes G(\overline{z}))(\mathbf{1}\otimes\mathbf{1})\rangle &= |\langle v, G(z)\mathbf{1}\rangle|^2 = \left|\left\langle P^*(z)u, P^{-1}(z)R(1)\mathbf{1}\right\rangle\right|^2 = |\langle u, R(1)\mathbf{1}\rangle|^2 \\
&= \left|\int_{\mathbb{R}} u_t R_t(1)dt\right|^2 \leq \left|\int_{\mathbb{R}} u_t P_t(z)dt\right|^2 k(z) = \langle u, P(z)\mathbf{1}\rangle^2 k(z) \\
&= \langle v, 1\rangle^2 k(z) = \langle v\otimes\overline{v}, \mathbf{1}\otimes\mathbf{1}\rangle k(z).
\end{aligned}$$

The result follows by integrating this inequality along $\mathbb{R}$ for $z = i\lambda$. □

Example 1.4.7. *Consider the simplest* $COBL(1,1)$ *defined by*

$$dX(t) = \alpha(t)X(t)dt + \gamma(t)X(t)d\epsilon(t), t \geq 0, X(0) = X_0 \qquad (1.4.2)$$

Let $a = \sup\limits_{t\in\mathbb{R}} |\alpha(t)|$ *and* $\gamma = \sup\limits_{t\in\mathbb{R}} |R_t(1)|^2$. *Then by the residue theorem we have*

$$\int_{\mathbb{R}} \sup_{t} \frac{|R_t(1)|^2}{|i\lambda - \alpha(t)|^2} dF(\lambda) \leq \frac{\gamma}{2\pi}\int_{\mathbb{R}} \frac{1}{\lambda^2 - a^2} d\lambda = \frac{\gamma}{2a}.$$

So the sufficient condition for (1.4.2) *to has a regular solution is that* $\gamma < 2a$.

1.5 Applications

In this section, the general results of previous sections are particularized for the computation of the first and second- order moments totally describes the statistical properties of a Gaussian processes.

1.5.1 Second-order structure of $COBL\,(1,1)$ processes

Proposition 1.5.1. *Under the conditions of Proposition 1.4.6 we have*

$$E\left\{X(t)\right\} = \varphi_t\left(0\right)\left(f_0(0) + \int_0^t \varphi_s^{-1}\left(0\right)\mu(s)ds\right)$$

and

$$Cov(X(t), X(s)) = \varphi_t\left(0\right)\varphi_s^{-1}\left(0\right)K(s), t \geq s \geq 0$$

where $K(t) = \psi_t\left(K(0) + \int_0^t \psi_s^{-1}\left(\gamma(s)f_s(0) + \beta(s)\right)^2 ds\right)$ *with* $\psi_t = \exp\left\{\int_0^t \left(2\alpha(s) + \gamma^2(s)\right)ds\right\}$

Proof. From the representation (1.2.2) we have $E\left\{X(t)\right\} = f_t(0)$, so the expression of $E\left\{X(t)\right\}$ follows from the first equation of the recursion (1.3.5). On the other hand, for any $t \geq 0$, let $X(t) - f_t(0) = \varphi_t\left(0\right)Y(t)$ where

$$Y(t) = \sum_{r\geq 1}\int_{\mathbb{R}^r}\left(f_0(\lambda_{(r)}) + r\int_0^t \varphi_s^{-1}\left(\underline{\lambda}_{(r)}\right)\left(\gamma(s)f_s(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta(s)\right)ds\right)dZ(\lambda_{(r)}),$$

since $K(t) = Cov(X(t), X(t)) = \varphi_t^2\left(0\right)Cov(Y(t), Y(t))$, then we have

$$dK(t) = 2\alpha(t)\varphi_t^2\left(0\right)Cov(Y(t), Y(t))dt + \varphi_t^2\left(0\right)dCov(Y(t), Y(t))$$

where

$$dCov(Y(t), Y(t)) = \varphi_t^{-2}\left(\gamma(t)f_t(0) + \beta(t)\right)^2 dt + \varphi_t^{-2}\gamma^2(t)dt$$

which implies the differential equation $dK(t) = \left(2\alpha(t) + \gamma^2(t)\right)K(t)dt + \left(\gamma(t)f_t(0) + \beta(t)\right)^2 dt$ and its solution given by

$$K(t) = \psi_t\left(K(0) + \int_0^t \psi_s^{-1}\left(\gamma(s)f_s(0) + \beta(s)\right)^2 ds\right)$$

with $\psi_t = \exp\left\{\int_0^t \left(2\alpha(s) + \gamma^2(s)\right)ds\right\}$. □

Remark 1.5.2. *In stationary case, i.e., when the functions* $\alpha(.)$, $\mu(.)$, $\gamma(.)$ *and* $\beta(.)$ *are constants, we have*

$E\left\{X(t)\right\} = -\frac{\mu}{\alpha}, K(0) = Var\left\{X(t)\right\} = \frac{(\alpha\beta - \mu\gamma)^2}{\alpha^2\left|2\alpha + \gamma^2\right|}$ *and* $Cov(X(t), X(t+h)) = K(0)e^{\alpha|h|}$, $h \in \mathbb{R}$.

The result of the Proposition 1.5.1 shows that the covariance function of $COBL\,(1,1)$ processes defined by (1.3.2) has the same form as that of a $CAR\,(1)$. So we have the following result due to Lebreton and Musiela [45].

Proposition 1.5.3. *There exists a wide-sense Wiener process* $(\epsilon^*(t), t \geq 0)$ *uncorrelated with* $X(0)$ *such that* $(X(t), t \geq 0)$ *admits the* $CAR\,(1)$ *representation, i.e.,*

$$dX(t) = \left(\alpha(t)X(t) + \mu(t)\right)dt + \left\{\gamma^2(t)K(t) + \left(\gamma(t)m(t) + \beta(t)\right)^2\right\}^{1/2} d\epsilon^*(t), t \geq 0, X(0) = X_0.$$

Remark 1.5.4. *To draw the conclusion that the second-order properties of $COBL\,(1,1)$ cannot be distinguished from an $CAR\,(1)$ process. This makes it necessary for us to look into third-order cumulant in order to distinguish the nonlinear random processes.*

1.5.2 Third-order structure of $COBL\,(1,1)$ processes

For the sake of convenience and simplicity, we shall assume constant the coefficients $\alpha(t), \mu(t)$ and $\gamma(t), \beta(t)$ with $\mu(t) = 0$ in Equation (1.3.2). Moreover, we assume the process solution is given by the Wiener-Itô representation in the form $X(t) = \sum_{r=1}^{+\infty} \int_{\mathbb{R}^r} g\left(\lambda_{(r)}\right) e^{it\underline{\lambda}_{(r)}} dZ\left(\lambda_{(r)}\right)$. Using the above representation, we can obtain the following approximation

$$X(t) = \int_{\mathbb{R}} g\left(\lambda_1\right) e^{it\lambda_1} dZ\left(\lambda_1\right) + \int_{\mathbb{R}^2} g\left(\lambda_{(2)}\right) e^{it\underline{\lambda}_{(2)}} dZ\left(\lambda_{(2)}\right) + \xi(t) = X^{(1)}(t) + X^{(2)}(t) + \xi(t),$$

where $\xi(t)$ is a second-order stationary process which it is orthogonal to the first two terms. The transfer functions $g(\lambda_1)$, $g\left(\lambda_{(2)}\right)$ are given by

$$g(\lambda_1) = (i\lambda_1 - \alpha)^{-1}\beta,$$

$$g(\lambda_1, \lambda_2) = \gamma\left(i(\lambda_1 + \lambda_2) - \alpha\right)^{-1}\left(i\lambda_1 - \alpha\right)^{-1}\beta.$$

It can be shown that

$$\begin{aligned} C_X(s,u) &= E\left\{X(t)X(t+s)X(t+u)\right\} \\ &= \left[E\left\{X^{(1)}(t)X^{(1)}(t+s)X^{(2)}(t+u)\right\} + E\left\{X^{(1)}(t)X^{(2)}(t+s)X^{(1)}(t+u)\right\}\right] \\ &+ \left[E\left\{X^{(2)}(t)X^{(1)}(t+s)X^{(1)}(t+u)\right\}\right] + O\left(1\right). \end{aligned}$$

We calculate $E\left\{X^{(1)}(t)X^{(1)}(t+s)X^{(2)}(t+u)\right\}$, and the other terms can be obtained by symmetry. First we observe that

$$\begin{aligned} & E\left\{X^{(1)}(t)X^{(1)}(t+s)X^{(2)}(t+u)\right\} \\ &= E\left[\int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)e^{it\lambda_1 + i(t+s)\lambda_2} dZ(\lambda_{(2)}) \times \int_{\mathbb{R}^2} g(\lambda_3,\lambda_4)e^{i(t+u)(\lambda_3+\lambda_4)} dZ(\lambda_3,\lambda_4)\right] \\ &= 2! \int_{\mathbb{R}^2} Sym\left\{g(\lambda_1)g(\lambda_2)e^{it\lambda_1 + i(t+s)\lambda_2}\right\} \overline{Sym\left\{g(\lambda_1,\lambda_2)e^{i(t+u)(\lambda_1+\lambda_2)}\right\}} dF\left(\lambda_{(2)}\right) \\ &= 2\int_{\mathbb{R}^2} \gamma g(\lambda_1)g(\lambda_2)g(-\lambda_1-\lambda_2)\frac{1}{\beta} Sym\left\{g(-\lambda_1)\right\} Sym\left\{e^{is\lambda_1}\right\} e^{-iu(\lambda_1+\lambda_2)} \frac{d\lambda_1\lambda_2}{(2\pi)^2}. \end{aligned}$$

We calculate $E\left\{X^{(1)}(t)X^{(2)}(t+s)X^{(1)}(t+u)\right\}$,we get

$$\begin{aligned} & E\left\{X^{(1)}(t)X^{(2)}(t+s)X^{(1)}(t+u)\right\} \\ &= 2\int_{\mathbb{R}^2} \gamma g(\lambda_1)g(\lambda_2)g(-\lambda_1-\lambda_2)\frac{1}{\beta} Sym\left\{g(-\lambda_1)\right\} Sym\left\{e^{iu\lambda_1}\right\} e^{-s(\lambda_1+\lambda_2)} \frac{d\lambda_1\lambda_2}{(2\pi)^2}. \end{aligned}$$

It remains to compute $E\left\{X^{(2)}(t)X^{(1)}(t+s)X^{(1)}(t+u)\right\}$, we have

$$\begin{aligned}
&E\left\{X^{(2)}(t)X^{(1)}(t+s)X^{(1)}(t+u)\right\}\\
&=E\left[\int_{\mathbb{R}^2} g(\lambda_1,\lambda_2)e^{it(\lambda_1+\lambda_2)}Z(d\lambda_{(2)})\times\int_{\mathbb{R}^2} g(\lambda_3)g(\lambda_4)e^{i(t+s)\lambda_3+i(t+u)\lambda_4}dZ(\lambda_3,d\lambda_4)\right]\\
&=2!\int_{\mathbb{R}^2} Sym\left\{g(\lambda_1,\lambda_2)e^{it(\lambda_1+\lambda_2)}\right\}\overline{Sym\left\{g(\lambda_1)g(\lambda_2)e^{i(t+s)\lambda_1+i(t+u)\lambda_2}\right\}}dF\left(\lambda_{(2)}\right)\\
&=2\int_{\mathbb{R}^2}\gamma g(\lambda_1+\lambda_2)\frac{1}{\beta}Sym\left\{g(\lambda_1)\right\}g(-\lambda_1)g(-\lambda_2)Sym\left\{e^{-is\lambda_1-iu\lambda_2}\right\}\frac{d\lambda_1\lambda_2}{(2\pi)^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
C_X(s,u)=E\left\{X(t)X(t+s)X(t+u)\right\}=2\frac{\gamma}{\beta}\int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1-\lambda_2)Sym\left\{g(-\lambda_1)\right\}\\
\times Sym\left[e^{i(s-u)\lambda_1-u\lambda_2}+e^{e^{i(u-s)\lambda_1-s\lambda_2}}+e^{i(s\lambda_1+u\lambda_2)}\right]\frac{d\lambda_1\lambda_2}{(2\pi)^2}.
\end{aligned}$$

By taking Fourier transforms (omitting the terms of $O(1)$), the bispectral density function $f(\lambda_1,\lambda_2)$ can be shown to be

$$f(\lambda_1,\lambda_2)=2\frac{\gamma}{\beta}\frac{1}{(2\pi)^2}\left\{S(\lambda_1\lambda_2)+S(\lambda_2,-\lambda_1-\lambda_2)+S(\lambda_1,-\lambda_1-\lambda_2)\right\}$$

where $S(\lambda_1,\lambda_2)=g(\lambda_1)g(\lambda_2)g(-\lambda_1-\lambda_2)Sym\left\{g(-\lambda_1)\right\}$. It is clear that the bispectrum is zero for all frequencies λ_1 and λ_1 if and only if the process is linear ($\gamma=0$) (and Gaussian).

1.6 Conclusion

In this chapter, we have extended some results of Terdik [64] on time-invariant bilinear SDE to time-varying one. So, we have analyzed the probabilistic structure of general nonlinear continuous-time processes via Wiener's chaos. In particular, necessary and sufficient conditions for the existence of regular second-order solutions are given for a $COBL(1,1)$ driven by a standard Brownian motion with explicit expression in terms of higher-order evolutionary transfer functions. The main advantage of the frequency approach is that beside its adaptation with nonlinear effects, it preserves the mathematically tractable $CARMA$ structure. In particular, it was seen in Section 4, that the spectrum (second-order properties) does not generally provide sufficient information about the structure of the process. Hence, it is necessary to look at third-order cumulant in order to discriminate the $COBL(1,1)$ from a $CAR(1)$ process which seem to be too difficult and tedious. However, specific tools, for instance wavelet methods as an alternative to Fourier methods should be adapted to analysis the general bilinear SDE with time-varying coefficients. We leave this important issue for future researches.

Chapter 2

A study of the first-order continuous-time bilinear processes driven by fractional Brownian motion[2]

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Abstract

The continuous-time bilinear ($COBL$) process has been used to model non linear and/or non Gaussian datasets. In this chapter, the first-order continuous-time bilinear $COBL\,(1,1)$ model driven by a fractional Brownian motion (fBm for short) process is presented. The use of fBm processes with certain Hurst parameter permits to obtain a much richer class of possibly long-range dependent property which are frequently observed in financial econometrics, and thus can be used as a power tool for modelling irregularly series having memory. So, the existence of Itô's solutions and there chaotic spectral representations for time-varying $COBL\,(1,1)$ processes driven by fBm are studied. The second-order properties of such solutions are analyzed and the long-range dependency property are studied.

2.1 COBL(1,1) driven by fractional Brownian motion

In discrete-time series analysis, the assumption of linearity and/or Gaussianity is frequently made. Unfortunately these assumption lead to models that fail to capture certain phenomena commonly observed in practice such as limit cycles, asymmetric distribution, leptokurtosis, etc..., Motived by these deficiencies, non linear parametric modelling of time series has attracted considerable attention in recent years. Indeed, one of the most useful class of non-linear time series models is the bilinear specification obtained by adding to an $ARMA$ model one or more interaction components between the observed series and the innovations. However, it is observed that these models are not be able to give full information about some datasets exhibit unequally spaced observations and hence the resort to a continuous-time version is crucial. So, in this chapter we consider a continuous-time bilinear processes $(X(t))_{t\in\mathbb{R}}$ defined on some complete probability space $(\Omega,\mathcal{A},P)$ equipped with a filtration $(\mathcal{A}_t)_{t\geq 0}$ and subjected to be a solution of the following affine time-varying stochastic differential equation (SDE)

$$dX(t) = (\alpha(t)X(t) + \mu(t))\,dt + (\beta(t) + \gamma(t)X(t))\,\,dW^h(t), t \geq t_0, X(t_0) = X_0 \qquad (2.1.1)$$

denoted hereafter $COBL(1,1)$. The parameters $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are differentiable complex deterministic functions subject to the following assumption
Condition 2.1

A1 For all $T > t_0$, $\int_{t_0}^{T} |\alpha(t)|\,dt < \infty$, $\int_{t_0}^{T} |\mu(t)|\,dt < \infty$, $\int_{t_0}^{T} |\gamma(t)|^2\,dt < \infty$. $\int_{t_0}^{T} |\beta(t)|^2\,dt < \infty$.

A2 $\alpha(t), \mu(t), \beta(t) \in \mathbb{C}$ and $\Re e\,(\gamma(t)) = 0$ and $\Re e\,\{\alpha(t)\} < 0$, for all $t \geq t_0$.

In (2.1.1), $(W^h(t))_{t\in\mathbb{R}}$ is a real fBm with Hurst parameter $h \in \left]0, \frac{1}{2}\right[$ defined on a basic given filtered stochastic probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, P)$, its covariance kernel is $Cov\left(W^h(t), W^h(s)\right) = \frac{\kappa(h)}{2}(|t|^{2h+1} + |s|^{2h+1} - |t-s|^{2h+1})$, for all $t, s \geq 0$, where $\kappa(h) = \frac{\Gamma(1-2h)}{h(2h+1)\pi} cos\left(\frac{\pi}{2}(1-2h)\right)$ and admits a spectral representation $W^h(t) = \int\limits_{\mathbb{R}} \phi_t(\lambda)\,(i\lambda)^{-h} dZ(\lambda)$ where $\phi_t(\lambda) = \frac{e^{it\lambda}-1}{i\lambda}$ and $dZ(.)$ is a complex$-$valued Gaussian spectral measure defined on $(\Omega, \mathcal{A}, P)$ with zero mean, variance $E\left\{|dZ(\lambda)|^2\right\} = dG(\lambda) = \frac{d\lambda}{2\pi}$ and where the principal value of $\frac{1}{2\pi}\int\limits_{\mathbb{R}} \phi_t(\lambda)\,d(\lambda)$ is 0. Note that the initial state $X(t_0)$ is a random variable, defined on the same probability space $(\Omega, \mathcal{A}, P)$ independent of $\sigma(W(t), t_0 \leq t \leq T)$ such that $E\{X(t_0)\} = m(t_0)$ and $Var\{X(t_0)\} = R(t_0) < +\infty$. It is well known that if $h = 0$, then the corresponding fBm reduces to the usual Brownian motion, otherwise, $\left(W^h(t)\right)_{t\geq 0}$ is neither a Markovian nor a semimartingales processes and hence the usual calculus cannot be used, so a different calculus is required. This non Markovian processes have not an independent stationary increments and are well suited for modelling data exhibiting a long-range dependency. For an in-depth detailed mathematical framework of the pertinent properties of fBm, we refer the reader to Mishura [51] and the references therein.
The SDE (2.1.1) is called time-invariant when $\alpha(t)$, $\mu(t)$, $\gamma(t)$ are complex deterministic constant functions, i.e., there is some constants complex α, μ, γ such that $\alpha(t) = \alpha$, $\mu(t) = \mu$, $\gamma(t) = \gamma$ and for all t.
The SDE (2.1.1) encompasses many commonly used models in the literature. Some examples among others are

1. First-order continuous-time Autoregressive processes ($CAR(1)$ for short): This classes of SDE may be obtained by assuming $\gamma(t) = 0$ for all t. (see Duncan at al. [22] and the reference therein).

2. Gaussian Ornstein-Uhlenbeck (OU) process: The Gaussian OU process is defined as $dX(t) = (\mu(t) - \alpha(t)X(t))\,dt + \beta(t)\,dW^h(t)$, with $\beta(t) > 0$ for all $t \geq 0$. So it can be obtained from (2.1.1) by assuming $\gamma(t) = 0$ for all t. (see Shen et al. [60] and the reference therein).

3. Nelson's diffusion process: In the diffusion process of Nelson (see Swishchuk [63], chapter 2), the time-varying volatility process may be defined as the second-order solution process $(V(t))_{t\geq 0}$ of $dV(t) = \lambda(t)(\mu(t) - V(t))\,dt + \gamma(t)\,V(t)dW^{(h)}(t)$ in which $\lambda(t), \mu(t)$ and $\gamma(t)$ are positive deterministic functions. This SDE can be obtained easily from (2.1.1).

4. Geometric Brownian motion (GBM): This class of processes is defined as a $\mathbb{R}-$valued solution process $(X(t))_{t\geq 0}$ of $dX(t) = \alpha(t)X(t)dt + \gamma(t)X(t)dW^{(h)}(t), t \geq 0$. So it can be obtained from (2.1.1) by assuming $\beta(t) = \mu(t) = 0$ for all t. (see Bender et al. [8] and the reference therein).

It is worth noting that beside the above mentioned particular cases, the equation (2.1.1) may be extended to vectorial case, i.e., when $X(t)$ is $\mathbb{R}^d-$valued process, so other particular models can be deduced.

2.2 The solution processes of $COBL(1,1)$

Let $\Im^{(h)} = \Im(W^{(h)}) := \sigma(W^{(h)}(t), t \geq t_0)$ (resp $\Im_t^{(h)} := \sigma(W^{(h)}(s), t_0 \leq s \leq t)$) be the σ -algebra generated by $\left(W^{(h)}(t)\right)_{t\geq 0}$ (resp. generated by $W^{(h)}(s)$ up to time t) and let $\mathbb{L}_2(\Im^{(h)}) = \mathbb{L}_2(\mathbb{C}, \Im^{(h)}, P)$ (resp. $\mathbb{L}_2(\Im_t^{(h)})$) be the Hilbert space of nonlinear $\mathbb{L}_2-$functional of $\left(W^{(h)}(t)\right)_{t\geq 0}$. In this section, we are interested in solving the SDE (2.1.1) in $\mathbb{L}_2(\Im_t^{(h)})$. As already pointed by several authors (see for instance Duncan [21] for further discussions), that there is no general theory for the solution of SDE driven by an fBm if $h \neq 0$. Nevertheless, recently some studies was investigated the existence of such solutions for various families of SDE driven by an fBm. Indeed,

2.2.1 The Itô approach

Our first approach is based on the Itô formula with respect to fBm and the general results on SDE to prove the uniqueness of the solution. First, we start by the fractional Itô's formula which is a powerful tool for dealing the solution. Consider the following stochastic differential equation driven by fBm

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW^h(t),\ X(t_0) = X_0 \tag{2.2.1}$$

in which $a(.,.)$, $b(.,.)$ are known continuous functions that represents the drift and diffusion respectively of the SDE (2.2.1) supposed to be smooth enough, and set $Y(t) = U(t, X(t))$ for some differentiable function $U : \mathbb{R} \to R$. Then Dai and Heyde [20] have shown that the Itô formula with respect to fBm is given by

$$dY(t) = \left\{\frac{\partial U}{\partial t}(t, X(t)) + a(t,w)\frac{\partial U}{\partial x}(t, X(t))\right\}dt + b(t,w)\frac{\partial U}{\partial x}(t, X(t))dW^h(t). \tag{2.2.2}$$

Therefore, from the SDE (2.2.1) and the Itô formula (2.2.2) we obtain

$$dY(t) = \frac{\partial U}{\partial t}(t, X(t))dt + \frac{\partial U}{\partial x}(t, X(t))dX(t) \tag{2.2.3}$$

So, the Itô$'s$ solution of the SDE (2.1.1) is given by

Theorem 2.2.1. *Under the assumption 2.1, the unique Itô's solution of SDE (2.1.1) in $\mathbb{L}_2(\Im^{(h)})$ is given by*

$$X(t) = \Phi_h(t,t_0)\left\{X(t_0) + \int_{t_0}^{t}\Phi_h^{-1}(s,t_0)\mu(s)ds + \int_{t_0}^{t}\Phi_h^{-1}(s,t_0)\beta(s)dW^h(s)\right\}, t \geq t_0 \tag{2.2.4}$$

where $\Phi_h(t,t_0) = \exp\left\{\int\limits_{t_0}^{t} \alpha(s)\,ds + \int\limits_{t_0}^{t} \gamma(s)\,dW^h(s)\right\}$ *with* $\Phi_h(t_0,t_0) = 1$ *and the stochastic integral* $\int\limits_{t_0}^{t} \gamma(s)\,dW^h(s)$ *is defined in Riemann's sense in probability.*

Proof. First it is no difficult to see that $\Phi_h(t,t_0)$ is the unique solution of stochastic differential equation

$$d\Phi_h(t,t_0) = \alpha(t)\Phi_h(t,t_0)dt + \gamma(t)\Phi_h(t,t_0)dW^h(t).$$

Now, set $\Phi_h(t,t_0) = \exp\{Y(t)\}$, $Z(t) = X(0) + \int\limits_{t_0}^{t} e^{-Y(s)}\mu(s)ds + \int\limits_{t_0}^{t} e^{-Y(s)}\beta(s)dW^h(s)$ and let $X(t) = U(Y(t), Z(t))$, where U is the function defined by $U(x,y) = e^x y$. The fractional Itô formula (2.2.2) and the expression (2.2.3) gives

$$\begin{aligned} dX(t) &= \frac{\partial U}{\partial x}(Y(t),Z(t))dY(t) + \frac{\partial U}{\partial y}(Y(t),Z(t))dZ(t) \\ &= e^{Y(t)}Z(t)dY(t) + e^{Y(t)}dZ(t) \\ &= X(t)dY(t) + e^{Y(t)}dZ(t) \\ &= X(t)\left(\alpha(t)dt + \gamma(t)dW^h(t)\right) + e^{Y(t)}\left(e^{-Y(t)}\mu(t) + e^{-Y(t)}\beta(t)dW^h(t)\right)dt \\ &= \left(\alpha(t)X(t) + \mu(t)\right)dt + \left(\gamma(t)X(t) + \beta(t)\right)dW^h(t). \end{aligned}$$

and hence the result follows. □

Remark 2.2.2. *If* $\beta(t) = 0$*, then the Itô solution of SDE* (2.1.1) *reduces to*

$$X(t) = \Phi_h(t,t_0)\left\{X(t_0) + \int\limits_{t_0}^{t} \Phi_h^{-1}(s,t_0)\mu(s)ds\right\}, t \geq t_0 \quad (2.2.5)$$

and when $\gamma(t) = 0$ *and* $\beta(t) \neq 0$*, this is provides a solution of Gaussian OU process, therefore if we are interested in non-Gaussian solution of* (2.1.1)*, it is necessary to assume that* $|\mu(t)|^2 + |\beta(t)|^2 > 0$ *and* $\gamma(t) \neq 0$*.*

Remark 2.2.3. *In time-invariant case, with* $\Re e\{\gamma\} = 0$ *and* $\Re e\{\alpha\} < 0$ *, then the Itô solution of SDE* (2.1.1) *can be written as*

$$X(t) = \mu\int_{-\infty}^{t} \exp\left\{\alpha(t-s) + i\gamma\left(W^h(t) - W^h(s)\right)\right\}ds + \beta\int_{-\infty}^{t} \exp\{\alpha(t-s)\}\,dW^h(s).$$

Remark 2.2.4. *For any* $t \geq t_0$*, let* $-\xi(t) = \int\limits_{t_0}^{t} \alpha(s)\,ds + \int\limits_{t_0}^{t} \gamma(s)\,dW^h(s)$ *and* $\eta^h(t) = \int\limits_{t_0}^{t} \mu(s)\,ds + \int\limits_{t_0}^{t} \beta(s)\,dW^h(s)$*, then the solution process* (2.2.4) *may be rewritten as*

$$X(t) = e^{-\xi(t)}\left\{X(t_0) + \int\limits_{t_0}^{t} e^{\xi(s)}d\eta^h(s)\right\}, t \geq t_0 \quad (2.2.6)$$

is the solution process of generalized Ornstein-Uhlenbeck (GOU) *process driven by an* fBm *defined by* $dX(t) = -\xi(t)X(t)dt + d\eta^h(t), t \geq t_0, X(t_0) = X_0$*.*

2.2.2 The frequency approach

In this subsection, we discuss a second approach to solve the *SDE* (2.1.1) based on the spectral representation. Indeed, it is now well known that for any regular second-order process $(X(t))_{t\geq t_0}$ (i.e., $X(t)$ is $\Im_t^{(h)}-$measurable not necessary stationary, belonging to $\mathbb{L}_2(\Im^{(h)})$) admits the so-called Wiener-Itô (or Stratonovich) spectral representation, i.e.,

$$X(t)=g_t(0)+\sum_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r} g_t(\underline{\lambda}_{(r)})e^{it\Sigma\underline{\lambda}_{(r)}}\prod_{j=1}^{r}(i\lambda_j)^{-h}dZ\left(\underline{\lambda}_{(r)}\right). \tag{2.2.7}$$

where $\underline{\lambda}_{(r)}=(\lambda_1,...,\lambda_r)$, $\Sigma\underline{\lambda}_{(r)}=\sum_{i=1}^r\lambda_i$ and $dZ\left(\underline{\lambda}_{(r)}\right)=\prod_{j=1}^r dZ\left(\lambda_i\right)$. (see Bibi and Merahi [11] for more details). The representation (2.2.7) is unique up to the permutation of the arguments of the evolutionary transfer functions $g_t(\underline{\lambda}_{(r)})$, $r\geq 2$ and $g_t(\underline{\lambda}_{(r)})\in\mathbb{L}_2\left(G^h\right)=\mathbb{L}_2\left(\mathbb{C}^n,B_{\mathbb{C}^n},G^h\right)$ for all $t\geq t_0$, with $dG^h(\lambda_{(r)})=\dfrac{1}{(2\pi)^r}\prod\limits_{i=1}^{r}|\lambda_i|^{-2h}d\lambda_{(r)}$ and such that

$$\sum_{r\geq 0}\frac{1}{r!}\int_{\mathbb{R}^r}\left|g_t(\underline{\lambda}_{(r)})\right|^2 dG^h(\underline{\lambda}_{(r)})<\infty \text{ for all } t\geq t_0. \tag{2.2.8}$$

Let us recall here the so-called the diagram formula for Wiener–Itô representation (2.2.7) which play an important role in some subsequent proofs and that state that for all g and f defined on $\mathbb{R}$ and on $\mathbb{R}^r$ respectively such that $(g,f)\in\mathbb{L}_2\left(\mathbb{R}\right)\times\mathbb{L}_{2r}\left(\mathbb{R}^r\right)$, if f is symmetric then

$$\begin{aligned}\int_{\mathbb{R}} g(\lambda)dZ(\lambda)\int_{\mathbb{R}^r} f(\underline{\lambda}_{(r)})dZ(\underline{\lambda}_{(r)}) &= \int_{\mathbb{R}^{r+1}} g(\lambda_{r+1})f(\underline{\lambda}_{(r)})dZ(\underline{\lambda}_{(r+1)}) \\ &\quad+\frac{r}{2\pi}\int_{\mathbb{R}^{r-1}}\left\{\int_{\mathbb{R}}\overline{g(\lambda_r)}f(\underline{\lambda}_{(r)})d\lambda_r\right\}dZ(\underline{\lambda}_{(r-1)}).\end{aligned}$$

The spectral representation of the solution process of *SDE* (2.1.1) is given in the following theorem

Theorem 2.2.5. *Assume that the process* $(X(t))_{t\geq t_0}$ *generated by the SDE* (2.1.1) *has a regular second-order solution. Then, the evolutionary symmetrized transfer functions* $\left(\widetilde{g}_t(\lambda_{(r)})\right)_{t\geq t_0}, r\in\mathbb{N}$ *of such solution are given by the symmetrization of the solution of the following first order ordinary differential equations*

$$g_t^{(1)}(\underline{\lambda}_{(r)})=\begin{cases}\alpha(t)g_t(0)+\mu(t)+\dfrac{\gamma(t)}{2\pi}\int_{\mathbb{R}} g_t(\lambda)\left|\lambda\right|^{-2h}d\lambda,\ r=0\\ \left(\alpha(t)-i\Sigma\underline{\lambda}_{(r)}\right)g_t(\underline{\lambda}_{(r)})+r\delta_{[r=1]}\beta(t)\\ +\gamma(t)\left(rg_t(\underline{\lambda}_{(r-1)})+\dfrac{1}{2\pi}\int_{\mathbb{R}} g_t(\underline{\lambda}_{(r+1)})\left|\lambda_{r+1}\right|^{-2h}d\lambda_{r+1}\right), r\geq 1\end{cases} \tag{2.2.9}$$

where the superscript $^{(j)}$ *denotes* $j-$*fold differentiation with respect to* t *and where* $\Sigma\underline{\lambda}_{(r)}=\sum_{i=1}^r\lambda_i$.

Proof. First, applying of the diagram formula for the nonlinear term $X(t)\dfrac{dW^h(t)}{dt}$ we get

$$X(t)\frac{dW^h(t)}{dt} = \int_{\mathbb{R}} g_t(0)e^{it\lambda}(i\lambda)^{-h}dZ(\lambda) + \sum_{r=1}^{\infty}\frac{1}{r!}\int_{\mathbb{R}^{r+1}} \widetilde{g}_t(\underline{\lambda}_{(r)})e^{it\Sigma\underline{\lambda}_{(r+1)}}\prod_{l=1}^{r+1}(i\lambda_l)^{-h}\, dZ(\underline{\lambda}_{(r+1)})$$
$$+\sum_{r=1}^{\infty}\frac{1}{(r-1)!}\int_{\mathbb{R}^{r-1}} e^{it\Sigma\underline{\lambda}_{(r-1)}}\left(\frac{1}{2\pi}\int_{\mathbb{R}} g_t(\underline{\lambda}_{(r)})\left|\lambda_r\right|^{-2h} d\lambda_r\right)\prod_{l=1}^{r-1}(i\lambda_l)^{-h}\, dZ(\underline{\lambda}_{(r-1)}).$$

Second, we insert the spectral representation (2.2.7) of the process $(X(t))_{t\geq t_0}$ and the last expression of $X(t)dW^h(t)$ in the equation (2.1.1) the results follows. □

Remark 2.2.6. *The existence and uniqueness of the solution* (2.2.9) *is ensured by general results on linear ordinary differential equations, so*

$$g_t(\underline{\lambda}_{(r)}) = \begin{cases} \varphi_t\left(0\right)\left(g_{t_0}(0)+\int_{t_0}^{t}\varphi_s^{-1}\left(0\right)\left(\mu(s)+\gamma(s)\frac{1}{2\pi}\int_{\mathbb{R}} g_s(\lambda)\left|\lambda\right|^{-2h}d\lambda\right)ds\right), r=0 \\ \varphi_t\left(\lambda\right)\left(g_{t_0}(\lambda)+\int_{t_0}^{t}\varphi_s^{-1}\left(\lambda\right)\left\{\beta(s)+\gamma(s)\left(g_s(0)+\frac{1}{2\pi}\int_{\mathbb{R}} g_s(\underline{\lambda}_{(2)})\left|\lambda_2\right|^{-2h}d\lambda_2\right)\right\}ds\right), r=1 \\ \varphi_t(\underline{\lambda}_{(r)})\left(g_{t_0}(\underline{\lambda}_{(r)})+\int_{t_0}^{t}\varphi_s^{-1}(\underline{\lambda}_{(r)})\gamma(s)\left(rg_s(\underline{\lambda}_{(r-1)})+\frac{1}{2\pi}\int_{\mathbb{R}} g_s(\underline{\lambda}_{(r+1)})\left|\lambda_{r+1}\right|^{-2h}d\lambda_{r+1}\right)ds\right), r\geq 2 \end{cases} \quad (2.2.10)$$

in which $\varphi_t\left(\underline{\lambda}_{(r)}\right) = \exp\left\{\int_{t_0}^{t}\left(\alpha(s)-i\Sigma\underline{\lambda}_{(r)}\right)ds\right\}.$

Remark 2.2.7. *Noting that beside the condition* (2.2.8) *a necessary conditions for that the evolutionary transfer functions* $\left(g_t(\underline{\lambda}_{(r)}), r\in\mathbb{N}\right)$ *defined by* (2.2.10) *determines a second-order process are*

$$\int_{\mathbb{R}}\left|\int_{\mathbb{R}} g_t(\underline{\lambda}_{(r+1)})\left|\lambda_{r+1}\right|^{-2h}d\lambda_{r+1}\right|^2\left|\lambda_r\right|^{-2h}d\lambda_r < +\infty \text{ and } \int_{\mathbb{R}}\left|g_t(\underline{\lambda}_{(r+1)})\right|\left|\lambda_{r+1}\right|^{-2h}d\lambda_{r+1} < +\infty$$

for all $t\geq t_0$. *These conditions are extremely difficult to be verified, except in time-invariant case when an explicit formula for the transfer functions are given (see for instance Iglói and Terdik [38]).*

It is worth noting that if $\Re e\left\{\gamma(t)\right\}\neq 0$, the SDE (2.1.1) may be haven't a second-order solution, but it does if $\gamma(t)$ is purely imaginary. So in what follows, we consider the particular SDE

$$dX(t) = \left(\alpha\left(t\right)X(t)+\mu\left(t\right)\right)dt + i\gamma\left(t\right)X(t)dW^h(t), t\geq t_0, X(t_0) = X_0. \quad (2.2.11)$$

and assume that

A3. $\alpha(t)$, $\mu(t)\in\mathbb{C}$, $\gamma(t)\in\mathbb{R}$ and $\Re e\left\{\alpha(t)\right\}<0$, $\gamma(t)\neq 0$ for all $t\geq t_0$.

Under the condition **A3**, the Itô's solution of (2.2.11) reduces to

$$X(t) = \Phi_h(t,t_0)\left\{X(t_0)+\int_{t_0}^{t}\Phi_h^{-1}(s,t_0)\mu(s)ds\right\},$$

in which the function $\gamma(t)$ is replaced by $i\gamma(t)$. The spectral representation of equation (2.2.11) is given in the following lemma

Lemma 2.2.8. *Assume that the process $(X(t))_{t\geq t_0}$ generated by the model (2.2.11) has a regular second-order solution. Then, the symmetrized evolutionary transfer functions $\left(\widetilde{g}_t(\lambda_{(r)})\right)_{t\in\mathbb{R}}, r\in\mathbb{N}$ of such solution may be obtained by the symmetrization of the following functions*

$$g_t(\lambda_{(r)}) \tag{2.2.12}$$
$$= \begin{cases} \varphi_t(0)\left(g_{t_0}(0)+\int\limits_{t_0}^{t}\varphi_s^{-1}(0)\left(\mu(s)+i\gamma(s)\frac{1}{2\pi}\int\limits_{\mathbb{R}}g_s(\lambda)\left|\lambda\right|^{-2h}d\lambda\right)ds\right), r=0\\ \varphi_t(\underline{\lambda}_{(r)})\left(g_{t_0}(\lambda_{(r)})+i\int\limits_{t_0}^{t}\varphi_s^{-1}(\underline{\lambda}_{(r)})\gamma(s)\left(r\ g_s(\lambda_{(r-1)})+\frac{1}{2\pi}\int\limits_{\mathbb{R}}g_s(\lambda_{(r+1)})\left|\lambda_{r+1}\right|^{-2h}d\lambda_{r+1}\right)ds\right), r\geq 1\end{cases}$$

Remark 2.2.9. *In time-invariant case we obtain*

$$g(\lambda_{(r)})=\begin{cases} g(\lambda_{(r)})=-\dfrac{1}{\alpha}\left\{\mu+\dfrac{i\gamma}{2\pi}\int\limits_{\mathbb{R}}g(\lambda)\left|\lambda\right|^{-2h}d\lambda\right\} & \text{if } r=0\\ \dfrac{-i\gamma}{\left(\alpha-i\underline{\lambda}_{(r)}\right)}\left\{r\ g(\lambda_{(r-1)})+\dfrac{1}{2\pi}\int\limits_{\mathbb{R}}g(\lambda_{(r+1)})\left|\lambda_{r+1}\right|^{-2h}d\lambda_{r+1}\right\} & \text{if } r\geq 1\end{cases}$$

so, its symmetrized version may be written as

$$\widetilde{g}(\lambda_{(r)})=Sym\left\{g(\lambda_{(r)})\right\}=\mu\left(i\gamma\right)^r\int\limits_0^{\infty}\exp\left\{\alpha u-\frac{\gamma^2}{2}k(h)u^{2h+1}\right\}\prod_{j=1}^{r}\frac{1-e^{-iu\lambda_j}}{i\lambda_j}du.$$

2.3 The moments properties and the second-order structure

In this section, we analyze the spectrum i.e., the second-order structure of the process $(X(t))_{t\geq t_0}$ solution of the SDE (2.1.1). For this purpose let $(\Psi_h(t,t_0))_{t\geq t_0}$ be the mean function of the process $(\Phi_h(t,t_0))_{t\geq t_0}$, and set $W_h(t,u,s,v)=h(2h+1)\kappa(h)\int_u^t\int_v^s\gamma(v_1)\overline{\gamma(v_2)}|v_1-v_2|^{2h-1}dv_2dv_1$, $u\leq t$, $v\leq s$. Then, we have

Lemma 2.3.1. *Under the conditions of 2.1, we have the following assertions*

1. $\Psi_h(t,t_0)=\exp\left\{\int_{t_0}^t\alpha(v_1)dv_1+h(2h+1)\frac{\kappa(h)}{2}\int_{t_0}^t\int_{t_0}^t\gamma(v_1)\gamma(v_2)|v_1-v_2|^{2h-1}dv_1dv_2\right\}$ *for* $t\geq t_0$.
2. $E\left\{\Phi_h(t,t_0)\Phi_h^{-1}(u,t_0)\right\}=\Psi_h(t,u)$ *for* $t\geq u$.
3. $E\left\{\Phi_h(t,t_0)\overline{\Phi_h(s,t_0)}\right\}=\Psi_h(t,t_0)\overline{\Psi_h(s,t_0)}\exp\left\{W_h(t,t_0,s,t_0)\right\}$ *for* $t\geq s$.
4. $E\left\{\Phi_h(t,t_0)\overline{\Phi_h(s,t_0)}\overline{\Phi_h^{-1}(v,t_0)}\right\}=\Psi_h(t,t_0)\overline{\Psi_h(s,v)}\exp\left\{W_h(t,t_0,s,v)\right\}$ *for* $t\geq s\geq v$.
5. $E\left\{\Phi_h(t,t_0)\overline{\Phi_h(s,t_0)}\Phi_h^{-1}(u,t_0)\overline{\Phi_h^{-1}(v,t_0)}\right\}=\Psi_h(t,u)\overline{\Psi_h(s,v)}\exp\left\{W_h(t,u,s,v)\right\}$ *for* $t\geq s\geq v$.

Proof. The assertions of the Lemma 2.3.1 follows upon observation that by using the expectation of exponential Gaussian process, we have

$$
\begin{aligned}
& \Psi_h(t,t_0) \\
= \;& \exp\left\{\int_{t_0}^{t}\alpha(v_1)dv_1 + \frac{1}{2}E\left\{\left(\int_{t_0}^{t}\gamma(v_1)dW^h(v_1)\right)^2\right\}\right\} \\
= \;& \exp\left\{\int_{t_0}^{t}\alpha(v_1)dv_1 + h(2h+1)\frac{\kappa(h)}{2}\int_{t_0}^{t}\int_{t_0}^{t}\gamma(v_1)\gamma(v_2)|v_1-v_2|^{2h-1}dv_1dv_2\right\}
\end{aligned}
$$

and for $t \geq u$

$$
\begin{aligned}
& E\left\{\Phi_h(t,t_0)\Phi_h^{-1}(u,t_0)\right\} \\
= \;& \exp\left\{\int_{u}^{t}\alpha(v_1)dv_1 + \frac{1}{2}E\left\{\left(\int_{u}^{t}\gamma(v_1)dW^h(v_1)\right)^2\right\}\right\} \\
= \;& \exp\left\{\int_{u}^{t}\alpha(v_1)dv_1 + h(2h+1)\frac{\kappa(h)}{2}\int_{u}^{t}\int_{u}^{t}\gamma(v_1)\gamma(v_2)|v_1-v_2|^{2h-1}dv_1dv_2\right\} \\
= \;& \Psi_h(t,u).
\end{aligned}
$$

and so on the rest are immediate. □

Lemma 2.3.2. *Under the condition of Lemma 2.3.1, the mean function* $(m_h(t) = E\{X(t)\})_{t\geq t_0}$ *is given by*

$$
m_h(t) = \Psi_h(t,t_0)m(t_0) + \int_{t_0}^{t}\Psi_h(t,u)\mu(u)du,\; t \geq t_0.
$$

and the covariance function $\left(R_h(t,s) = E\left\{(X(t)-m_h(t))\,\overline{(X(s)-m_h(s))}\right\}\right)_{t\geq s}$ *is given by*

$$
\begin{aligned}
R_h(t,s) = \;& \Psi_h(t,t_0)\overline{\Psi_h(s,t_0)}\exp\left\{W_h\left(t,t_0,s,t_0\right)\right\}R(t_0) \\
& + \Psi_h(t,t_0)\overline{\Psi_h(s,t_0)}\left[\exp\left\{W_h\left(t,t_0,s,t_0\right)\right\}-1\right]|m(t_0)|^2 \\
& + m(t_0)\int_{t_0}^{s}\Psi_h(t,t_0)\overline{\Psi_h(s,v)}\left[\exp\left\{W_h\left(t,t_0,s,v\right)\right\}-1\right]\overline{\mu(v)}dv \\
& + \overline{m(t_0)}\int_{t_0}^{t}\overline{\Psi_h(s,t_0)}\Psi_h(t,u)\left[\exp\left\{W_h\left(t,u,s,t_0\right)\right\}-1\right]\mu(u)du \\
& + \int_{t_0}^{t}\int_{t_0}^{s}\Psi_h(t,u)\overline{\Psi_h(s,v)}\left[\exp\left\{W_h\left(t,u,s,v\right)\right\}-1\right]\overline{\mu(v)}\mu(u)dvdu \\
& + h(2h+1)\kappa(h)\int_{t_0}^{t}\int_{t_0}^{s}\Psi_h(t,u)\overline{\Psi_h(s,v)}\exp\left\{W_h\left(t,u,s,v\right)\right\}\overline{\beta(v)}\beta(u)|u-v|^{2h-1}dvdu.
\end{aligned}
$$

Proof. From the Itô's solution (2.2.4), we can obtain

$$
\begin{aligned}
m_h(t) = E\{X(t)\} &= E\left\{\Phi_h(t,t_0)X(t_0)\right\} + \int_{t_0}^{t}E\left\{\Phi_h(t,t_0)\Phi_h^{-1}(u,t_0)\right\}\mu(u)du \\
&= \Psi_h(t,t_0)m(t_0) + \int_{t_0}^{t}\Psi_h(t,u)\mu(u)du.
\end{aligned}
$$

Since $W^h(t)$ independent of $X(t_0)$, then $E\left\{\Phi_h(t,t_0)X(t_0)\right\} = E\left\{\Phi_h(t,t_0)\right\}E\left\{X(t_0)\right\} = \Psi_h(t,t_0)m_h(t_0)$. In order to evaluate the expression of $R_h(t,s)$ we use the Itô's solution (2.2.4) to obtain

$$E\left\{X(t)\overline{X(s)}\right\} = E\left\{\Phi_h(t,t_0)\overline{\Phi_h(s,t_0)}\right\}E\left\{|X(t_0)|^2\right\} + m(t_0)\int_{t_0}^{s} E\left\{\Phi_h(t,t_0)\overline{\Phi_h(s,t_0)}\,\overline{\Phi_h^{-1}(v,t_0)}\right\}\overline{\mu(v)}dv$$
$$+\overline{m(t_0)}\int_{t_0}^{t} E\left\{\overline{\Phi_h(s,t_0)}\Phi_h(t,t_0)\Phi_h^{-1}(u,t_0)\right\}\mu(u)du$$
$$+\int_{t_0}^{t}\int_{t_0}^{s} E\left\{\Phi_h(t,t_0)\overline{\Phi_h(s,t_0)}\Phi_h^{-1}(u,t_0)\overline{\Phi_h^{-1}(v,t_0)}\right\}\overline{\mu(v)}\mu(u)dvdu$$
$$+h(2h+1)\kappa(h)\int_{t_0}^{t}\int_{t_0}^{s} E\left\{\Phi_h(t,t_0)\overline{\Phi_h(s,t_0)}\Phi_h^{-1}(u,t_0)\overline{\Phi_h^{-1}(v,t_0)}\right\}\overline{\beta(v)}\beta(u)|u-v|^{2h-1}dvdu,$$

In other hand

$$\begin{aligned} m_h(t)\overline{m_h(s)} &= \Psi_h(t,t_0)\overline{\Psi_h(s,t_0)}|m(t_0)|^2 + m(t_0)\int_{t_0}^{s}\Psi_h(t,t_0)\overline{\Psi_h(s,v)\mu(v)}dv \\ &\quad +\overline{m(t_0)}\int_{t_0}^{t}\overline{\Psi_h(s,t_0)}\Psi_h(s,u)\mu(u)du + \int_{t_0}^{t}\int_{t_0}^{s}\Psi_h(t,u)\overline{\Psi_h(s,v)\mu(v)}\mu(u)dvdu, \end{aligned}$$

the fact that $R_h(t,s) = E\left\{X(t)\overline{X(s)}\right\} - m_h(t)\overline{m_h(s)}$, the expression for $R_h(t,s)$ follows. □

Lemma 2.3.3. *Consider the time-invariant process $(X(t))_{t\geq t_0}$ generated by SDE (2.1.1). Then under the condition (2.1), the mean and covariance functions of the solution process $(X(t))_{t\geq t_0}$ are given by*

$$m_h = \mu\int_0^{\infty} K_h(u)du,$$
$$R_h(|\tau|) = |\mu|^2\int_0^{\infty}\int_0^{\infty} K_h(u_1)\overline{K_h(u_2)}\left(\exp\left\{-\frac{\gamma^2}{2}\kappa(h)W_{(\tau)}^h(u_1,u_2)\right\} - 1\right)du_1du_2$$
$$+|\beta|^2 h(2h+1)\kappa(h)\int_0^{\infty}\int_0^{\infty} K_h(u_1)\overline{K_h(u_2)}\exp\left\{-\frac{\gamma^2}{2}\kappa(h)W_{(\tau)}^h(u_1,u_2)\right\}du_1du_2,$$

where
$$W_{(\tau)}^h(u_1,u_2) = |\tau|^{2h+1} - |\tau-u_1|^{2h+1} - |\tau+u_2|^{2h+1} + |\tau-u_1+u_2|^{2h+1},$$

and $K_h(t) = \exp\left\{\alpha t - \frac{\gamma^2}{2}\kappa(h)t^{2h+1}\right\}$.

Proof. Straightforward and hence omitted. □

Corollary 2.3.1. *Consider the time-invariant version of the SDE (2.2.11), then* $\lim\limits_{\tau\to+\infty}\dfrac{R(\tau)}{c\tau^{-\delta}} = 1$ *for some constant c and $0<\delta<1$, this means that the solution process exhibits long range dependence. In this case the dependence between $X(t)$ and $X(t+\tau)$ decays slowly as $\tau\to+\infty$ and* $\int_{\mathbb{R}} R(|\tau|)d\tau = \infty$.

Proof. First we have

$$\exp\left\{-\frac{\gamma^2}{2}\kappa(h)\left(|\tau|^{2h+1}-|\tau-u_1|^{2h+1}-|\tau+u_2|^{2h+1}+|\tau-u_1+u_2|^{2h+1}\right)\right\}$$
$$=\exp\left\{-\frac{\gamma^2}{2}\kappa(h)|\tau|^{2h+1}\left(1-|1-\frac{u_1}{\tau}|^{2h+1}-|1+\frac{u_2}{\tau}|^{2h+1}+|1+\frac{u_2-u_1}{\tau}|^{2h+1}\right)\right\},$$

and

$$\left(1-\frac{u_1}{\tau}\right)^{2h+1}=1-(2h+1)\frac{u_1}{\tau}+\frac{(2h+1)(2h)}{2}\frac{u_1^2}{\tau^2}+...\tau\to+\infty$$
$$\left(1+\frac{u_2}{\tau}\right)^{2h+1}=1+(2h+1)\frac{u_2}{\tau}+\frac{(2h+1)(2h)}{2}\frac{u_2^2}{\tau^2}+...\tau\to+\infty$$
$$\left(1+\frac{u_2-u_1}{\tau}\right)^{2h+1}=1+(2h+1)\frac{(u_2-u_1)}{\tau}+\frac{(2h+1)(2h)}{2}\frac{(u_2-u_1)^2}{\tau^2}+...\tau\to+\infty.$$

Let $\delta=-(2h-1)$, it is clear $0<\delta<1$ because $0<h<\frac{1}{2}$, then we have

$$\lim_{\tau\to+\infty}\frac{\exp\left\{-\frac{\gamma^2}{2}\kappa(h)W_\tau^h(u_1,u_2)\right\}-1}{\tau^{-\delta}}=\lim_{\tau\to+\infty}\frac{\exp\left\{\frac{\gamma^2}{2}\kappa(h)h(2h+1)u_1u_2\tau^{2h-1}\right\}-1}{\tau^{2h-1}}$$
$$=\frac{\gamma^2}{2}h(2h+1)u_1u_2.$$

It follows that

$$\lim_{\tau\to+\infty}\frac{R(\tau)}{\tau^{-\delta}}$$
$$=|\mu|^2\int_0^\infty\int_0^\infty K_h(u_1)\overline{K_h(u_2)}\lim_{\tau\to+\infty}\tau^\delta\left\{\exp\left\{-\frac{\gamma^2}{2}\kappa(h)W_\tau^h(u_1,u_2)\right\}-1\right\}du_1du_2$$
$$=\frac{\gamma^2}{2}\kappa(h)h(2h+1)|\mu|^2\int_0^\infty u_1K_h(u_1)du_1\overline{\int_0^\infty u_1K_h(u_2)du_2}$$
$$=\frac{\gamma^2}{2}\kappa(h)h(2h+1)|\mu|^2\left|\int_0^\infty uK_h(u)du\right|^2=c<\infty,$$

Hence, the process $(X(t))_{t\geq0}$ generated by the *SDE* (2.2.11) with time-invariant parameters is a long memory process. □

2.3.1 Third-order structure of COBL(1,1) process

For the sake of convenience and simplicity, we shall consider the time-invariant version of the *SDE* (2.1.1). Moreover, we assume the process solution admits the spectral representation (2.2.7) in which the symmetrized version of transfer functions $g\left(\underline{\lambda}_{(r)}\right)$ may be written as

$$g(\lambda_{(r)})=\mu\,(i\gamma)^r\int_0^\infty K_h(u)\prod_{j=1}^r\frac{1-e^{-iu\lambda_j}}{i\lambda_j}du,\ \forall r\geq0.$$

Then using the representation (2.2.7) we can obtain the following approximation

$$\begin{aligned} X(t) &= g(0) + \int_{\mathbb{R}} g(\lambda_1) e^{it\lambda_1} dZ(\lambda_1) + \int_{\mathbb{R}^2} g\left(\lambda_{(2)}\right) e^{it\underline{\lambda}_{(2)}} dZ\left(\lambda_{(2)}\right) + \xi(t) \\ &= X^{(1)}(t) + X^{(2)}(t) + \xi(t), \end{aligned}$$

where $\xi(t)$ is a second-order stationary process which it is orthogonal to the first two terms. The symmetrized transfer functions $\widetilde{g}(\lambda_1)$ and $\widetilde{g}(\lambda_{(2)})$ are given by

$$g(\lambda_1) = \mu(i\gamma) \int_0^\infty K_h(u) \frac{1 - e^{-iu\lambda_1}}{i\lambda_1} du \text{ and } g(\lambda_1, \lambda_2) = \mu(i\gamma)^2 \int_0^\infty K_h(u) \prod_{j=1}^{2} \frac{1 - e^{-iu\lambda_2}}{i\lambda_2} du$$

It can be shown that

$$\begin{aligned} &C_h(s,u) = E\left\{(X(t) - g(0))\left((X(t+s) - g(0)\right)\left((X(t+u) - g(0)\right)\right\} \\ &= E\left\{X^{(1)}(t)X^{(1)}(t+s)X^{(2)}(t+u)\right\} + E\left\{X^{(1)}(t)X^{(2)}(t+s)X^{(1)}(t+u)\right\} \\ &+ E\left\{X^{(2)}(t)X^{(1)}(t+s)X^{(1)}(t+u)\right\} + O(1). \end{aligned}$$

We calculate $E\left\{X^{(1)}(t)X^{(1)}(t+s)X^{(2)}(t+u)\right\}$, and the other terms can be obtained by symmetry. First we observe that

$$\begin{aligned} &E\left\{X^{(1)}(t)X^{(1)}(t+s)X^{(2)}(t+u)\right\} \\ &= E\left\{\int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)e^{it\lambda_1 + i(t+s)\lambda_2} dZ(\lambda_{(2)}) \int_{\mathbb{R}^2} g(\lambda_3, \lambda_4) e^{i(t+u)(\lambda_3+\lambda_4)} dZ(\lambda_3, \lambda_4)\right\} \\ &= 2! \int_{\mathbb{R}^2} sym\left\{g(\lambda_1)g(\lambda_2)e^{it\lambda_1 + i(t+s)\lambda_2}\right\} \overline{sym\left\{g(\lambda_1, \lambda_2)e^{i(t+u)(\lambda_1+\lambda_2)}\right\}} dF\left(\lambda_{(2)}\right) \\ &= 2 \int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1, -\lambda_2) sym\left\{e^{is\lambda_1}\right\} e^{-iu(\lambda_1+\lambda_2)} \frac{d\lambda_1\lambda_2}{(2\pi)^2} \\ &= \frac{1}{(2\pi)^2}\left\{\int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1, -\lambda_2)e^{is\lambda_1}e^{-iu(\lambda_1+\lambda_2)} d\lambda_1\lambda_2 + \int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1, -\lambda_2)e^{is\lambda_2}e^{-iu(\lambda_1+\lambda_2)} d\lambda_1\lambda_2\right\} \\ &= \frac{1}{(2\pi)^2}\left\{\int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1, -\lambda_2)e^{i(s-u)\lambda_1 - iu\lambda_2} d\lambda_1\lambda_2 + \int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1, -\lambda_2)e^{-iu\lambda_1 + i(s-u)\lambda_2} d\lambda_1\lambda_2\right\}. \end{aligned}$$

Moreover we have

$$\begin{aligned} &E\left\{X^{(1)}(t)X^{(2)}(t+s)X^{(1)}(t+u)\right\} \\ &= E\left\{\int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)e^{it\lambda_1 + i(t+u)\lambda_2} dZ(\lambda_{(2)}) \int_{\mathbb{R}^2} g(\lambda_3, \lambda_4) e^{i(t+s)(\lambda_3+\lambda_4)} dZ(\lambda_3, \lambda_4)\right\} \\ &= 2! \int_{\mathbb{R}^2} sym\left\{g(\lambda_1)g(\lambda_2)e^{it\lambda_1 + i(t+u)\lambda_2}\right\} \overline{sym\left\{g(\lambda_1, \lambda_2)e^{i(t+s)(\lambda_1+\lambda_2)}\right\}} dF\left(\lambda_{(2)}\right) \\ &= 2 \int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1, -\lambda_2) sym\left\{e^{iu\lambda_1}\right\} e^{-is(\lambda_1+\lambda_2)} \frac{d\lambda_1\lambda_2}{(2\pi)^2} \\ &= \frac{1}{(2\pi)^2}\left\{\int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1, -\lambda_2)e^{iu\lambda_1}e^{-is(\lambda_1+\lambda_2)} d\lambda_1\lambda_2 + \int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1, -\lambda_2)e^{iu\lambda_2}e^{-is(\lambda_1+\lambda_2)} d\lambda_1\lambda_2\right\} \\ &= \frac{1}{(2\pi)^2}\left\{\int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1, -\lambda_2)e^{i(u-s)\lambda_1 - is\lambda_2} d\lambda_1\lambda_2 + \int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1, -\lambda_2)e^{-is\lambda_1 + i(u-s)\lambda_2} d\lambda_1\lambda_2\right\}. \end{aligned}$$

It remains to compute $E\left\{X^{(2)}(t)X^{(1)}(t+s)X^{(1)}(t+u)\right\}$, then

$$E\left\{X^{(2)}(t)X^{(1)}(t+s)X^{(1)}(t+u)\right\}$$
$$= E\left\{\int_{\mathbb{R}^2} g(\lambda_3)g(\lambda_4)e^{i(t+s)\lambda_3+i(t+u)\lambda_4}Z(d\lambda_3,d\lambda_4)\int_{\mathbb{R}^2} g(\lambda_1,\lambda_2)e^{it(\lambda_1+\lambda_2)}Z(d\lambda_{(2)})\right\}$$
$$= 2!\int_{\mathbb{R}^2} sym\left\{g(\lambda_1)g(\lambda_2)e^{i(t+s)\lambda_1+i(t+u)\lambda_2}\right\}\overline{sym\left\{g(\lambda_1,\lambda_2)e^{it(\lambda_1+\lambda_2)}\right\}}dF\left(\lambda_{(2)}\right)$$
$$= 2\int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1,-\lambda_2))sym\left\{e^{is\lambda_1+iu\lambda_2}\right\}\frac{d\lambda_1\lambda_2}{(2\pi)^2}$$
$$= \frac{1}{(2\pi)^2}\left\{\int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1,-\lambda_2)e^{is\lambda_1+iu\lambda_2}d\lambda_1\lambda_2 + \int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1,-\lambda_2)e^{iu\lambda_1+is\lambda_2}d\lambda_1\lambda_2\right\}.$$

Hence

$$C_h(s,u) = 2\int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1,-\lambda_2)sym\left\{e^{i(s-u)\lambda_1-u\lambda_2} + e^{e^{i(u-s)\lambda_1-s\lambda_2}} + e^{i(s\lambda_1+u\lambda_2)}\right\}\frac{d\lambda_1\lambda_2}{(2\pi)^2}.$$

By taking Fourier transforms (omitting the terms of $O(1)$), the bispectral density function $f(\lambda_1,\lambda_2)$ can be shown to be $f(\lambda_1,\lambda_2) = \frac{2}{(2\pi)^2}\left\{S(\lambda_1,\lambda_2) + S(\lambda_2,-\lambda_1-\lambda_2) + S(\lambda_1,-\lambda_1-\lambda_2)\right\}$ where $S(\lambda_1,\lambda_2) = g(\lambda_1)g(\lambda_2)g(-\lambda_1,-\lambda_2)$. It is clear from the above that the bispectrum is zero for all frequencies λ_1 and λ_2 if and only if the process is linear ($\gamma = 0$) (and Gaussian).

2.4 Conclusion

This chapter describes some basic probabilistic properties of continuous-time bilinear process driven by an $(f)\,Bm$. Our main aim was focused firstly on the existence of the solution in time-frequency domain and secondary to prove that the use of fBm as innovation we led to a long-range dependency property.

Chapter 3

Transfer functions solution for continuous-time bilinear stochastic processes[3]

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Abstract

In the present chapter we study some probabilistic and statistical properties of continuous-time version of the well known bilinear processes driven by standard Brownian motion. This class of processes, which includes many popular growth curve processes, were defined as a nonlinear stochastic differential equation which has raised considerable interest in the last few years. So, the $\mathbb{L}_2-$structure is studied and the covariance function of the process and its powers are given. The presence of the Taylor property and its relationship with respect to leptokurtosis effect is analyzed.

3.1 Introduction

Discrete-time series analysis has been well developed within the framework of linear and/or Gaussian models. Unfortunately these hypothesis lead to models that fail to capture certain phenomena commonly observed in practice such as limit cycles, self-excitation, asymmetric distribution, leptokurtosis and sudden jumping behaviour. So, in recent times we have become more aware of the fact that there are many datasets that cannot be modelled as discrete-time linear models. Wegman et al. [69] provide a rich source of examples emanating from the oceanographic and meteorological sciences which are clearly non-linear. One of the classes of non-linear models which has attracted considerable attention is the classes of bilinear one, initially discussed by Granger and Andersen [30]. The version of continuous-time of these models have been widely studied and considered in time series analysis and in theory of stochastic differential equations. For instance, Lebreton and Musiela [45] and Bibi and Merahi [11] have considered a processes $(X(t))_{t\geq 0}$ generated by the following time-varying stochastic differential equation (SDE)

$$dX(t) = (\alpha(t)X(t) + \mu(t))\,dt + (\gamma(t)X(t) + \beta(t))\,dw(t), t \geq 0, X(0) = X_0 \tag{3.1.1}$$

denoted hereafter $COBL\,(1,1)$ in which $(w(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}$ defined on some basic filtered space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, P)$ and with spectral representation $w(t) = \int_{\mathbb{R}} \frac{e^{it\lambda}-1}{i\lambda} dZ(\lambda)$, where $Z(\lambda)$ is an orthogonal complex-valued stochastic measure on $\mathbb{R}$ with zero mean, $E\left\{|dZ(\lambda)|^2\right\} = dF(\lambda) = \frac{d\lambda}{2\pi}$ and uniquely determined by $Z([a,b[) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-i\lambda a}-e^{-i\lambda b}}{i\lambda} dw(\lambda)$, for all $-\infty < a < b < +\infty$. The SDE (3.1.1) is called time-invariant if there exists some constants α, μ, γ and β such that for all t, $\alpha(t) = \alpha$, $\mu(t) = \mu$, $\gamma(t) = \gamma$ and $\beta(t) = \beta$. The initial state $X(0)$ is a random variable defined on $(\Omega, \mathcal{A}, P)$ independent of w such that $E\{X(0)\} = m_1(0)$ and $Var\{X(0)\} = K_X(0)$. The parameters $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are measurable deterministic functions and subject to the following assumption:

Assumption 1. *$\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are differentiable functions such that* $\forall T > 0$, $\int_0^T |\alpha(t)|\, dt < \infty$, $\int_0^T |\mu(t)|\, dt < \infty$, $\int_0^T |\gamma(t)|^2\, dt < \infty$ *and* $\int_0^T |\beta(t)|^2 \, dt < \infty$.

The SDE (3.1.1) encompasses many commonly used models in the literature. Some specific examples among others are:

1. $COGARCH(1,1)$: This classes of processes is defined as a SDE by $dX\,(t) = \sigma\,(t)\, dB_1(t)$ with $d\sigma^2\,(t) = \left(\mu\,(t) - \alpha\,(t)\,\sigma^2\,(t)\right) dt + \gamma\,(t)\,\sigma^2\,(t)\, dB_2\,(t)$, $t > 0$ where B_1 and B_2 are independent Brownian motions and $\mu\,(t) > 0$, $\alpha\,(t) \geq 0$ and $\gamma\,(t) \geq 0$. So, the stochastic volatility equation can be regarded a particular case of (3.1.1) by assuming constant the function $\beta(t) = 0$ for all t. (see Kluppelberg et al. [42] and the reference therein).

2. $CAR(1)$: This classes of SDE may be obtained by assuming $\gamma\,(t) = 0$ for all t. (see Brockwell [15] and the reference therein)

3. Gaussian Ornstein-Uhlenbeck (GOU) process: The GOU process is defined as $dX(t) = (\mu\,(t) - \alpha\,(t)\,X(t))\, dt + \beta\,(t)\, dw(t)$, with $\beta\,(t) > 0$ for all $t \geq 0$. So it can be obtained from (3.1.1) by assuming $\gamma(t) = 0$ for all t. (see Brockwell [15] and the reference therein).

4. Geometric Brownian motion (GBM): This class of processes is defined as a $\mathbb{R}-$valued solution process $(X(t))_{t\geq 0}$ of $dX(t) = \alpha\,(t)\,X(t)dt + \gamma\,(t)\,X(t)dw(t), t \geq 0$. So it can be obtained from (3.1.1) by assuming $\beta(t) = \mu\,(t) = 0$ for all t. (see Øksendal [7] and the reference therein).

The existence of solution process of equation (3.1.1), was investigated by several authors, for instance, Iglói and Terdik [38] have studied the same model driven by fractional Brownian innovation with time-invariant coefficients. A class of bilinear SDE with time-varying coefficients was studied by Lebreton and Musiela [45], Bibi and Merahi [11] and Leon et. all [47]. Terdik [64] and Lebreton and Musiela [46] have considered a general bilinear SDE driven by a Brownian motion. In this chapter, we shall investigate some probabilistic and statistical properties of second-order solution process of equation (3.1.1) which are also causal (or regular), i.e., $X(t)$ is $\sigma\{w(s), s \leq t\}-$measurable. For this purpose, let $\mathbb{L}_r(F)$ be the real Hilbert space of complex valued functions $f_t\left(\lambda_{(r)}\right)$ defined on $\mathbb{R}^r$ such that $f_t(-\lambda_{(r)}) = \overline{f_t(\lambda_{(r)})}$ with a inner product $\langle f_t, g_t\rangle_F = r!\int_{\mathbb{R}^r} Sym\left\{f_t(\lambda_{(r)})\right\}\overline{Sym\left\{g_s(\lambda_{(r)})\right\}} dF(\lambda_{(r)})$ where $\lambda_{(r)} = (\lambda_1, ..., \lambda_r) \in \mathbb{R}^r$, $Sym\left\{f_t(\lambda_{(r)})\right\} = \frac{1}{r!}\sum_{\pi\in\mathcal{P}} f\left(\lambda_{\pi(1),}, ..., \lambda_{\pi(r)}\right) \in \mathbb{L}_r(F)$ with $\mathcal{P}$ denotes the group of all permutations of the set $\{1,...,r\}$ and $dF(\lambda_{(r)})=\prod_{i=1}^{r}dF(\lambda_i)$. It is well known that if $(X(t))_{t\geq 0}$ is second-order and causal process (see Major [50] and Dobrushin [23] for further discussions) then it admits the so-called Wiener-Itô representation, i.e.,

$$X(t)=f_t(0)+\sum_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r}e^{it\underline{\lambda}_{(r)}}f_t(\lambda_{(r)})dZ(\lambda_{(r)}), \tag{3.1.2}$$

where $\underline{\lambda}_{(r)}=\sum_{i=1}^{r}\lambda_i$ and the integrals are multiple Wiener-Itô stochastic integrals with respect to the stochastic measure $dZ\,(\lambda)$, $f_t(0)=E\{X(t)\}$, $dZ(\lambda_{(r)})=\prod_{i=1}^{r}dZ(\lambda_i)$ and $f_t(\lambda_{(r)})$ are referred as the $r-th$ evolutionary transfer functions of $(X(t))_{t\geq 0}$, uniquely determined up to symmetrization and fulfill the condition

$$\sum_{r\geq 0}\frac{1}{r!}\int_{\mathbb{R}^r}\left|f_t(\lambda_{(r)})\right|^2dF(\lambda_{(r)})<\infty \text{ for all } t. \tag{3.1.3}$$

As a property of the representation (3.1.2) is that for any $f_t(\lambda_{(n)})$ and $f_s(\lambda_{(m)})$, we have

$$E\left\{\int_{\mathbb{R}^n}f_t(\lambda_{(n)})dZ(\lambda_{(n)})\overline{\int_{\mathbb{R}^m}f_s(\lambda_{(m)})dZ(\lambda_{(m)})}\right\}=\delta_n^m n!\int_{\mathbb{R}^n}Sym\left\{f_t(\lambda_{(n)})\right\}\overline{Sym\left\{f_s(\lambda_{(n)})\right\}}dF(\lambda_{(n)}) \tag{3.1.4}$$

where δ_n^m is the delta function. Another property linked with (3.1.2) is the diagram formula which state that

$$\begin{aligned}&\int_{\mathbb{R}}f_t(\lambda)dZ(\lambda)\int_{\mathbb{R}^n}g_s\left(\lambda_{(n)}\right)dZ(\lambda_{(n)})\\&=\int_{\mathbb{R}^{n+1}}g_s\left(\lambda_{(n)}\right)f_t\left(\lambda_{n+1}\right)dZ(\lambda_{(n+1)})+\sum_{k=1}^{n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}g_s\left(\lambda_{(n)}\right)\overline{f_t\left(\lambda_k\right)}dF\left(\lambda_k\right)dZ(\lambda_{(n\backslash k)})\end{aligned}$$

where $Z(d\lambda_{(n\backslash k)})=Z\left(d\lambda_1\right)...Z\left(d\lambda_{k-1}\right).Z\left(d\lambda_{k+1}\right)...Z\left(d\lambda_n\right)$.

The main aim of the chapter is to use the transfer functions approach associated with the solution process of (3.1.1) to establish some characterizations and properties for a such process. So, in section 2 we study the conditions ensuring the existence of the processes $(X(t))_{t\in\mathbb{R}_+}$ and $(X^2(t))_{t\in\mathbb{R}_+}$ using their spectral representation. In section 3 we give the conditions of stability of SDE (3.1.1) based on the associated transfer functions. In section 4 we analyze the presence of the Taylor property of equation (3.1.1) and some simulations are given to confirm our theoretical results for different values of parameters.

3.2 $COBL\,(1,1)$ equation and their solutions

The existence and uniqueness of the Itô solution process $(X(t))_{t\geq 0}$ of equation (3.1.1) in time domain is ensured by the general results on SDE and under the Assumption 1 (see e.g., [3]) which is given by

$$X(t)=\Phi(t)\left\{X(0)+\int_0^t\Phi^{-1}(s)\left(\mu(s)-\gamma\left(s\right)\beta\left(s\right)\right)ds+\int_0^t\Phi^{-1}(s)\beta\left(s\right)dw(s)\right\},\ a.e., \tag{3.2.1}$$

where $\Phi(t) = \exp\left\{\int_0^t \left(\alpha(s) - \frac{1}{2}\gamma^2(s)\right) ds + \int_0^t \gamma(s)\, dw(s)\right\}$. The solution (3.2.1) is however Markovian when $\beta(t) \neq 0$ for all t, otherwise the solution process is neither a Markov process nor a martingale.

Remark 3.2.1. *When everywhere $\gamma(t) = 0$, $\alpha(t) < 0$ and $\beta(t) \neq 0$, this provides a second-order solution processes for GOU or GAR(1) equations. If we are interested in second-order non-Gaussian solution of COGARCH(1,1) or GBM equations, it is necessary to assume that everywhere $\mu^2(t) + \beta^2(t) > 0$, $\gamma(t) \neq 0$ and not only $\alpha(t) < 0$ but $2\alpha(t) + \gamma^2(t) < 0$ as well. Moreover, the condition $\gamma(t)\mu(t) \neq \alpha(t)\beta(t)$ for all t, must be hold, otherwise the equation (3.1.1) has only a degenerate solution,i.e., $X(t) = -\dfrac{\beta(t)}{\gamma(t)} = -\dfrac{\mu(t)}{\alpha(t)}$.*

The solution based on Wiener-Itô representation (3.1.2) is discussed along the following sections. For this purpose, we recalling the following two theorems due to Bibi and Merahi [11].

Theorem 3.2.2. *Assume that everywhere*

$$2\alpha(t) + \gamma^2(t) < 0, \tag{3.2.2}$$

then the process $(X(t))_{t\geq 0}$ generated by the SDE (3.1.1) has a regular second-order solution given by the Wiener-Itô representation (3.1.2). The evolutionary symmetrized transfer functions $f_t(\lambda_{(r)}), (t,r) \in \mathbb{R}_+ \times \mathbb{N}$ of this solution are given by the symmetrization of the following differential equations

$$f_t^{(1)}(\lambda_{(r)}) = \begin{cases} \alpha(t) f_t(0) + \mu(t), & \text{if } r = 0 \\ \left(\alpha(t) - i\underline{\lambda}_{(r)}\right) f_t(\lambda_{(r)}) + r\left(\gamma(t) f_t(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta(t)\right), & \text{if } r \geq 1 \end{cases} \tag{3.2.3}$$

where the superscript $^{(j)}$ denotes $j-$fold differentiation with respect to t.

Proof. See Bibi and Merahi [11]. □

Remark 3.2.3. *The existence and uniqueness of the evolutionary symmetrized transfer functions $f_t(\lambda_{(r)}), (t,r) \in \mathbb{R}\times\mathbb{N}$ given by (3.2.3) is ensured by general results on linear ordinary differential equations (see, e.g., [40], ch. 1), so*

$$f_t(\lambda_{(r)}) = \begin{cases} \varphi_t(0)\left(f_0(0) + \int_0^t \varphi_s^{-1}(0)\mu(s)ds\right) & \text{if } r = 0 \\ \varphi_t\left(\underline{\lambda}_{(r)}\right)\left(f_0(\lambda_{(r)}) + r\int_0^t \varphi_s^{-1}\left(\underline{\lambda}_{(r)}\right)\left(\gamma(s) f_s(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta(s)\right) ds\right) & \text{if } r \geq 1 \end{cases} \tag{3.2.4}$$

where $\varphi_t\left(\underline{\lambda}_{(r)}\right) = \exp\left\{\int_0^t \left(\alpha(s) - i\underline{\lambda}_{(r)}\right) ds\right\}$.

Corollary 3.2.1. *Assume that $\alpha(t), \mu(t), \beta(t)$ and $\gamma(t)$ are constant (time-invariant SDE), then the transfer functions $f(\lambda_{(r)})$ for all $r \in \mathbb{N}$ are given by*

$$f(\lambda_{(r)}) = -\frac{\mu}{\alpha}\delta_{\{r=0\}} + r\left(i\underline{\lambda}_{(r)} - \alpha\right)^{-1}\left(\gamma f(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta\right) \text{ for any } r \geq 0$$

or also $f(\lambda_{(r)}) = \gamma^{r-1} r! \left(\beta - \frac{\mu}{\alpha}\gamma\right) \prod_{j=1}^{r} \left(i\underline{\lambda}_{(j)} - \alpha\right)^{-1}$, *and hence the symmetrized version can be rewritten as*

$$Sym\left\{f(\lambda_{(r)})\right\} = (\mu\gamma - \alpha\beta)\,\gamma^{r-1} \int_{0}^{+\infty} \exp\left\{\alpha\lambda\right\} \prod_{j=1}^{r} \frac{1-\exp\left\{-i\lambda\lambda_j\right\}}{i\lambda_j} d\lambda.$$

Moreover, if $\gamma \neq 0, \alpha\beta \neq \mu\gamma$ *and* $2\alpha + \gamma^2 < 0$, *then the solution process* $(X(t))_{t\geq 0}$ *is strictly stationary and there exists a unique invariant probability distribution* $\Pi(\theta)$ *with* $\theta = (\gamma, \beta, \alpha, \mu)$ *for the associated solution given by* (3.2.1) *such that* $\Pi(\theta)$ *is the the distribution of the inverse* $\Gamma\left(1 - \frac{2\alpha}{\gamma^2}, \frac{\gamma^3}{2(\gamma\mu - \alpha\beta)}\right)$ *distribution.*

Proof. See Lebreton and Musiela [45]. □

The following lemma give Wiener-Itô representation for the quadratic process $\left(q(t) = X^2(t)\right)_{t\geq 0}$.

Lemma 3.2.4. *Suppose that* $X(t)$ *is* $\sigma\left\{w(s), s \leq t\right\}$ *−measurable satisfying equation* (3.1.1) *with* $E\left\{q^2(t)\right\} < +\infty$ *. Then* $(q(t))_{t\geq 0}$ *has a Wiener-Itô representation, i.e.,*

$$q(t) = f_t^{[2]}(0) + \sum_{r\geq 1} \frac{1}{r!} \int_{\mathbb{R}^r} e^{it\underline{\lambda}_{(r)}} f_t^{[2]}(\lambda_{(r)}) dZ(\lambda_{(r)}),$$

where the transfer functions $f_t^{[2]}(\lambda_{(r)})$, $r \geq 0$ *are given by the following recursive formula*

$$f_t^{[2](1)}(\lambda_{(r)}) = \begin{cases} \left(2\alpha(t) + \gamma^2(t)\right) f_t^{[2]}(0) + 2\left(\gamma(t)\beta(t) + \mu(t)\right) f_t(0) + \beta^2(t) \text{ if } r = 0 \\ \left(\left(2\alpha(t) + \gamma^2(t)\right) - i\underline{\lambda}_{(r)}\right) f_t^{[2]}(\lambda_{(r)}) + 2\left(\gamma(t)\beta(t) + \mu(t)\right) f_t\left(\lambda_{(r)}\right)) \\ + 2r\left(\gamma(t) f_t^{[2]}(\lambda_{(r-1)}) + \beta(t) f_t(\lambda_{(r-1)})\right) \text{ if } r \geq 1 \end{cases} \quad (3.2.5)$$

Proof. The proof follows upon the observation that the process $(q(t))_{t\geq 0}$ satisfying the following stochastic differential equation

$$dq(t) = \left[\left(2\alpha(t) + \gamma^2(t)\right) q(t) + 2\left(\gamma(t)\beta(t) + \mu(t)\right) X(t) + \beta^2(t)\right] dt + 2\left(\gamma(t)q(t) + \beta(t)X(t)\right) dw(t), \text{ a.e.,} \quad (3.2.6)$$

So, using the diagram formula the result follows. □

Remark 3.2.5. *The existence and uniqueness of the evolutionary symmetrized transfer functions* $f_t^{[2]}(\lambda_{(r)}), (t, r) \in \mathbb{R} \times \mathbb{N}$ *given by* (3.2.5) *is ensured by general results on linear ordinary differential equations (see, e.g., [40], ch. 1), so*

$$f_t^{[2]}(\lambda_{(r)}) = \begin{cases} \phi_t(0)\left(f_0^{[2]}(0) + \int_0^t \phi_s^{-1}(0)\,\mu_s(0) ds\right) \text{ if } r = 0 \\ \phi_t\left(\underline{\lambda}_{(r)}\right)\left(f_0^{[2]}(\lambda_{(r)}) + \int_0^t \phi_s^{-1}\left(\underline{\lambda}_{(r)}\right) \mu_s(\lambda_{(r)}) ds\right) \text{ if } r \geq 1 \end{cases} \quad (3.2.7)$$

in which $\phi_t\left(\underline{\lambda}_{(r)}\right) = \exp\left\{\int_0^t \left(2\alpha(s) + \gamma^2(s) - i\underline{\lambda}_{(r)}\right) ds\right\}$ *and*

$$\mu_t(\lambda_{(r)}) = 2\left(\gamma(t)\beta(t) + \mu(t)\right) f_t\left(\lambda_{(r)}\right) + \beta^2(t)\delta_{\{r=0\}} + 2r\left(\gamma(t) f_t^{[2]}(\lambda_{(r-1)}) + \beta(t) f_t(\lambda_{(r-1)})\right).$$

Remark 3.2.6. *In time-invariant case, the transfer functions* $f^{[2]}(\lambda_{(r)})$ *associated with this case reduces to*

$$f^{[2]}(\lambda_{(r)}) = \begin{cases} \dfrac{2\left(\gamma\beta+\mu\right)f\left(0\right)+\beta^2}{|2\alpha+\gamma^2|} & \text{if } r=0 \\ \dfrac{2\left(\gamma\beta+\mu\right)f(\lambda_{(r)})+2r\gamma f^{[2]}(\lambda_{(r-1)})+2r\beta f(\lambda_{(r-1)})}{i\underline{\lambda}_{(r)}-(2\alpha+\gamma^2)} & \text{if } r\geq 1 \end{cases}$$

where $f\left(0\right)=E\left\{X(t)\right\}=-\dfrac{\mu}{\alpha}$. *In particular, when* $\beta=0$ *and* $\mu\neq 0$, *the transfer functions reduces to*

$$f^{[2]}(0)=\frac{-2\mu^2}{\alpha|2\alpha+\gamma^2|} \text{ and } f^{[2]}(\lambda_{(r)})=\frac{2\mu f(\lambda_{(r)})+2r\gamma f^{[2]}(\lambda_{(r-1)})}{i\underline{\lambda}_{(r)}-(2\alpha+\gamma^2)} \text{ if } r\geq 1.$$

which are similar to the results already obtained by Subba and Terdik [65].

3.3 Second-order properties of $(X(t))_{t\geq 0}$ and $(X^2(t))_{t\geq 0}$

In theorem 3.2.2 and remark 3.2.3 a recursive formula is derived for the evolutionary transfer functions of regular solution of $COBL(1,1)$. Condition (3.2.2) give sufficient condition for that these transfer functions determine a solution process given by the Wiener-Itô representation (3.1.2) for equation (3.1.1). In this section we examine the second-order properties of the solution processes $(X(t))_{t\geq 0}$ and $(q(t))_{t\geq 0}$.

3.3.1 Linear case

In linear case, i.e., $\gamma(t)=0$, for all $t\geq 0$,we have

$$dX(t)=\left(\alpha(t)X(t)+\mu(t)\right)dt+\beta(t)dw(t),\ a.e., \tag{3.3.1}$$

its Wiener-Itô representation (3.1.2) reduces to $X(t)=f_t(0)+\int_{\mathbb{R}}e^{it\lambda}f_t(\lambda)dZ(\lambda)$, and the evolutionary transfer functions associated with regular solution are uniquely given by

$$f_t(\lambda)=\varphi_t\left(\lambda\right)\left(f_0(\lambda)+\int_0^t\varphi_s^{-1}\left(\lambda\right)\beta(s)ds\right) \tag{3.3.2}$$

where $\varphi_t\left(\lambda\right)$ is given in remark 3.2.3.

Thus, for equation (3.3.1) we have

Theorem 3.3.1. *If* $\alpha(t)<0$, *then the solution process* $(X(t))_{t\geq 0}$ *of equation* (3.3.1) *is Gaussian if and only if* X_0 *is normally distributed or constant. The mean* $m_1\left(t\right)$, *variance* $K_X(t)$ *and covariance* $K_X(t,s)$ *functions are given by*

$$m_1\left(t\right)=\varphi_t\left(0\right)\left(m_1(0)+\int_0^t\varphi_s^{-1}\left(0\right)\mu(s)ds\right),$$

$$K_X(t) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} |f_t(\lambda)|^2\, d\lambda,\ K_X(t,s) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{i\lambda(t-s)} f_t(\lambda)\,\overline{f_s(\lambda)}d\lambda.$$

Moreover, the solution process $(X(t))_{t\geq 0}$ *has independent increments if and only if* X_0 *is constant or* $\alpha(t) = 0$ *(i.e.,* $\varphi_t(0) = 1$*) for all* $t \geq 0$.

Proof. See Arnold [3] (see Theorem 8.2.10 and Theorem 8.2.12, Chapter 8). □

Now we are in position to state the following theorem.

Theorem 3.3.2. *The solution process of equation* (3.3.1) *is stationary Gaussian process if* $\alpha(t)$, $\mu(t)$ *and* $\beta(t)$ *are constant functions, with* $\alpha(t) = \alpha < 0$, $\mu(t) = 0$, $\beta(t) = \beta$, *and* X_0 *is normally distributed with zero mean and* $K_X(0) = -\dfrac{\beta^2}{2\alpha}$. *In this case the covariance function* $Cov(X(t), X(t+h)) = K_X(0)e^{\alpha|h|} \to 0$ *as* $|h| \to +\infty$.

Proof. From theorem 3.3.1, It is clear that the solution process $(X(t))_{t\geq 0}$ of the equation (3.3.1) is Gaussian process, furthermore a necessary and sufficient condition for second-order stationarity is that $E\{X(t)\}$ is a constant and $Cov(X(s), X(t)) = K_X(0)e^{\alpha|t-s|}$. These conditions are certainly satisfied if $\mu(t) = 0$, $\alpha(t) = \alpha$ and $\beta(t) = \beta$. In this case and from the recursion (3.2.4) we can see that $f(0) = 0$ and $f(\lambda) = \dfrac{\beta}{i\lambda - \alpha}$. From the representation (3.3.1), we have $E\{X(t)\} = 0$ and by the property (3.1.4) we get

$$\begin{aligned}K_X(0) &= E\left\{\int_{\mathbb{R}} e^{it\lambda_1} f(\lambda_1)dZ(\lambda_1)\int_{\mathbb{R}} e^{it\lambda_2} f(\lambda_2)dZ(\lambda_2)\right\}\\ &= \int_{\mathbb{R}} |f(\lambda)|^2\, dF(\lambda) = \beta^2\left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{\lambda^2+\alpha^2}d\lambda\right) = \frac{\beta^2}{2\,|\alpha|}.\end{aligned}$$

and

$$\begin{aligned}K_X(h) &= E\left\{\int_{\mathbb{R}} e^{it\lambda_1} f(\lambda_1)dZ(\lambda_1)\int_{\mathbb{R}} e^{i(t+h)\lambda_2} f(\lambda_2)dZ(\lambda_2)\right\} = \int_{\mathbb{R}} |f(\lambda)|^2\, e^{-ih\lambda}dF(\lambda)\\ &= \beta^2\left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{\lambda^2+\alpha^2}e^{-ih\lambda}d\lambda\right) = \frac{\beta^2}{2\,|\alpha|}e^{\alpha|h|} = K_X(0)e^{\alpha|h|}.\square\end{aligned}$$

□

Now we examine the second-order properties of the quadratic process $(q(t))_{t\geq 0}$.

Theorem 3.3.3. *Under the conditions of the theorem* 3.3.2*, the process* $(q(t))_{t\geq 0}$ *is strict stationary and its distribution has the form* $\dfrac{\beta^2}{2|\alpha|}\chi_{(1)}$ *where* $\chi_{(1)}$ *is the chi-squared distribution with 1 degree of freedom. Moreover* $(q(t))_{t\geq 0}$ *is a second order stationary process. The mean* $m_2(0)$ *, variance* $K_q(0)$ *and covariance* $K_q(h)$ *functions are given by*

$$m_2(0) = \frac{\beta^2}{2|\alpha|},\ K_q(0) = 2\left(\frac{\beta^2}{2\alpha}\right)^2,\ K_q(h) = K_q(0)e^{2\alpha|h|},\ h \in \mathbb{R}$$

Proof. The proof of this theorem follows from the fact that $(X(t))_{t\geq 0}$ is stationary Gaussian stochastic process and the definition of the chi-squared distribution. □ □

3.3.2 Bilinear case

In this subsection we consider the SDE (3.1.1), first recalling the following theorem which is due to Lebreton and Musiela [45] where we give the proof using the transfer functions approach..

Theorem 3.3.4. *Under the condition (3.2.2), the mean, variance and covariance functions for* $COBL\,(1,1)$ *are given respectively by the expressions*

$$m_1(t) = \varphi_t(0)\left(f_0(0) + \int_0^t \varphi_s^{-1}(0)\mu(s)ds\right). \tag{3.3.3}$$

$$K_X(t) = \phi_t\,(0)\left(K_X(0) + \int\limits_0^t \phi_s^{-1}\,(0)\,(\gamma(s)f_s(0) + \beta(s))^2\,ds\right). \tag{3.3.4}$$

$$K_X(t,s) = \varphi_t(0)\varphi_s^{-1}(0)K_X(s), t \geq s \geq 0, \tag{3.3.5}$$

where $\varphi_t(0) = \exp\left\{\int_0^t \alpha(u)du\right\}$ *and* $\phi_t(0) = \exp\left\{\int_0^t \left(2\alpha(u) + \gamma^2(u)\right)du\right\}$ *(see remark 3.2.3 and remark 3.2.5).*

Proof. From the formula (3.2.4) it follows

$$f_t(0) = \varphi_t\,(0)\left(f_0(0) + \int\limits_0^t \varphi_s^{-1}\,(0)\,\mu(s)ds\right),$$

since $E\{X(t)\} = m_1(t) = f_t(0)$, $\forall t \geq 0$, the expression (3.3.3) holds.

To prove (3.3.5) we have $Var\{X(t)\} = K_X(t) = E\{X^2(t)\} - (E\{X(t)\})^2$, but $E\{X(t)\} = m_1(t) = f_t(0)$, and $E\{X^2(t)\} = m_2(t) = f_t^{[2]}(0)$, $\forall t \geq 0$ which implies $K_X(t) = f_t^{[2]}(0) - (f_t(0))^2$, then by differentiating with respect to t and from the formulae (3.2.3),(3.2.5) we substitute respectively $\dfrac{df_t(0)}{dt}$, $\dfrac{df_t^{[2]}(0)}{dt}$ we find

$$\begin{aligned}
\frac{dK_X(t)}{dt} &= \frac{df_t^{[2]}(0)}{dt} - 2f_t(0)\frac{df_t(0)}{dt} \\
&= \left[\left(2\alpha\,(t) + \gamma^2(t)\right)f_t^{[2]}\,(0) + 2\,(\gamma(t)\beta(t) + \mu(t))\,f_t\,(0) + \beta^2(t)\right] - 2f_t(0)\,[\alpha(t)f_t(0) + \mu(t)] \\
&= \left(2\alpha\,(t) + \gamma^2(t)\right)f_t^{[2]}\,(0) + 2\gamma(t)\beta(t)f_t\,(0) + \beta^2(t) - 2\alpha(t)\,(f_t\,(0))^2 \\
&= \left(2\alpha\,(t) + \gamma^2(t)\right)\left(K_X(t) + (f_t(0))^2\right) + 2\gamma(t)\beta(t)f_t\,(0) + \beta^2(t) - 2\alpha(t)\,(f_t\,(0))^2 \\
&= \left(2\alpha\,(t) + \gamma^2(t)\right)K_X(t) + (\gamma(t)f_t(0) + \beta(t))^2 \\
&= \left(2\alpha\,(t) + \gamma^2(t)\right)K_X(t) + (\gamma(t)m_1(0) + \beta(t))^2\,,
\end{aligned}$$

hence the expression (3.3.5) is ensured by applying the general results on linear ordinary differential equations.

Finally it remains to prove (3.3.4), first we remark that for all $t \geq s$

$$K_X(t,s) = Cov(X(t), X(s)) = \sum_{r\geq 1}\frac{1}{(r!)^2}E\left\{\int_{\mathbb{R}^r} f_t\left(\lambda_{(r)}\right)e^{it\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})\overline{\int_{\mathbb{R}^r} f_s\left(\lambda_{(r)}\right)e^{is\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})}\right\}$$

then differentiating with respect to t and the use of the formula (3.2.3) we obtain

$$\frac{dK_X(t,s)}{dt} = \sum_{r\geq 1}\frac{1}{(r!)^2}E\left\{\int_{\mathbb{R}^r}\frac{d\left(f_t\left(\lambda_{(r)}\right)e^{it\underline{\lambda}_{(r)}}\right)}{dt}dZ(\lambda_{(r)})\overline{\int_{\mathbb{R}^r}f_s\left(\lambda_{(r)}\right)e^{is\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})}\right\}$$

$$= \sum_{r\geq 1}\frac{1}{(r!)^2}E\left\{\int_{\mathbb{R}^r}\left(f_t^{(1)}\left(\lambda_{(r)}\right)+i\underline{\lambda}_{(r)}f_t\left(\lambda_{(r)}\right)\right)e^{it\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})\overline{\int_{\mathbb{R}^r}f_s\left(\lambda_{(r)}\right)e^{is\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})}\right\}$$

$$= \sum_{r\geq 1}\frac{1}{(r!)^2}E\left\{\int_{\mathbb{R}^r}\left(\alpha(t)f_t\left(\lambda_{(r)}\right)+r\left[\gamma(t)f_t\left(\lambda_{(r-1)}\right)+\delta_{\{r=1\}}\beta(t)\right]\right)e^{it\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})\overline{\int_{\mathbb{R}^r}f_s\left(\lambda_{(r)}\right)e^{is\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})}\right\},$$

Now apply the property of orthogonality (3.1.4) to get

$$\frac{dK_X(t,s)}{dt} = \alpha(t)\sum_{r\geq 1}\frac{1}{(r!)^2}E\left\{\int_{\mathbb{R}^r}f_t\left(\lambda_{(r)}\right)e^{it\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})\overline{\int_{\mathbb{R}^r}f_s\left(\lambda_{(r)}\right)e^{is\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})}\right\}$$
$$= \alpha(t)K_X(t,s), \text{ for all } t\geq s,$$

and the expression (3.3.4) holds.

□ □

Corollary 3.3.1. *In time-invariant case and under the condition (3.2.2), we have the following results*

$$m_1(0) = -\frac{\mu}{\alpha},\ K_X(0) = \frac{(\alpha\beta-\mu\gamma)^2}{\alpha^2\left|2\alpha+\gamma^2\right|},\ K_X(h) = K_X(0)e^{\alpha|h|},\ h\in\mathbb{R}$$

Now, we examine the second order properties of the quadratic process $(q(t))_{t\geq 0}$ which satisfy the *SDE* (3.2.6) in which the condition

$$2\alpha(t)+3\gamma^2(t)<0, \text{ for all } t\geq 0 \tag{3.3.6}$$

must be imposed.

Theorem 3.3.5. *Consider the quadratic version of the SDE (3.1.1), then under the condition (3.3.6), the mean, covariance and variance functions are given respectively by*

$$m_2(t) = \phi_t(0)\phi_s^{-1}(0)\left\{m_2(s)+\int_s^t\phi_s(0)\phi_u^{-1}(0)\left[2\left(\gamma(u)\beta(u)+\mu(u)\right)m_1(u)+\beta^2(u)\right]du\right\}\ ,\ t\geq s. \tag{3.3.7}$$

$$K_q(t,s) = \phi_t(0)\phi_s^{-1}(0) \tag{3.3.8}$$
$$\times\left\{K_q(s)+2\int_s^t\phi_s(0)\phi_u^{-1}(0)\left(\gamma(u)\beta(u)+\mu(u)\right)\left[Cum(X(u),X(s),X(s))+2K_X(u,s)m_1(s)\right]du\right\}$$

$$K_q(t) = \psi(t)\psi^{-1}(s)K_q(s)+4\int_s^t\psi(t)\psi^{-1}(u) \tag{3.3.9}$$
$$\times\left[\gamma^2(u)m_2^2(u)+\left(\mu(u)+3\gamma(u)\beta(u)\right)m_3(u)-\left(\mu(u)+\gamma(u)\beta(u)\right)m_1(u)m_2(u)+\beta^2(u)m_2(u)\right]du,$$

where $\phi_t(0) = \exp\left\{\int_0^t\left(2\alpha(u)+\gamma^2(u)\right)du\right\}$ *and* $\psi(t) = \exp\left\{2\int_0^t\left(2\alpha(u)+3\gamma^2(u)\right)du\right\}$.

Proof. The formula (3.3.7) follows immediately from (3.2.6). To obtain the variance of the quadratic process $q(t)$ we need to compute the moments of $X(t)$, that are $m_k(t) = E\left\{X^k(t)\right\}, k \geq 2$. Apply Itô formula to SDE (3.1.1), with $f(x) = x^k$, we find

$$dX^k(t) = \left(a_k(t)X^k(t) + b_k(t)X^{k-1}(t) + c_k(t)X^{k-2}(t)\right) dt + \left(\gamma(t)kX^k(t) + \beta(t)kX^{k-1}(t)\right) dw(t), \tag{3.3.10}$$

with

$$a_k(t) = \alpha(t)k + \frac{1}{2}\gamma^2(t)k(k-1),\ b_k(t) = \mu(t)k + \gamma(t)\beta(t)k(k-1) \text{ and } c_k(t) = \frac{1}{2}\beta^2(t)k(k-1). \tag{3.3.11}$$

We can write (3.3.10) as

$$X^k(t) = X^k(0) + \int_0^t \left(a_k(t)X^k(s) + b_k(t)X^{k-1}(s) + c_k(t)X^{k-2}(s)\right) ds + \int_0^t \left(\gamma(t)kX^k(s) + \beta(t)kX^{k-1}(s)\right) dw(s). \tag{3.3.12}$$

Therefore, taking the expected value of each side of (3.3.12), if we put $m_k(t) = E\left\{X^k(t)\right\}$ we find

$$m_k(t) = m_k(0) + \int_0^t (a_k(s)m_k(s) + b_k(s)m_{k-1}(s) + c_k(s)m_{k-2}(s))\, ds. \tag{3.3.13}$$

Differentiating with respect to t we obtain

$$\frac{dm_k(t)}{dt} = a_k(t)m_k(t) + b_k(t)m_{k-1}(t) + c_k(t)m_{k-2}(t), \text{ for } k \geq 2,\ t > 0 \tag{3.3.14}$$

$$m_k(0) = E\left\{X^k(0)\right\}, \text{ for } k \geq 2 \text{ and } m_0(t) = 1,\ \forall t \geq 0. \tag{3.3.15}$$

We will solve the above of differential equations, finding $m_k(t)$ for $k \geq 2$. Since $K_q(t) = m_4(t) - (m_2(t))^2$ and by differentiating with respect to t we find

$$\frac{dK_q(t)}{dt} = \frac{dm_4(t)}{dt} - 2m_2(t)\frac{dm_2(t)}{dt},$$

we use (3.3.14) for $k \in \{2, 4\}$ and the fact that $m_4(t) = K_q(t) + m_2^2(t)$ we abtain the following differential equation

$$\frac{dK_q(t)}{dt} = a_4(t)K_q(t) + (a_4(t) - 2a_2(t))\, m_2^2(t) + b_4(t)m_3(t) - 2b_2(t)m_1(t)m_2(t) + (c_4(t) - 2c_2(t))\, m_2(t),$$

from (3.3.11) the coefficients of this equation can be given as

$$a_4(t) = 2\left(2\alpha(t) + 3\gamma^2(t)\right),\ a_4(t) - 2a_2(t) = 4\gamma^2(t),\ b_4(t) = 4\left(\mu(t) + 3\gamma(t)\beta(t)\right),$$
$$-2b_2(t) = -4\left(\mu(t) + \gamma(t)\beta(t)\right),\ c_4(t) - 2c_2(t) = 4\beta^2(t),$$

by general results on linear ordinary differential equations (see, e.g., [40], ch. 1) the expression (3.3.9) holds.

To prove the formula (3.3.8), we observe that for any $t \geq s$

$$dE\{q(t)q(s)\} = \left(2\alpha(t)+\gamma^2(t)\right)E\{q(t)q(s)\}\,dt + 2\left(\gamma(t)\beta(t)+\mu(t)\right)E\{X(t)X(s)X(s)\}\,dt + \beta^2(t)m_2(s)dt$$

By general results on linear ordinary differential equations (see, e.g., [40], ch. 1) we obtain

$$E\{q(t)q(s)\} = \phi_t(0)\phi_s^{-1}(0)\ E\left\{q^2(s)\right\} + \int_s^t \phi_t(0)\phi_u^{-1}(0)\left[2\left(\gamma(u)\beta(u)+\mu(u)\right)E\{X(u)X(s)X(s)\} + \beta^2(u)m_2(s)\right]du.$$

Since $E\{q(t)q(s)\} = Cov(q(t),q(s)) + E\{q(t)\}E\{q(s)\} = K_q(t,s) + m_2(t)m_2(s)$, $E\left\{q^2(s)\right\} = K_q(s) + m_2^2(s)$ and

$$\begin{aligned}E\{X(u)X(s)X(s)\} &= Cum\left((X(u),X(s),X(s)\right) + m_1(u)m_2(s) + 2m_1(s)E\{X(u)X(s)\} - 2m_1(u)m_1^2(s)\\ &= Cum\left((X(u),X(s),X(s)\right) + m_1(u)m_2(s) + 2m_1(s)K_X(u,s),\end{aligned}$$

then using the formula (3.3.7) we obtain

$$\begin{aligned}&K_q(t,s) + m_2(t)m_2(s) = \phi_t(0)\phi_s^{-1}(0)\\ &\times\left\{K_q(s) + 2\int_s^t \phi_s(0)\phi_u^{-1}(0)\left(\gamma(u)\beta(u)+\mu(u)\right)\left[Cum\{X(u),X(s),X(s)\} + 2m_1(s)K_X(u,s)\right]du\right\}\\ &+ \phi_t(0)\phi_s^{-1}(0)\left\{m_2(s) + \int_s^t \phi_s(0)\phi_u^{-1}(0)\left[2\left(\gamma(u)\beta(u)+\mu(u)\right)m_1(u) + \beta^2(u)\right]du\right\}\times m_2(s)\\ &= \phi_t(0)\phi_s^{-1}(0)\\ &\times\left\{K_q(s) + 2\int_s^t \phi_s(0)\phi_u^{-1}(0)\left(\gamma(u)\beta(u)+\mu(u)\right)\left[Cum\{X(u),X(s),X(s)\} + 2m_1(s)K_X(u,s)\right]du\right\}\\ &+ m_2(t)m_2(s)\end{aligned}$$

and the expression (3.3.8) holds.□ □

The second-order stationarity of the process $(q(t))_{t\geq 0}$ is characterized in the theorem

Theorem 3.3.6. *The quadratic process $(q(t))_{t\geq 0}$ generated by the SDE (3.2.6) is second order stationary if and only if one of the following assertions hold true*

***A**. $(q(t))_{t\geq 0}$ is deterministic with $q(0) = m_2(0)$, a.e., and*

$$\left(2\alpha(t)+\gamma^2(t)\right)m_2(0) + 2\left(\gamma(t)\beta(t)+\mu(t)\right)m_1(0) + \beta^2(t) = 2\left(\gamma(t)m_2(0) + \beta(t)m_1(0)\right) = 0$$

,

***B**. the SDE (3.2.6) is time-invariant such that*

$$\left(2\alpha+\gamma^2\right)m_2(0) + 2\left(\gamma\beta+\mu\right)m_1(0) + \beta^2 = 0, \tag{3.3.16}$$

and

$$\left(2\alpha+3\gamma^2\right)K_q(0) + 2\gamma^2 m_2^2(0) + 2\left(\mu+3\gamma\beta\right)m_3(0) - 2\left(\mu+\gamma\beta\right)m_1(0)m_2(0) + 2\beta^2 m_2(0) = 0. \tag{3.3.17}$$

In this case, the covariance function of the process $(q(t))_{t\geq 0}$ is given by

$$K_q(t,s) = K_q(0)e^{\left(2\alpha+\gamma^2\right)(t-s)} + \frac{2(\gamma\beta+\mu)\left(C_X(0,0)+2K_X(0)m_1(0)\right)}{\alpha+\gamma^2}\left(e^{\left(2\alpha+\gamma^2\right)(t-s)} - e^{\alpha(t-s)}\right) \tag{3.3.18}$$

where $C_X(.\ ,\ .)$ denotes the third order cumulant of the process $(X(t))_{t\geq 0}$.

Proof. In deterministic case with $m_2(0) = q(0)$ and $K_q(0) = 0$ the process $(q(t))_{t\geq 0}$ is obviously second-order stationary if and only if $m_2(t) = m_2(0)$ and $K_q(t) = 0$, $\forall t \geq 0$, which implies from (3.3.7) and (3.3.9), $(q(t))_{t\geq 0}$ is second order stationary if and only if

$$\left(2\alpha(t)+\gamma^2(t)\right)m_2(0) + 2\left(\gamma(t)\beta(t)+\mu(t)\right)m_1(0) + \beta^2(t) = 2\left(\gamma(t)m_2(0)+\beta(t)m_1(0)\right) = 0 \text{ a.e.},$$

If the process $(q(t))_{t\geq 0}$ is not deterministic i.e., $K_q(0) > 0$, it follows from the theorem 3.3.5 that the conditions (3.3.16) − (3.3.18) ensure the second order stationarity and the fact that $C_X(u-s,u-s) = e^{\alpha(u-s)}C_X(0,0)$, $K_X(u,s) = e^{\alpha(u-s)}K_X(0)$ for all $u \geq s$. Conversely, if $(q(t))_{t\geq 0}$ is second-order stationary, then $K_q(t) = K_q(0)$, which implies from (3.3.8) that for $t \geq s \geq 0$,

$$\begin{aligned} K_q(t,s) &= e^{\left(2\alpha+\gamma^2\right)(t-s)}\left\{K_q(0) + 2\left(\gamma\beta+\mu\right)\int_s^t e^{\left(2\alpha+\gamma^2\right)(s-u)}\left(C_X(u-s,u-s)+2K_X(u,s)m_1(0)\right)du\right\} \\ &= K_q(0)e^{\left(2\alpha+\gamma^2\right)(t-s)} + \frac{2(\gamma\beta+\mu)\left(C_X(0,0)+2K_X(0)m_1(0)\right)}{\alpha+\gamma^2}\left(e^{\left(2\alpha+\gamma^2\right)(t-s)} - e^{\alpha(t-s)}\right). \end{aligned}$$

So that, there exists a constants α, γ and β such that $\alpha(t) = \alpha$, $\gamma(t) = \gamma$ and $\beta(t) = \beta$ a.e., and since $m_2(t) = m_2(0), t \geq 0$, the formula (3.3.7) leads to $\left(2\alpha+\gamma^2\right)m_2(0) + 2\left(\gamma\beta+\mu\right)m_1(0) + \beta^2 = 0$ a.e., Furthermore, since $K_q(t) = K_q(0), t \geq 0$, formula (3.3.9) ensure that (3.3.17) holds.□ □

Remark 3.3.7. *It is easy to show that if the process $(X(t))_{t\geq 0}$ is deterministic with $X(0) = m_1(0)$ a.e., and $\alpha(t)m_1(0)+\mu(t) = \gamma(t)m_1(0)+\beta(t) = 0$ a.e., then the quadratic process $(q(t))_{t\geq 0}$ is deterministic with $q(0) = m_2(0)$ a.e., and*

$$\left(2\alpha(t)+\gamma^2(t)\right)m_2(0) + 2\left(\gamma(t)\beta(t)+\mu(t)\right)m_1(0) + \beta^2(t) = 2\left(\gamma(t)m_2(0)+\beta(t)m_1(0)\right) = 0$$

Example 3.3.8. *Assume that $\gamma(t) \neq 0, \beta(t) \neq 0, \mu(t) \neq 0$ and $\gamma(t)\beta(t)+\mu(t) = 0$, then for all $t \geq 0$*

$$m_2(t) = \phi_t(0)\left\{m_2(0) + \int_0^t \phi_s^{-1}(0)\,\beta^2(s)ds\right\},$$

$$\begin{aligned} K_q(t) &= \psi(t)\left\{K_q(0) + 4\int_0^t \psi^{-1}(s)\left(\gamma^2(s)m_2^2(s) + 2\gamma(s)\beta(s)m_3(s) + \beta^2(s)m_2(s)\right)ds\right\} \\ &= \psi(t)\times \\ &\left\{K_q(0) + 4\int_0^t \psi^{-1}(s)\left(2\gamma(s)\beta(s)Cov(q(s),X(s)) + \beta^2(s)K_X(s) + \left(\beta(s)m_1(s)+\gamma(s)m_2(s)\right)^2\right)ds\right\}, \end{aligned}$$

$$K_q(t,s) = \exp\left\{\int_s^t \left(2\alpha(v)+\gamma^2(v)\right)dv\right\}K_q(s),\ t \geq s.$$

In particular in time-invariant case, with $2\alpha+3\gamma^2 < 0$, $(q(t))_{t\geq 0}$ is second-order stationary process with

$$m_2(0) = \frac{\beta^2}{|2\alpha+\gamma^2|},\ K_q(0) = 2\left(\frac{2\gamma\beta m_3(0) + \beta^2K_X(0) + \beta^2m_1^2(0) + \gamma^2m_2^2(0)}{|2\alpha+3\gamma^2|}\right),\ K_q(h) = e^{\left(2\alpha+\gamma^2\right)|h|}K_q(0).$$

Its transfer functions associated with Itô-Wiener representation are given by the symmetrization of the following functions

$$f^{[2]}(\lambda_{(r)}) = \begin{cases} \dfrac{\beta^2}{|2\alpha+\gamma^2|} & \text{if } r=0 \\ \dfrac{2r\left(\gamma f^{[2]}(\lambda_{(r-1)}) + \beta f(\lambda_{(r-1)})\right)}{i\underline{\lambda}_{(r)} - (2\alpha+\gamma^2)} & \text{if } r\geq 1 \end{cases}$$

Example 3.3.9. *If* $\gamma(t)=0$*, the process* $(X(t))_{t\geq 0}$ *is solution of the linear SDE (3.3.1) and the quadratic process* $(q(t))_{t\geq 0}$ *satisfies the following SDE,* $dq(t) = \left(2\alpha(t)q(t) + 2\mu(t)X(t) + \beta^2(t)\right)dt + 2\beta(t)X(t)dw(t)$*. It follows from theorem 3.3.6 that the quadratic process* $(q(t))_{t\geq 0}$ *is second order stationary if and only if either it is deterministic with* $q(0)=m_2(0)$ *a.e., and* $2\alpha(t)m_2(0) + 2\mu(t)m_1(0) + \beta^2(t) = 2\beta(t)m_1(0) = 0$ *a.e., or there exist some constants* α, μ *and* β *such that* $\alpha(t)=\alpha$, $\mu(t)=\mu$, $\beta(t)=\beta$ *and* $2\alpha m_2(0) + 2\mu m_1(0) + \beta^2 = 0$ *and hence* $\alpha K_q(0) + \mu m_3(0) - \mu m_1(0)m_2(0) + \beta^2 m_2(0) = 0$*. Under the above conditions the covariance function of the process* $(q(t))_{t\geq 0}$ *is given by*

$$K_q(t,s) = e^{2\alpha(t-s)}K_q(0) + \frac{4\mu K_X(0)m_1(0)}{\alpha}\left(e^{2\alpha(t-s)} - e^{\alpha(t-s)}\right),\ t\geq s$$

where $K_q(0) = \dfrac{\mu(m_3(0) - m_1(0)m_2(0)) + \beta^2 m_2(0)}{|\alpha|}$, $(\alpha<0)$*. On the other hand* $(X(t))_{t\geq 0}$ *admits the Wiener-Itô representation, i.e.,* $X(t) = f_t(0) + \int_{\mathbb{R}} f_t(\lambda)e^{it\lambda}dZ(\lambda)$*, with* $f_t(0) = -\dfrac{\mu}{\alpha}$, $f_t(\lambda) = \dfrac{\beta}{i\lambda - \alpha}$ *(see corollary 3.2.1) and the quadratic process* $(q(t))_{t\geq 0}$ *has the following spectral representation*

$$q(t) = f_t^{[2]}(0) + \int_{\mathbb{R}} f_t^{[2]}(\lambda)e^{it\lambda}dZ(\lambda) + \frac{1}{2}\int_{\mathbb{R}^2} f_t^{[2]}(\lambda_{(2)})e^{it\underline{\lambda}_{(2)}}dZ(\lambda_{(2)}),$$

where $m_2(t) = f_t^{[2]}(0)$ *and the evolutionary symmetrized transfer functions* $f_t^{[2]}(\lambda_{(r)})$*, for* $r = 0,1,2$ *of the quadratic process* $(q(t))_{t\geq 0}$ *are given by the following differential equations*

$$f_t^{[2](1)}(\lambda_{(r)}) = \begin{cases} 2\left(\alpha(t)f_t^{[2]}(0) + \mu(t)f_t(0)\right) + \beta^2(t) & \text{if } r=0 \\ (2\alpha(t) - i\lambda_1)\, f_t^{[2]}(\lambda_1) + 2\left(\mu(t)f_t(\lambda_1) + \beta(t)f_t(0)\right) & \text{if } r=1 \\ \left(2\alpha(t) - i\underline{\lambda}_{(2)}\right) f_t^{[2]}(\lambda_{(2)}) + 4\beta(t)f_t(\lambda_1) & \text{if } r=2. \end{cases}$$

In particular, when $(X(t))_{t\geq 0}$ *is time-invariant, we have*

$$f^{[2]}(0) = \frac{2\mu^2 - \alpha\beta^2}{2\alpha^2}\ ,\ f^{[2]}(\lambda_1) = \frac{2\mu\beta}{\alpha(\alpha - i\lambda_1)} \text{ and } f^{[2]}(\lambda_{(2)}) = \frac{4\beta^2}{(\alpha - i\lambda_1)(2\alpha - i\underline{\lambda}_{(2)})}.$$

Moreover, if $X(0)$ *is normally distributed with mean* $m_1(0)$ *and variance* $K_X(0) = \dfrac{\beta^2}{2|\alpha|}$*, then the process* $(q(t))_{t\geq 0}$ *is strict stationary and its distribution has the form* $\dfrac{\beta^2}{2|\alpha|}\chi_{(1)} + 2m_1(0)U + m_1^2(0)$ *where* $U \rightsquigarrow \mathcal{N}\left(0, \dfrac{\beta^2}{2|\alpha|}\right)$ *and* $\chi_{(1)}$ *is the chi-squared distribution with* 1 *degree of freedom.*

3.4 Taylor's property of $COBL(1,1)$ process

In this section we consider the time-invariant version of SDE (3.1.1) with $\gamma \neq 0$, $\mu \neq 0$ and $\gamma\beta + \mu = 0$ (see Example 3.3.8). In order to ensure the second-order stationarity of both processes $(X(t))_{t\geq 0}$ and $(q(t))_{t\geq 0}$, and to give conditions ensuring the inequality $\rho_X(h) > \rho_{X^2}(h)$, $h \in \mathbb{R}$ (called Taylor's property), where $\rho_X(h)$ and $\rho_{X^2}(h)$ denote, respectively, the autocorrelations of the processes $(X(t))_{t\geq 0}$ and $\left(X^2(t)\right)_{t\geq 0}$, we shall assume that the condition (3.3.6) holds true. Noting here that this property was studied by Goncalves et. al. [29] for some discrete-time bilinear models. It follows from the previous section that $\rho_X(h) = e^{\alpha|h|}$ and $\rho_{X^2}(h) = e^{\left(2\alpha+\gamma^2\right)|h|}$, $\forall h \in \mathbb{R}$. So, the Taylor's property is present for values of α in the interval $]-\infty, -\gamma^2[$, for a fixed γ. In Fig1 below, we can see that the Taylor's property is not present because $\alpha \notin]-\infty, -\gamma^2[$, for $\alpha = -3$, $\gamma = -2$, but in Fig2 , for $-3 \leq \alpha \leq -1.5$ and fixed $\gamma = 1$, $\alpha \in]-\infty, -\gamma^2[$ which implies the Taylor's property is achieved.

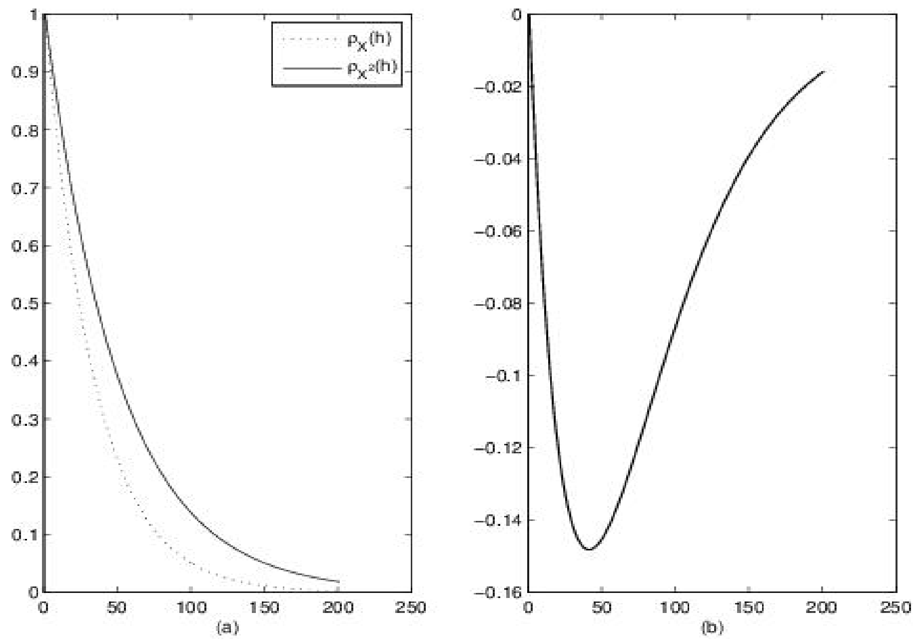


Figure 3.1: (a) :The plot of $\rho_X(h)$ and $\rho_{X^2}(h)$, (b) :The plot of $\rho_X(h) - \rho_{X^2}(h)$

Now, we analyze the relationship between the Taylor's property and leptokurtosis of the corresponding time-invariant $COBL(1,1)$ process. First recall that the kurtosis of a process X is defined by $Kur(X) = \dfrac{E\left\{(X(t) - m_1(0))^4\right\}}{K_X^2(0)}$ for our model, the kurtosis is given by

$$Kur(X) = \frac{m_4(0) - 4m_3(0)m_1(0) + 6m_2(0)m_1^2(0) - 3m_1^4(0)}{K_X^2(0)}$$

with $m_1(0) = -\dfrac{\mu}{\alpha}$, $K_X(0) = \dfrac{(\alpha\beta - \mu\gamma)^2}{\alpha^2|2\alpha + \gamma^2|}$, so the kurtosis increases and the Taylor's property holds when $\alpha \in [-2, -\gamma^2[$, but when $\alpha \in [-\gamma^2, -\frac{\gamma^2}{2}[$, the Taylor's property is not occurs and the kurtosis is a decreasing function of α. In particular for $E\{X\} = 0$ i.e., $\mu = 0$, the kurtosis

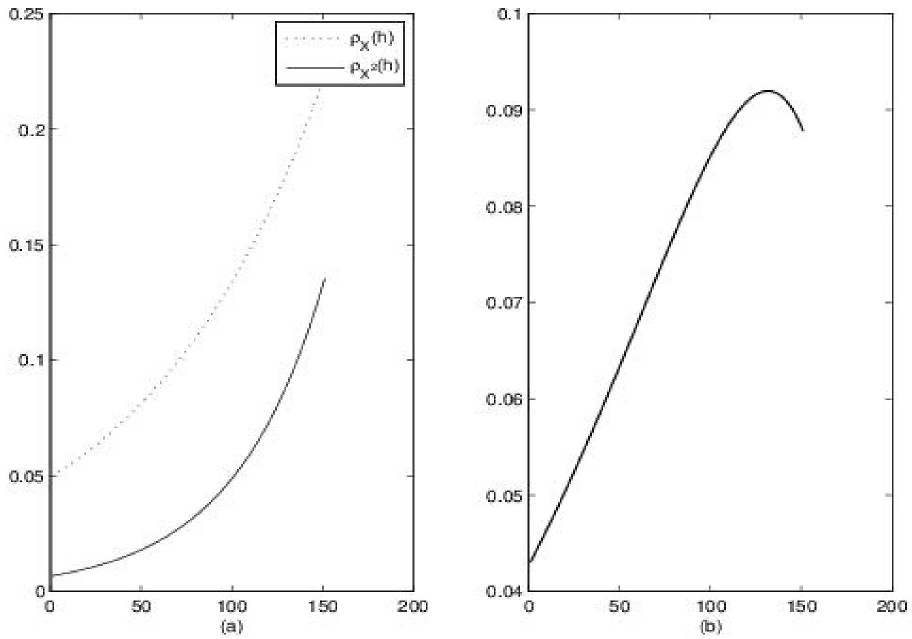


Figure 3.2: (a) :The plot of $\rho_X(h)$ and $\rho_{X^2}(h)$, (b) :The plot of $\rho_X(h) - \rho_{X^2}(h)$

reduces to $Kur(X) = 6\dfrac{(5\eta - 11)}{(\eta - 3)(\eta - 4)} + 3 > 3$ where $\eta = 1 - \dfrac{2\alpha}{\gamma^2}$ and hence the condition (3.3.6) imply that $\eta > 4$.

3.5 Conclusion

In this chapter, we have studied some properties of continuous-time bilinear processes in both domain, frequency and time domain. The transfer functions of the bilinear process which is given by SDE with time-varying coefficients are computed by A. Bibi and F. Merahi [11], a similar result are given here for the quadratic process. In order to analyze the Taylor property of the process, we need to compute the covariance function of the quadratic process, for this purpose we have studied the second order properties of this quadratic process and we calculated its second order moments. For the particular case in example 3.3.8 when $\gamma \neq 0$, $\mu \neq 0$, $2\alpha + \gamma^2$ and $\gamma\beta + \mu = 0$, we observe that the existence of Taylor property depends on the values of α and γ^2, i.e. in this case the bilinear process have the Taylor property only if $\alpha < -\gamma^2$, and the leptokurtosis induce the Taylor property. In general, when $\gamma\beta + \mu \neq 0$, we can conclude that all parameters affect on the Taylor property, it means if $\alpha < -\gamma^2$ and $(\gamma\beta + \mu)\,(C_X(0,0) + 2K(0)m_1(0)) > 0$, the Taylor property holds.

Part II

STATISTICAL STUDY

Chapter 4

Moment method estimation of first-order continuous-time bilinear processes[4]

4. Ce chapitre est accepté par : Communications in Statistics - Simulation and Computation.

Abstract

In the present chapter, we propose an estimation method of the first order continuous-time bilinear ($COBL$) process based on Euler-Maruyama discretization of the Itô solution associated with the stochastic differerential equation (SDE) defining the process, and we suggest a standard moment method (MM) estimates of the unknown parameters involving in $COBL$ process. So, some relationships linking the parameters and the theoretical moments of the process and its quadratic version are given. These relationships we allow to construct two algorithms to estimate the parameters based on MM. Using the fact that the incremented processes are strongly mixing with exponential rate whenever certain conditions are fulfilled, we show that the resulting estimators are strongly consistent and asymptotically normal. The theory can be applied to the $COGARCH(1,1)$, Gaussian Ornstein-Uhlenbeck (OU) models and among other specifications. Finite sample properties are also considered throught Monte-Carlo experimencts. In end, this algorithm is then used to model the exchanges rate of the Algerian Dinar against the US-dollar and against the single European currency.

4.1 Introduction

Stochastic differential equations (SDE) plays an important role in various field such as in control, financial engineering, biology and among others. So, several authors have been studied the probabilistic and statistical structure of linear and nonlinear SDE (see for instance Brockwell et al. [14], [15], [16], Kluppelberg et al. [42], Le Breton and Musiela [45], Iglòi and Terdik [38] and Subba Rao and Terdik [62] and the reference therein). Of course, these SDE depends on some unknown parameters, and however a growing literature on different methods have been proposed for their identification. For an in-depth detailed mathematical inference we refer the interested reader to the nomography by Prakasa Rao [58]. Recently, discretization methods becomes an appealing tool for the parameters estimation and continue to gain popularity especially in diffusion processes. Indeed, based on a general quasi-likelihood distribution Aït Sahalia [1] proposed a class of discretized quasi-maximum likelihood estimator ($QMLE$) for stationary diffusion process and compared their asymptotic efficiency with different QM distribution assumptions (see also Tsai an Chan [67] for further discussions). The no tractability of $QMLE$ conducting however Haug et al. [32] to suggest method of moment (MM) for estimating the continuous-time $GARCH(1,1)$ based on Euler-Maruyama discretization and established their asymptotic properties. Kallsen et al. [39] established the consistency and the asymptotic normality of MM for estimating the time-changed Lévy models.

In this chapter, we consider a first-order continuous-time bilinear process (often called Black-Scholes Model) governed by the following SDE

$$dX(t) = (\alpha X(t) + \mu)dt + (\gamma X(t) + \beta)\, dW(t), t \geq 0, X(0) = X_0, \tag{4.1.1}$$

denoted hereafter $COBL\,(1,1)$ in wherein $(W(t))_{t\geq 0}$ is a real standard Brownian motion defined on some basic filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}\, P)$, the initial state $X(0) = X_0$ is a random variable, defined on $(\Omega, \mathcal{A}, P)$ independent of W with $E\{X(0)\} = m_X$ and $Var\{X(0)\} = K(0)$ and the parameters of interest are gathered in vector $\underline{\theta} = (\alpha, \mu, \gamma, \beta)' \in \mathbb{R}^4$. The SDE (4.1.1) was introduced firstly in control theory literature by Mohler [52] and has been widely applied in engineering and finance (see for instance Rémillar [59] and the reference therein). It is worth noting the SDE (4.1.1) encompasses many commonly used models in the literature. Some specific examples among others are:

1. $COGARCH(1,1)$: This classes of processes is defined as an SDE by $dX\,(t) = \sigma\,(t)\, dB_1(t)$ where $(\sigma\,(t))_{t\geq 0}$ the volatility process, satisfies $d\sigma^2\,(t) = \left(\mu - \alpha\sigma^2\,(t)\right) dt + \gamma\sigma^2\,(t)\, dB_2\,(t)$, $t > 0$, B_1 and B_2 are independent Brownian motions and $\mu > 0$, $\alpha \geq 0$ and $\gamma \geq 0$. So, the volatility equation can be regarded as a particular case of (4.1.1) by assuming $\beta = 0$. (see Haug et al. [32] and the reference therein).

2. $CARMA(1,0)$ or $CAR(1)$: This classes of SDE may be obtained by assuming $\gamma = 0$. (see Brockwell [15]).

3. Gaussian Ornstein-Uhlenbeck (OU) process: The Gaussian OU process is defined as $dX(t) = (\mu - \alpha X(t))\, dt + \beta dW(t)$, with $\beta > 0$. So it can be obtained from (4.1.1) by assuming $\gamma = 0$. (see Brockwell [15]).

4. Geometric Brownian motion (GBM): This class of processes is defined as a $\mathbb{R}-$valued solution process $(X(t))_{t\geq 0}$ of $dX(t) = \alpha X(t)dt + \gamma X(t)dW(t), t \geq 0$. So it can be obtained from (4.1.1) by assuming $\beta = \mu = 0$ (see Øksendal [7]).

An outline of the chapter can be given as follows. In the next section we investigate the stationarity and moments properties of the $COBL(1,1)$, its quadratic and their discretized schemes. The obtained results we allow to construct two methods of moment estimation which are described in section 3. Section 4 deals with the asymptotic properties of the proposal methods, while the finite sample properties are evaluated in Section 5 by means of Monte Carlo simulations followed by an application to model the exchanges rate of the Algerian Dinar against the US-dollar and against the single European currency. Section 6 concludes the chapter. Finally, Appendix **A** collects the main proofs of our theoretical results.

4.2 Moment properties of $COBL(1,1)$, its quadratic and their discretized schemes

The existence and the uniqueness of the solution process $(X(t))_{t\geq 0}$ of SDE (4.1.1) is ensured by general results on stochastic differential equations (see, e.g; [49], Ch.4). In order to ensure the stationarity of the solution process, we require that the parameters α and γ subject to the following condition (see Le Breton and Musiela [45], Lemma 2.3 and Lemma 2.4)

A1. $2\alpha + \gamma^2 < 0$

So, by the Itô formula (see e.g., [3], Theorem 8.4.2) the solution process $(X(t))_{t\geq 0}$ of equation (4.1.1) is given by

$$X(t) = \Phi(t)\left\{X(0) + (\mu - \gamma\beta)\int_0^t \Phi^{-1}(s)ds + \beta\int_0^t \Phi^{-1}(s)dW(s)\right\} \tag{4.2.1}$$

where $\Phi(t) = \exp\left\{\left(\alpha - \frac{1}{2}\gamma^2\right)t + \gamma W(t)\right\}$ its mean function is $\Psi(t) = \exp\{\alpha t\}$, $t \geq 0$, yielding under the condition **A1**., that the solution process $(X(t))_{t\geq 0}$ is strictly stationary with

$$\begin{aligned} m_X &= E\{X(t)\} = -\frac{\mu}{\alpha}, \\ K_X(0) &= Var\{X(t)\} = \frac{-(\alpha\beta - \mu\gamma)^2}{\alpha^2(2\alpha + \gamma^2)}, \\ K_X(h) &= Cov(X(t), X(t+h)) = K_X(0)e^{\alpha|h|}, h \neq 0 \end{aligned}$$

Remark 4.2.1. *It is worth noting that if $\mu\gamma = \alpha\beta$, Equation (4.1.1) has only a degenerate solution given by $X(t) = -\beta/\gamma = -\mu/\alpha$, even, if $\beta = 0$, the solution process (4.2.1) is neither a standardized diffusion process nor a martingale. Moreover, if $\gamma = 0$ and $\beta \neq 0$, the solution process (4.2.1) provides a Gaussian OU process. So if we are interested in stationary non Gaussian solution of (4.1.1), it is necessary to assume that $\mu^2 + \beta^2 > 0$ and $\gamma \neq 0$.*

Remark 4.2.2. *Le Breton and Musiela [45] have showed that under the condition* **A1**.*, the second-order structure the solution process (4.2.1) is similar to an $CAR(1)$ process, and hence there exists some real Brownian motion $(W^*(t))_{t\geq 0}$ uncorrelated with $X(0)$ such that $(X(t))_{t\geq 0}$ admits the following representation*

$$dX(t) = (\alpha X(t) + \mu)dt + \left(\gamma^2 K_X(0) + (\gamma m_X + \beta)^2\right)^{1/2} dW^*(t)$$

Remark 4.2.3. *For any $t \geq 0$, let $-\xi(t) = \left(\alpha - \frac{1}{2}\gamma^2\right)t + \gamma W(t)$ and $\eta(t) = (\mu - \gamma\beta)t + \beta W(t)$, then the solution process (4.2.1) may be rewritten as*

$$X(t) = e^{-\xi(t)}\left\{X(0) + \int_0^t e^{\xi(s)}d\eta(s)\right\}, t \geq 0.$$

that is the solution process of the celebrated generalized OU (GOU) process defined by $dX(t) = -\xi(t)X(t)dt + d\eta(t), t \geq 0, X(0) = X_0$.

In the sequel, we shall assume without the loss of generality that

A2. $\gamma\beta + \mu = 0, \gamma \neq 0$

The Assumption **A2.,** is quite technical, it means that the parameter β may be expressed via μ and γ. Otherwise, it can be assumed that $\beta = 0$, because this assumption can be fulfilled by the following transformation $Y(t) = \dfrac{\mu}{\gamma\mu - \alpha\beta}(\beta + \gamma X(t))$.

4.2.1 Quadratic process

The quadratic process $\left(X^2(t)\right)_{t\geq 0}$ is denoted hereafter $(q(t))_{t\geq 0}$ and satisfy the following SDE

$$dq(t) = \left(\left(2\alpha + \gamma^2\right) q(t) + 2\left(\gamma\beta + \mu\right) X(t) + \beta^2\right) dt + 2\left(\gamma q(t) + \beta X(t)\right) dW(t), \tag{4.2.2}$$

Proposition 4.2.4. *Under the conditions* ***A1****. and* ***A2****., the covariance function of the process* $(q(t))_{t\geq 0}$ *is given for any* $h \geq 0$ *by*

$$K_q(h) = K_q(0)e^{\left(2\alpha+\gamma^2\right)h}. \tag{4.2.3}$$

so $K_q(h) \to 0$ *at an exponential rate as* $h \to +\infty$*, and hence* $(q(t))_{t\geq 0}$ *has a short term memory property.*

4.2.2 Discretized schemes

In this subsection, we consider the Euler-Maruyama discretization, for this purpose, let the incremented process $X^{(r)}(t) = X(t) - X(t-r)$ for any $t \geq r > 0$ and let $\left(X_j^{(r)}\right)_{j\in\mathbb{N}}$ describes an equidistant sequence of the process $(X(t))_{t\geq 0}$ associated with $X^{(r)}(t)$, $t \geq r \geq 0$, i.e.,

$$\begin{aligned} X_j^{(r)} &= X^{(r)}(rj) \\ &= \int_{(j-1)r}^{jr} ((\alpha X(s) + \mu)ds + (\gamma X(s) + \beta)\, dW(s)) \\ &= \alpha_j^{(r)} X_{j-1}^{(r)} + e_j^{(r)} \end{aligned} \tag{4.2.4}$$

where (see remark 4.2.3) $\alpha_j^{(r)} = \exp\left\{-\left(\xi(jr) - \xi((j-1)r)\right)\right\}$ and $e_j^{(r)} = \exp\left\{-\xi(jr)\right\} \int_{(j-1)r}^{jr} e^{-\xi(u)} d\eta(u)$. So, Equation (4.2.4) is an $AR(1)$ form with martingale error in particular for OU process with $\mu = 0$, we have $E\left\{e_j^{(r)}\right\} = 0$ and $Var\left\{e_j^{(r)}\right\} = \dfrac{\beta^2}{2\alpha}\left(1 - e^{-2\alpha r}\right)$. Also $\left(q_j^{(r)}\right)_{j\in\mathbb{N}}$ describes an equidistant sequence of the quadratic process $(q(t))_{t\geq 0}$ associated with the process $q^{(r)}(t) = q(t) - q(t-r)$, $t \geq r \geq 0$.

Proposition 4.2.5. *Let* $(X(t))_{t\geq 0}$ *be the stationary solution process of* SDE (4.1.1) *with incremented process* $\left(X^{(r)}(t)\right)_{t\geq 0}$*. Then under the condition* ***A1****.,* $E\left\{X^2(t)\right\} < \infty$ *for all* t*, and* *for any $h \geq r > 0$ we have*

$$m_X^{(r)} = E\left\{X^{(r)}(t)\right\} = 0,$$

$$K_X^{(r)}(0) = E\left\{\left(X^{(r)}(t)\right)^2\right\} = 2\left(1 - e^{\alpha r}\right) K_X(0) \tag{4.2.5}$$

$$K_X^{(r)}(h) = Cov(X^{(r)}(t), X^{(r)}(t+h)) = \left(2 - e^{-\alpha r} - e^{\alpha r}\right) e^{\alpha h} K_X(0),\ \forall h > r \geq 0, \tag{4.2.6}$$

Furthermore, $E\left\{X^4(t)\right\} < \infty$ for all $t \geq 0$, and for every $h \geq r > 0$, we have

$$m_q^{(r)} = E\left\{q^{(r)}(t)\right\} = 0,$$

$$K_q^{(r)}(0) = E\left\{\left(q^{(r)}(t)\right)^2\right\} = 2\left(1 - e^{(2\alpha+\gamma^2)r}\right) K_q(0), \tag{4.2.7}$$

$$K_q^{(r)}(h) = Cov(q^{(r)}(t), q^{(r)}(t+h)) = K_q(0)\left(2 - e^{-(2\alpha+\gamma^2)r} - e^{(2\alpha+\gamma^2)r}\right) e^{(2\alpha+\gamma^2)h} \tag{4.2.8}$$

Remark 4.2.6. *From the expressions (4.2.5) and (4.2.6), the covariance function of the process $(X^{(r)}(t))_{t\geq 0}$ can be formed into the form*

$$K_X^{(r)}(h) = \frac{\left(2 - e^{-\alpha r} - e^{\alpha r}\right)}{2\left(1 - e^{\alpha r}\right)} K_X^{(r)}(0)\, e^{\alpha h},\ \text{for any } h \geq r > 0. \tag{4.2.9}$$

On the other hand, from the expressions (4.2.7), (4.2.8) and (4.2.9), the covariance function of the process $(q^{(r)}(t))_{t\geq 0}$ may be rewritten as

$$K_q^{(r)}(h) = \frac{\left(2 - e^{-(2\alpha+\gamma^2)r} - e^{(2\alpha+\gamma^2)r}\right)}{2\left(1 - e^{(2\alpha+\gamma^2)r}\right)} K_q^{(r)}(0)\, e^{(2\alpha+\gamma^2)h},\ \forall h \geq r > 0.$$

Now, for any integers $\tau \geq r > 0$, we denote by $R_X^{(r)}(\tau) = Cov(X_j^{(r)}, X_{j+\tau}^{(r)})$ (resp. $R_q^{(r)}(\tau) = Cov(q_j^{(r)}, q_{j+\tau}^{(r)})$) the covariance function of discretized process $\left(X_j^{(r)}\right)_{j\in\mathbb{N}}$ (resp. $\left(q_j^{(r)}\right)_{j\in\mathbb{N}}$) and by $\rho_X^{(r)}(\tau) = Corr(X_j^{(r)}, X_{j+\tau}^{(r)})$ (resp. $\rho_q^{(r)}(\tau) = Corr(q_j^{(r)}, q_{j+\tau}^{(r)})$) the correlation function of $\left(X_j^{(r)}\right)_{j\in\mathbb{N}}$ (rep. $\left(q_j^{(r)}\right)_{j\in\mathbb{N}}$). Then the above quantities are summarized in the following proposition.

Proposition 4.2.7. *Let $\left(X_j^{(r)}\right)_{j\in\mathbb{N}}$ (resp. $\left(q_j^{(r)}\right)_{j\in\mathbb{N}}$) be the discretized process of be the stationary solution process $(X(t))_{t\geq 0}$ (resp. $(q(t))_{t\geq 0}$) of the SDE (4.1.1) (resp. of SDE (4.2.2)). Then under the condition **A1**., we have for all $\tau \geq r > 0$*

$$R_X^{(r)}(0) = E\left\{\left(X^{(r)}(t)\right)^2\right\},$$

$$\rho_X^{(r)}(\tau) = \frac{\left(2 - e^{-\alpha r} - e^{\alpha r}\right)}{2\left(1 - e^{\alpha r}\right)} e^{\alpha\tau}, \forall \tau \geq r > 0$$

$$R_q^{(r)}(0) = E\left\{\left(q^{(r)}(t)\right)^2\right\},$$

$$\rho_q^{(r)}(\tau) = \frac{\left(2 - e^{-(2\alpha+\gamma^2)r} - e^{(2\alpha+\gamma^2)r}\right)}{2\left(1 - e^{(2\alpha+\gamma^2)r}\right)} e^{(2\alpha+\gamma^2)\tau},\ \forall \tau \geq r > 0$$

Proof. Straightforward and hence omitted. □

Corollary 4.2.1. *Under the condition of the proposition 4.2.7 the process* $\left(X_j^{(r)}\right)_{j\in\mathbb{N}}$ *(resp.* $\left(q_j^{(r)}\right)_{j\in\mathbb{N}}$*) has for each* $r>0$ *the correlation structure of an* $ARMA(1,1)$ *process.*

Remark 4.2.8. *It is worth noting that the quantities* $\rho_X^{(r)}(\tau)$ *and* $\rho_q^{(r)}(\tau)$ *are defined only for* $\tau \geq r > 0$*, and hence their values at* $\tau = 0$ *are different from* 1.

Example 4.2.9. *[OU process] We consider the diffusion equation* $dX(t) = -\alpha X(t)dt + \beta dW(t)$ *,* $\alpha, \beta > 0$*. The incremented process is given by* $X^{(r)}(t) = (e^{-\alpha r} - 1)\, X(t-r) + \beta e^{\alpha(r-t)} \int\limits_{t-r}^{t} e^{\alpha s} dW(s)$ *. So, the second-order properties of* $X^{(r)}(t)$ *and* $q^{(r)}(t)$ *are summarized in the following table*

	Variance	*Covariance*
$X^{(r)}(t)$	$2\left(1-e^{-\alpha r}\right) K_X(0)$	$K_X(0)\left(2-e^{-\alpha r}-e^{\alpha r}\right) e^{-\alpha h}$, $h \geq r > 0$
$q^{(r)}(t)$	$2\left(1-e^{-2\alpha r}\right) K_q(0)$	$K_q(0)\left(2-e^{-2\alpha r}-e^{2\alpha r}\right) e^{-2\alpha h}$, $h \geq r > 0$

Table(1): Second-order properties $X^{(r)}(t)$ *and* $q^{(r)}(t)$ *associated with OU process*

4.3 Method of moment estimation

There is an extensive literature devoted to the problem of estimating the unknown parameters in SDE. In most application the observations are equally discretized (financial mathematics and econometrics models). The first paper to deal with parametric estimation from a discretized stationary and ergodic process is due to Dacunha-Castelle et al. [19]. This technique allows the econometricians and/or statisticians to consider a lot of parametric methods commonly used in literature of discrete-time series models, for instance generalized method of moment (GMM) (e.g., Chan et al. [18] and the references therein). In what follows, we suppose that we can only observe the process at fixed, equally spaced sampling times. Let $r > 0$ be the sampling interval and $X = \{X(r), ..., X(nr)\}$ the observations from a second-order stationary $COBL(1,1)$ and we estimate $\underline{\theta} = (\alpha, \mu, \gamma, \beta)'$ using the method of moment based on X and its asymptotic properties described in the remainder of the paper.

4.3.1 First method of moment estimation

We first identify the vector $\underline{\theta}$ via the moments of the process $\left(X_t^{(r)}\right)_{t\in\mathbb{N}}$ and its quadratic version $\left(q_t^{(r)}\right)_{t\in\mathbb{N}}$ as follows

Lemma 4.3.1. *Consider the case* $r = 1$*, then under the conditions* **A**1.*, and* **A**2.*, the parameters* α, μ, γ *and* β *are uniquely determined by*

$$\alpha = log\left(1 - \frac{R_X^{(1)}(0)}{2K_X(0)}\right),\ \mu = -\alpha m_X,\ \gamma^2 = \delta - 2\alpha \text{ and } \beta = -\frac{\mu}{\gamma} \text{ where } \delta = log\left(1 - \frac{R_q^{(1)}(0)}{2K_q(0)}\right). \tag{4.3.1}$$

Remark 4.3.2. *From the expressions of α and δ, we remark that the parameter vector $\underline{\theta} = (\alpha, \mu, \gamma, \beta)'$ is a continuous function of the the moments m_X, $R_X^{(1)}(0)$, $K_X(0)$ and the parameter δ. Hence, by continuity, consistency of the moments will immediately imply consistency of the corresponding plug-in estimates $\underline{\theta} = (\alpha, \beta, \gamma, \mu)'$.*

Now, define the mapping $\mathcal{J}_0 : \mathbb{R}_+^2 \times \mathbb{R} \times \mathbb{R}_+^2 \longrightarrow \mathbb{R}_-^* \times \mathbb{R} \times \mathbb{R}_+$ by

$$\mathcal{J}_0(K_X(0), R_X^{(1)}(0), m_X, K_q(0), R_q^{(1)}(0)) \tag{4.3.2}$$
$$= \left(log\left(1 - \frac{R_X^{(1)}(0)}{2K_X(0)}\right), -m_X log\left(1 - \frac{R_X^{(1)}(0)}{2K_X(0)}\right), log\left(1 - \frac{R_q^{(1)}(0)}{2K_q(0)}\right) - 2log\left(1 - \frac{R_X^{(1)}(0)}{2K_X(0)}\right)\right)'.$$

Then the estimator $\left(\widehat{\alpha}, \widehat{\mu}, \widehat{\gamma}^2\right)'$ of $\left(\alpha, \mu, \gamma^2\right)'$ by the method of moment (MM) is thus

$$\left(\widehat{\alpha}, \widehat{\mu}, \widehat{\gamma}^2\right)' = \mathcal{J}_0(\widehat{K}_X(0), \widehat{R}_X^{(1)}(0), \widehat{m}_X, \widehat{K}_q(0), \widehat{R}_q^{(1)}(0)),$$

in which the quantities with hats in $\mathcal{J}_0$ are the empirical estimates of the corresponding quantity based on $\left(X_j^{(1)}\right)_{1\le j\le n}$, $\left(q_j^{(1)}\right)_{1\le j\le n}$, $(X(j))_{1\le j\le n}$ and $(q(j))_{1\le j\le n}$. In order to make the notation shorter, we have suppress the dependence of empirical estimators on the size n.

Now we turn to the second step which is the estimation of the parameters based on the correlations. We need the following lemma

Lemma 4.3.3. *Under the assumptions of the Theorem 4.3.1, the correlation functions of the discrete-time processes $\left(X_j^{(1)}\right)_{j\in\mathbb{N}}$ and its quadratic version $\left(q_j^{(1)}\right)_{j\in\mathbb{N}}$ are given respectively for any $\tau \in \mathbb{N}^*$, by*

$$\rho_X^{(1)}(\tau) = k(\alpha)\, e^{\alpha\tau} \text{ and } \rho_q^{(1)}(\tau) = k(\delta)\, e^{\delta\tau} \tag{4.3.3}$$

where k is the function defined by $k(x) = \dfrac{(2 - e^{-x} - e^{x})}{2(1 - e^{x})}$.

Remark 4.3.4. *It follow from the lemma (4.3.3) that the estimators of $k(\alpha)$ and $k(\delta)$ may be deduced from the estimators of α and δ, i.e.,*

$$\widehat{k}(\alpha) = \frac{\left(2 - e^{-\widehat{\alpha}} - e^{\widehat{\alpha}}\right)}{2\left(1 - e^{\widehat{\alpha}}\right)} \text{ and } \widehat{k}(\delta) = \frac{\left(2 - e^{-\widehat{\delta}} - e^{\widehat{\delta}}\right)}{2\left(1 - e^{\widehat{\delta}}\right)}.$$

Now, since the correlation functions $\rho_X^{(1)}(\tau)$ and $\rho_q^{(1)}(\tau)$ depends respectively on the parameters α and δ then we can follow the following algorithm to estimate these parameters

Algorithm 1

Step1 Calculate the empirical means $\widehat{m}_X = \frac{1}{n}\sum_{j=1}^n X(j)$ and $\widehat{m}_q = \frac{1}{n}\sum_{j=1}^n q(j)$ of m_X and m_q respectively.

Step2 Calculate the empirical variances $\widehat{K}_X(0) = \frac{1}{n}\sum_{j=1}^n (X(j) - \widehat{m}_X)^2$, $\widehat{K}_q(0) = \frac{1}{n}\sum_{j=1}^n (q(j) - \widehat{m}_q)^2$ of $K_X(0)$ and $K_q(0)$ respectively.

Step3 Calculate the empirical variances for discretized versions $\widehat{R}_X^{(1)}(0) = \dfrac{1}{n}\sum_{j=1}^{n}\left(X_j^{(1)}\right)^2$ and $\widehat{R}_q^{(1)}(0) = \dfrac{1}{n}\sum_{j=1}^{n}\left(q_j^{(1)}\right)^2$.

Step4 Define the mapping $J_1, J_2 : \mathbb{R}_+^2 \times \mathbb{R}_-^* \longrightarrow \mathbb{R}_-^* \times \mathbb{R}^*$ by

$$J_1(R_X^{(1)}(0), K_X(0), \alpha) = \begin{cases} \left(log\left(1 - \dfrac{R_X^{(1)}(0)}{2K_X(0)}\right), k(\alpha)\right)' & \text{if } \alpha < 0, \\ (0,0) \text{ otherwise.} \end{cases}$$

$$J_2(R_q^{(1)}(0), K_q(0), \delta) = \begin{cases} \left(log\left(1 - \dfrac{R_q^{(1)}(0)}{2K_q(0)}\right), k(\delta)\right)', & \text{if } \delta < 0, \\ (0,0)\,, \text{ otherwise .} \end{cases}$$

Then the MM estimators of $(\alpha, k(\alpha))'$ and of $(\delta, k(\delta))'$ are given by

$$(\widehat{\alpha}, k(\widehat{\alpha}))' = J_1(\widehat{R}_X^{(1)}(0), \widehat{K}_X(0), \widehat{\alpha}) \text{ and } \left(\widehat{\delta}, k(\widehat{\delta})\right)' = J_2(\widehat{R}_q^{(1)}(0), \widehat{K}_q(0), \widehat{\delta})$$

.

Step5 Defining the mapping $S_1, S_2 : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_-^* \times \mathbb{R}^*$ as a function of $\widehat{R}_X^{(1)}(0)$, $\widehat{K}_X(0)$ and $\widehat{R}_q^{(1)}(0)$, $\widehat{K}_q(0)$ respectively such that

$$(\widehat{\alpha}, k(\widehat{\alpha})) = S_1\left(\widehat{R}_X^{(1)}(0), \widehat{K}_X(0)\right) \text{ and } \left(\widehat{\delta}, k(\widehat{\delta})\right) = S_2\left(\widehat{R}_q^{(1)}(0), \widehat{K}_q(0)\right).$$

4.3.2 Second method of moment estimation

We shall note $k^+(.) = |k(.)|$, $\rho_X^{(1)+}(\tau) = \left|\rho_X^{(1)}(\tau)\right|$, $\rho_q^{(1)+}(\tau) = \left|\rho_q^{(1)}(\tau)\right|$, $R_X^{(1)+}(\tau) = \left|R_X^{(1)}(\tau)\right|$, $R_q^{(1)+}(\tau) = \left|R_q^{(1)}(\tau)\right|$ for $\tau = 1, 2, ..., d$ and let $\underline{\widehat{R}}_X^{(1)+} = \left(\widehat{R}_X^{(1)+}(0), \widehat{R}_X^{(1)+}(1), ..., \widehat{R}_X^{(1)+}(d)\right)'$, $\underline{\widehat{\rho}}_X^{(1)+} = \left(\widehat{\rho}_X^{(1)+}(1), \widehat{\rho}_X^{(1)+}(2), ..., \widehat{\rho}_X^{(1)+}(d)\right)'$. The estimation of α, δ, $k(\alpha)$ and $k(\delta)$ may be achieved according to the following algorithm.

Algorithm 2

Step1 Calculate the empirical means $\widehat{m}_X$ and $\widehat{m}_q$ of m_X and m_q as in Algorithm 1.

Step2 Calculate the empirical variances $\widehat{K}_X(0)$, $\widehat{K}_q(0)$ of $K_X(0)$ and for $K_q(0)$ as in Algorithm 1.

Step3 Calculate the empirical variances $\widehat{R}_X^{(1)}(0)$, $\widehat{R}_q^{(1)}(0)$ of $R_X^{(1)}(0)$ and for $R_q^{(1)}(0)$ associated for discretized versions and for fixed $d \geq 2$, let $\underline{\widehat{R}}_X^{(1)} = \left(\widehat{R}_X^{(1)}(0), \widehat{R}_X^{(1)}(1), ..., \widehat{R}_X^{(1)}(d)\right)^T$, $\underline{\widehat{R}}_q^{(1)} = \left(\widehat{R}_q^{(1)}(0), \widehat{R}_q^{(1)}(1), ..., \widehat{R}_q^{(1)}(d)\right)^T$ where $\widehat{R}_X^{(1)}(h) = \dfrac{1}{n}\sum_{j=1}^{n-h} X_{j+h}^{(1)} X_j^{(1)}$, and $\widehat{R}_q^{(1)}(h) = \dfrac{1}{n}\sum_{j=1}^{n-h} q_{j+h}^{(1)} q_j^{(1)}$, $h = 0, 1, ..., d$.

Step4 Compute the empirical autocorrelations $\widehat{\rho}_X^{(1)}(h) = \dfrac{\widehat{R}_X^{(1)}(h)}{\widehat{R}_X^{(1)}(0)}$ and $\widehat{\rho}_q^{(1)}(h) = \dfrac{\widehat{R}_q^{(1)}(h)}{\widehat{R}_q^{(1)}(0)}$, $h = 1, ..., d$ and set $\underline{\widehat{\rho}}_X^{(1)} = \left(\widehat{\rho}_X^{(1)}(1), \widehat{\rho}_X^{(1)}(2), ..., \widehat{\rho}_X^{(1)}(d)\right)^T$ and $\underline{\widehat{\rho}}_q^{(1)} = \left(\widehat{\rho}_q^{(1)}(1), \widehat{\rho}_q^{(1)}(2), ..., \widehat{\rho}_q^{(1)}(d)\right)^T$.

Step5 Let $\underline{\widehat{R}}_X^{(1)+} = \left(\widehat{R}_X^{(1)+}(0), \widehat{R}_X^{(1)+}(1), ..., \widehat{R}_X^{(1)+}(d)\right)'$ where $\widehat{R}_X^{(1)+}(\tau) = \left|\widehat{R}_X^{(1)}(\tau)\right|$ for $\tau = 0, 1, ..., d$.

Step6 Let $\underline{\widehat{R}}_q^{(1)+} = \left(\widehat{R}_q^{(1)+}(0), \widehat{R}_q^{(1)+}(1), ..., \widehat{R}_q^{(1)+}(d)\right)'$ where $\widehat{R}_q^{(1)+}(\tau) = \left|\widehat{R}_q^{(1)}(\tau)\right|$ for $\tau = 0, 1, ..., d$.

Step7 Let $\underline{\widehat{\rho}}_X^{(1)+} = \left(\widehat{\rho}_X^{(1)+}(1), \widehat{\rho}_X^{(1)+}(2), ..., \widehat{\rho}_X^{(1)+}(d)\right)'$ where $\widehat{\rho}_X^{(1)+}(\tau) = \left|\widehat{\rho}_X^{(1)}(\tau)\right|$ for $\tau = 1, ..., d$.

Step8 Let $\underline{\widehat{\rho}}_q^{(1)+} = \left(\widehat{\rho}_q^{(1)+}(1), \widehat{\rho}_q^{(1)+}(2), ..., \widehat{\rho}_q^{(1)+}(d)\right)'$ where $\widehat{\rho}_q^{(1)+}(\tau) = \left|\widehat{\rho}_q^{(1)}(\tau)\right|$ for $\tau = 1, ..., d$.

Step9 For fixed $d \geq 2$, define the two mapping H_1^+ and $H_2^+ : \mathbb{R}_+^d \times \mathbb{R}_+^* \times \mathbb{R}_-^* \longrightarrow \mathbb{R}_+$ by

$$H_1^+(\underline{\widehat{\rho}}_X^{(1)+}, k^+(\alpha), \alpha) = \sum_{\tau=1}^{d} \left(log(\widehat{\rho}_X^{(1)+}(\tau)) - log(k^+(\alpha)) - \alpha\tau\right)^2,$$
$$H_2^+(\underline{\widehat{\rho}}_q^{(1)+}, k^+(\delta), \delta) = \sum_{\tau=1}^{d} \left(log(\widehat{\rho}_q^{(1)+}(\tau)) - log(k^+(\delta)) - \delta\tau\right)^2.$$

and consider the least squares estimators

$$\left(\widetilde{\alpha}, \widetilde{k^+}(\alpha)\right) = Arg \min_{(k^+(\alpha),\alpha)} H_1^+(\underline{\widehat{\rho}}_X^{(1)+}, k^+(\alpha), \alpha)\ ,\ \left(\widetilde{\delta}, \widetilde{k^+}(\delta)\right) = Arg \min_{(k^+(\delta),\delta)} H_2^+(\underline{\widehat{\rho}}_q^{(1)+}, k^+(\delta), \delta),$$

their minimum are achieved for

$$\widetilde{\alpha}^\star = \frac{\sum_{\tau=1}^{d}\left(log\left(\widehat{\rho}_X^{(1)+}(\tau)\right) - \overline{log\left(\widehat{\rho}_X^{(1)+}\right)}\right)(\tau - x)}{\sum_{\tau=1}^{d}(\tau - x)^2},\ \widetilde{\delta}^\star = \frac{\sum_{\tau=1}^{d}\left(log\left(\widehat{\rho}_q^{(1)+}(\tau)\right) - \overline{log\left(\widehat{\rho}_q^{(1)+}\right)}\right)(\tau - x)}{\sum_{\tau=1}^{d}(\tau - x)^2},$$

and

$$\widetilde{k^+}(\alpha) = \exp\left\{\overline{log\left(\widehat{\rho}_X^{(1)+}\right)} - x\widetilde{\alpha}^\star\right\},\ \widetilde{k^+}(\delta) = \exp\left\{\overline{log\left(\widehat{\rho}_q^{(1)+}\right)} - x\widetilde{\delta}^\star\right\}$$

where $x = \dfrac{(d+1)}{2}$ and where $\overline{log\left(\widehat{\rho}_{\cdot}^{(1)+}\right)} = \dfrac{1}{d}\sum_{\tau=1}^{d} log\left(\widehat{\rho}_{\cdot}^{(1)+}(\tau)\right)$. Notice that $\widetilde{\alpha}^\star$ and $\widetilde{\delta}^\star$ may be positive, so we define the estimators of α and δ by this second method by $\widetilde{\alpha} = Min\left\{\widetilde{\alpha}^\star, 0\right\}$ and $\widetilde{\delta} = Min\left\{\widetilde{\delta}^\star, 0\right\}$

Step10 Define the mapping $S_1^+, S_2^+ : \mathbb{R}_+^{d+1} \longrightarrow \mathbb{R}_-^\star$ as a functions of $\underline{\widehat{R}}_X^{(1)+}$ and of $\underline{\widehat{R}}_q^{(1)+}$ such that $\widetilde{\alpha} = S_1^+\left(\underline{\widehat{R}}_X^{(1)+}\right)$, $\widetilde{\delta} = S_2^+\left(\underline{\widehat{R}}_q^{(1)+}\right)$. It follows from algorithm 1 that the estimate $(\widetilde{\alpha}, \widetilde{\mu}, \widetilde{\gamma}^2)$ are given by

$$(\widetilde{\alpha}, \widetilde{\mu}, \widetilde{\gamma}^2) = \left(S_1^+\left(\underline{\widehat{R}}_X^{(1)+}\right), -\widehat{m}_X S_1^+\left(\underline{\widehat{R}}_X^{(1)+}\right), S_2^+\left(\underline{\widehat{R}}_q^{(1)+}\right) - 2S_1^+\left(\underline{\widehat{R}}_X^{(1)+}\right)\right).$$

4.4 Asymptotic properties of the MM estimation

Let $(X(t))_{t\geq 0}$ be the strictly stationary bilinear process driven by the *SDE* (4.1.1) subject the condition **A**1.

4.4.1 Strong consistency

As already observed that Equation (4.1.1) may be regarded as a diffusion *GOU* process (see remark 4.2.3), then it follows from Oesook [54] that $(X(t))_{t\geq 0}$ is exponentially $\beta-$mixing process and hence α-mixing with an exponential decreasing rate. Since $\alpha-$mixing is invariant under continuous transformations, then the process $(q(t))_{t\geq 0}$ is α-mixing and for every $r > 0$, the processes $\left(X_t^{(r)}\right)_{t\in\mathbb{N}}$, $\left(q_t^{(r)}\right)_{t\in\mathbb{N}}$ are also strictly stationary, ergodic and $\alpha-$mixing with an exponential decreasing rate. So, since the estimators in algorithms 1 and 2 are continuously differentiable functions of empirical moments, then the strongly consistency and asymptotic normality will follow from ergodicity of the processes $\left(X_t^{(r)}\right)_{t\in\mathbb{N}}$ and $\left(q_t^{(r)}\right)_{t\in\mathbb{N}}$. In particular as $n \longrightarrow \infty$

$$\widehat{K}_X(0) \xrightarrow{a.s} K_X(0)\ ,\ \widehat{R}_X^{(1)}(0) \xrightarrow{a.s} R_X^{(1)}(0)\ ,\ \widehat{m}_X \xrightarrow{a.s} m_X \text{ and } \widehat{K}_q(0) \xrightarrow{a.s} K_q(0)\ ,\ \widehat{R}_q^{(1)}(0) \xrightarrow{a.s} R_q^{(1)}(0)$$

This implies the strong consistency of the estimators $\mathcal{J}_0(\widehat{K}_X(0), \widehat{R}_X^{(1)}(0), \widehat{m}_X, \widehat{K}_q(0), \widehat{R}_q^{(1)}(0))$ and $\mathcal{J}\left(\widehat{m}_X, \underline{\widehat{R}}_X^{(1)}, \underline{\widehat{R}}_q^{(1)}\right)$ where the mapping $\mathcal{J}_0$ is defined by (4.3.2) and $\mathcal{J}$ defined as

$$\mathcal{J}\left(m_X, \underline{R}_X^{(1)}, \underline{R}_q^{(1)}\right) = \left(S_1^+\left(\underline{R}_X^{(1)+}\right), -m_X S_1^+\left(\underline{R}_X^{(1)+}\right), S_2^+\left(\underline{R}_q^{(1)+}\right) - 2S_1^+\left(\underline{R}_X^{(1)+}\right)\right).$$

This finding we allow to

Theorem 4.4.1. *Under the condition* **A1** *and* **A2**, *the moment methods given in algorithms* **1** *and* **2** *are strongly consistent.*

4.4.2 Asymptotic normality of the estimator $(\widehat{\alpha}, \widehat{\mu}, \widehat{\gamma}^2)'$

Hereafter, we consider the following condition

A3. There exists a positive constant $\lambda > 0$ such that $E\left\{X_1^{8+\lambda}\right\} < \infty$.

Proposition 4.4.2. *Under the conditions* **A**1 − **A**3, *we have as* $n \longrightarrow \infty$,

$$\sqrt{n}\left(\left(\widehat{K}_X(0), \widehat{R}_X^{(1)}(0), \widehat{m}_X, \widehat{K}_q(0), \widehat{R}_q^{(1)}(0)\right)' - \left(K_X(0), R_X^{(1)}(0), m_X, K_q(0), R_q^{(1)}(0)\right)'\right) \overset{\mathcal{D}}{\rightsquigarrow} \mathcal{N}(0, \Sigma) \tag{4.4.1}$$

where the entries of the asymptotic variance−covariance matrix Σ *are given by*

$$\Sigma_{1,1} = Var\left\{(X(1)-\widehat{m}_X)^2\right\} + 2\sum_{i=1}^{\infty} Cov\left((X\,(1)-\widehat{m}_X)^2, (X\,(i+1)) - \widehat{m}_X)^2\right),$$

$$\Sigma_{2,2} = R_X^{(1)}(0) + 2\sum_{i=1}^{\infty} Cov\left(X_1^{(1)}, X_{i+1}^{(1)}\right);$$

$$\Sigma_{3,3} = R_X(0) + 2\sum_{i=1}^{\infty} Cov\left(X\,(1)\,, X(i+1)\right)$$

$$\Sigma_{4,4} = Var\left\{(q(1)-\widehat{m}_q)^2\right\} + 2\sum_{i=1}^{\infty} Cov\left((q(1)-\widehat{m}_q)^2, (q(i+1)-\widehat{m}_q)^2\right)$$

$$\Sigma_{5,5} = R_q^{(1)}(0) + 2\sum_{i=1}^{\infty} Cov\left(q_1^{(1)}, q_{i+1}^{(1)}\right)$$

and

$$\Sigma_{1,2} = \sum_{i=0}^{\infty} 2^{\delta_{\{i\geq 1\}}}\left\{Cov\left((X(1)-\widehat{m}_X)^2, X_{i+1}^{(1)}\right) + Cov\left(X_1^{(1)}, (X(i+1)-\widehat{m}_X)^2\right)\right\}$$

$$\Sigma_{1,3} = \sum_{i=0}^{\infty} 2^{\delta_{\{i\geq 1\}}}\left\{Cov\left((X(1)-\widehat{m}_X)^2, X(i+1)\right) + Cov\left(X(1), (X(i+1)-\widehat{m}_X)^2\right)\right\}$$

$$\Sigma_{1,4} = \sum_{i=0}^{\infty} 2^{\delta_{\{i\geq 1\}}}\left\{Cov\left((X(1)-\widehat{m}_X)^2, (q(i+1)-\widehat{m}_q)^2\right) + Cov\left((q(1)-\widehat{m}_q)^2, (X(i+1)-\widehat{m}_X)^2\right)\right\}$$

$$\Sigma_{1,5} = \sum_{i=0}^{\infty} 2^{\delta_{\{i\geq 1\}}}\left\{Cov\left((X(1)-\widehat{m}_X)^2, q_{i+1}^{(1)}\right) + Cov\left(q_1^{(1)}, (X(i+1)-\widehat{m}_X)^2\right)\right\}$$

and

$$\Sigma_{2,3} = \sum_{i=0}^{\infty} 2^{\delta_{\{i\geq 1\}}}\left\{Cov\left(X_1^{(1)}, X(i+1)\right) + Cov\left(X(1), X_{i+1}^{(1)}\right)\right\},$$

$$\Sigma_{2,4} = \sum_{i=0}^{\infty} 2^{\delta_{\{i\geq 1\}}}\left\{Cov\left(X_1^{(1)}, (q(i+1)-\widehat{m}_q)^2\right) + Cov\left((q(1)-\widehat{m}_q)^2, X_{i+1}^{(1)}\right)\right\}$$

$$\Sigma_{2,5} = \sum_{i=0}^{\infty} 2^{\delta_{\{i\geq 1\}}}\left\{Cov\left(X_1^{(1)}, q_{i+1}^{(1)}\right) + Cov\left(q_1^{(1)}, X_{i+1}^{(1)}\right)\right\}$$

$$\Sigma_{3,4} = \sum_{i=0}^{\infty} 2^{\delta_{\{i\geq 1\}}}\left\{Cov\left(X(1), (q(i+1)-\widehat{m}_q)^2\right) + Cov\left((q(1)-\widehat{m}_q)^2, X(i+1)\right)\right\}$$

$$\Sigma_{3,5} = \sum_{i=0}^{\infty} 2^{\delta_{\{i\geq 1\}}}\left\{Cov\left(X(1), q_{i+1}^{(1)}\right) + Cov\left(q_1^{(1)}, X(i+1)\right)\right\}$$

$$\Sigma_{4,5} = \sum_{i=0}^{\infty} 2^{\delta_{\{i\geq 1\}}}\left\{Cov\left((q(1)-\widehat{m}_q)^2, q_{i+1}^{(1)}\right) + Cov\left(q_1^{(1)}, (q(i+1)-\widehat{m}_q)^2\right)\right\}.$$

in which δ_Δ *is the indicator function of the set* Δ.

Now, we consider moments vector $\underline{U} = \left(K_X(0), R_X^{(1)}(0), m_X, K_q(0), R_q^{(1)}(0)\right)'$ associated with parameter vector $\underline{V} = \left(\alpha, \mu, \gamma^2\right)'$ their true values are indicated by $\underline{U}_0$ and $\underline{V}_0$ respectively. Then the asymptotic normality of our estimates is given by the following theorem

Theorem 4.4.3. *Let $\widehat{\underline{V}} = \left(\widehat{\alpha}, \widehat{\mu}, \widehat{\gamma^2}\right)'$ be the vector of estimates of $\underline{V} = \left(\alpha, \mu, \gamma^2\right)'$, then under the conditions* **A**1. – **A**3.*, we have as $n \longrightarrow \infty$, almost surely $\widehat{\underline{V}} \to \underline{V}_0$.and*

$$\sqrt{n}\left(\widehat{\underline{V}} - \underline{V}_0\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, J\Sigma J'\right), \tag{4.4.2}$$

where $J = \dfrac{\partial \mathcal{J}_0\left(\underline{U}\right)}{\partial \underline{U}}$ and Σ is as in Proposition 4.4.2.

4.4.3 Asymptotic normality of the estimator $(\widetilde{\alpha}, \widetilde{\mu}, \widetilde{\gamma}^2)'$

By an analogously way as the previous subsection, we study the asymptotic properties estimators $(\widetilde{\alpha}, \widetilde{\mu}, \widetilde{\gamma}^2)'$. Applying the central limit theorem for strongly mixing processes to obtain asymptotic normality of the empirical estimates.

Proposition 4.4.4. *Under the conditions* **A**1. – **A**3.*, we have as $n \longrightarrow \infty$,*

$$\sqrt{n}\left(\left(\widehat{m}_X, \widehat{\underline{R}}_X^{(1)\prime}, \widehat{\underline{R}}_q^{(1)\prime}\right)' - \left(m_X, \underline{R}_X^{(1)\prime}, \underline{R}_q^{(1)\prime}\right)'\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \Lambda\right), \tag{4.4.3}$$

where the covariance Λ has the components

$$\Lambda(k,j) = \begin{cases} Cov\left(X_1^{(1)}X_{j-1}^{(1)}, X_1^{(1)}X_{k-1}^{(1)}\right) + 2\sum\limits_{i=1}^{\infty} Cov\left(X_1^{(1)}X_{j-1}^{(1)}, X_{i+1}^{(1)}X_{k+i-1}^{(1)}\right), \ \text{if } 2 \le k, j \le d+2 \\ Cov\left(q_1^{(1)}q_{j-d-2}^{(1)}, q_1^{(1)}q_{k-d-2}^{(1)}\right) + 2\sum\limits_{i=1}^{\infty} Cov\left(q_1^{(1)}q_{j-d-2}^{(1)}, q_{i+1}^{(1)}q_{k+j-d-2}^{(1)}\right), \text{if } d+3 \le k, j \le 2d+3 \\ 2Cov\left(X_1^{(1)}X_{j-1}^{(1)}, q_1^{(1)}q_{k-d-2}^{(1)}\right) + 2\sum\limits_{i=1}^{\infty} Cov\left(X_1^{(1)}X_{j-1}^{(1)}, q_{i+1}^{(1)}q_{k+i-d-2}^{(1)}\right), \text{if } 2 \le j \le d+2,\ d+3 \le k \le 2d+3 \\ 2\sum\limits_{i=1}^{\infty} Cov\left(q_1^{(1)}q_{j-d-2}^{(1)}, X_{i+1}^{(1)}X_{k+i-1}^{(1)}\right), \text{if } d+3 \le j \le d+2,\ d \le k \le d+2 \end{cases}$$

$$\Lambda(1,j) = \begin{cases} Cov\left(X(1), X_1^{(1)}X_{j-1}^{(1)}\right) + 2\sum\limits_{i=1}^{\infty}\left\{Cov\left(X(1), X_{i+1}^{(1)}X_{j+i-1}^{(1)}\right) + Cov\left(X_1^{(1)}X_{j-1}^{(1)}, X(i+1)\right)\right\}, \ \text{if } 2 \le j \le d+2 \\ Cov\left(X(1), q_1^{(1)}q_{j-d-2}^{(1)}\right) + 2\sum\limits_{i=1}^{\infty}\left\{Cov\left(X(1), q_{i+1}^{(1)}q_{j+i-d-2}^{(1)}\right) + Cov\left(q_1^{(1)}q_{j-d-2}^{(1)}, X(i+1)\right)\right\}, \text{if} \\ d+3 \ \le j \le \ 2d+3 \end{cases}$$

and $\Lambda(1,1) = Cov(X(1), X(1)) + 2\sum\limits_{i=1}^{\infty} Cov(X(i), X(i+1)).$

By applying the delta method (see Theorem 3.1 van der Vaart [68]), we obtain

Corollary 4.4.1. *Under the same conditions of Proposition 4.4.4, we have*

$$\sqrt{n}\left(\widehat{\underline{\rho}}_X^{(1)} - \underline{\rho}_X^{(1)}\right) \overset{\mathcal{D}}{\rightsquigarrow} \mathcal{N}\left(0, \Sigma\right) \ \text{ and } \sqrt{n}\left(\widehat{\underline{\rho}}_q^{(1)} - \underline{\rho}_q^{(1)}\right) \overset{\mathcal{D}}{\rightsquigarrow} \mathcal{N}\left(0, \Sigma\right).$$

Now, we consider the true moments $\underline{U} = \left(m_X, \underline{R}_X^{(1)\prime}, \underline{R}_q^{(1)\prime}\right)'$ corresponding to the true parameter vector $\underline{V} = \left(\alpha, \mu, \gamma^2\right)'$ are indicated by $\underline{U}_0$ and $\underline{V}_0$ respectively. Then the asymptotic normality of our estimates is given by the following theorem

Theorem 4.4.5. *Let $\underline{\widetilde{V}}$ be the estimates vector of $\underline{V}$ by the second method of moment, then under the same conditions of proposition 4.4.4, almost surely $\underline{\widetilde{V}} \rightarrow \underline{V}_0$ as $n \longrightarrow \infty$ and*

$$\sqrt{n}\left(\underline{\widetilde{V}} - \underline{V}_0\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, I\Lambda I'\right)$$

where $I = \dfrac{\partial \mathcal{J}\left(\underline{U}\right)}{\partial \underline{U}}$ *and Λ is as in Proposition 4.4.4.*

4.5 Monte Carlo experiments

We provide in this section some simulations results for the moment method illustrated by Algorithms 4.3.1 and 4.3.2 and their asymptotic behavior given in the above section for estimating the unknown vector $\underline{\theta} = (\alpha, \mu, \gamma, \beta)$ involved in the model. For this purpose, we simulated 500 independent trajectories from a second-order stationary series according to $COBL(1,1)$ of length $n \in \{5000, 10000, 20000, 50000\}$ with standard Brownian motion. The graphic of trajectories of $X(t)$, $q(t)$ and their discretized schemes are presented in Fig.1 according to the parameters vector $\underline{\theta}_0 = (\alpha_0, \mu_0, \gamma_0, \beta_0)$ subjected to conditions **A1**. − **A3** listed table (2) below. The results of simulation experiments for estimating the vector $\underline{\theta}_0$ by the first and second method of moment described by Algorithms 4.3.1 and 4.3.2 are reported in table (2) in which we have indicated in the first column the length n of the series, the number d of lags used by the second method, the second and the third columns indicates the vector $\underline{\theta}$ and its true values $\underline{\theta}_0$ to be estimated and the column "Mean" correspond to the average of the parameters estimates over the 500 repetitions. In order to show the performance of MM, we have reported (results between bracket) the root-mean square errors ($RMSE$) of each algorithm. Figure 2, shows the box plot summary of the statistical properties of each estimates. On the other hand, the asymptotic distribution of estimated density by the two methods corresponding to each parameter are shown in Figure 3.

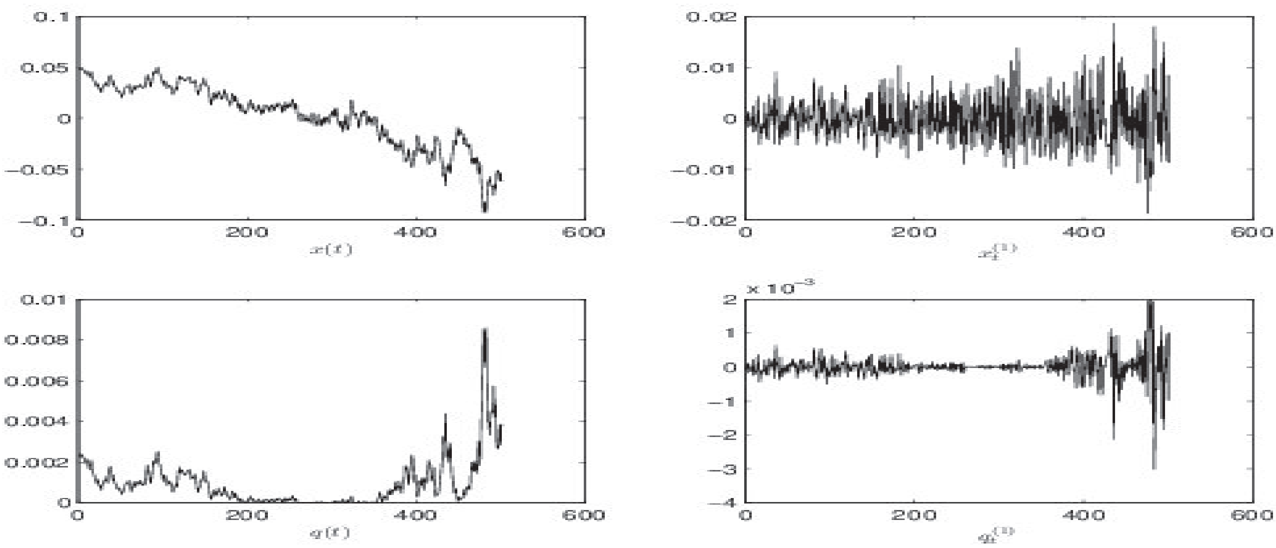


Fig1. The trajectories of $X(t)$, $q(t)$ and their discretized schemes

$\binom{n}{d}$	Parameters $\underline{\theta}$	True values $\underline{\theta}_0$	Means $\left(\widehat{\underline{\theta}}, \widetilde{\underline{\theta}}\right)$	
$\binom{5000}{d=32}$	α	-0.5	$-0.6949\,(0.0257)$	$-0.6947\,(0.0271)$
	μ	0.01	$0.0138\,(0.0021)$	$0.0140\,(0.0021)$
	γ^2	0.01	$0.0749\,(0.0596)$	$0.0690\,(0.0573)$
	$\lvert\beta\rvert$	0.1	$0.0822\,(0.1271)$	$0.0795\,(0.0674)$
$\binom{10000}{d=35}$	α	-0.5	$-0.6937\,(0.0169)$	$-0.6933\,(0.0180)$
	μ	0.01	$0.0138\,(0.0015)$	$0.0139\,(0.0015)$
	γ^2	0.01	$0.0524\,(0.0427)$	$0.0529\,(0.0383)$
	$\lvert\beta\rvert$	0.1	$0.0964\,(0.1091)$	$0.0886\,(0.0782)$
$\binom{20000}{d=40}$	α	-0.5	$-0.6931\,(0.0127)$	$-0.6936\,(0.0126)$
	μ	0.01	$0.0139\,(0.0011)$	$0.0139\,(0.0010)$
	γ^2	0.01	$0.0418\,(0.0327)$	$0.0420\,(0.0329)$
	$\lvert\beta\rvert$	0.1	$0.1063\,(0.1118)$	$0.1138\,(0.3112)$
$\binom{50000}{d=40}$	α	-0.5	$-0.6927\,(0.0080)$	$-0.6932\,(0.0080)$
	μ	0.01	$0.0138\,(0.0007)$	$0.0138\,(0.0007)$
	γ^2	0.01	$0.0353\,(0.0245)$	$0.0349\,(0.0235)$
	$\lvert\beta\rvert$	0.1	$0.1015\,(0.0749)$	$0.1165\,(0.2057)$

Table(2).The results of simulation by the first and second method of moments

The box plots summary of $\widehat{\underline{\theta}}_n$ and $\widetilde{\underline{\theta}_n}$ are showed in the following

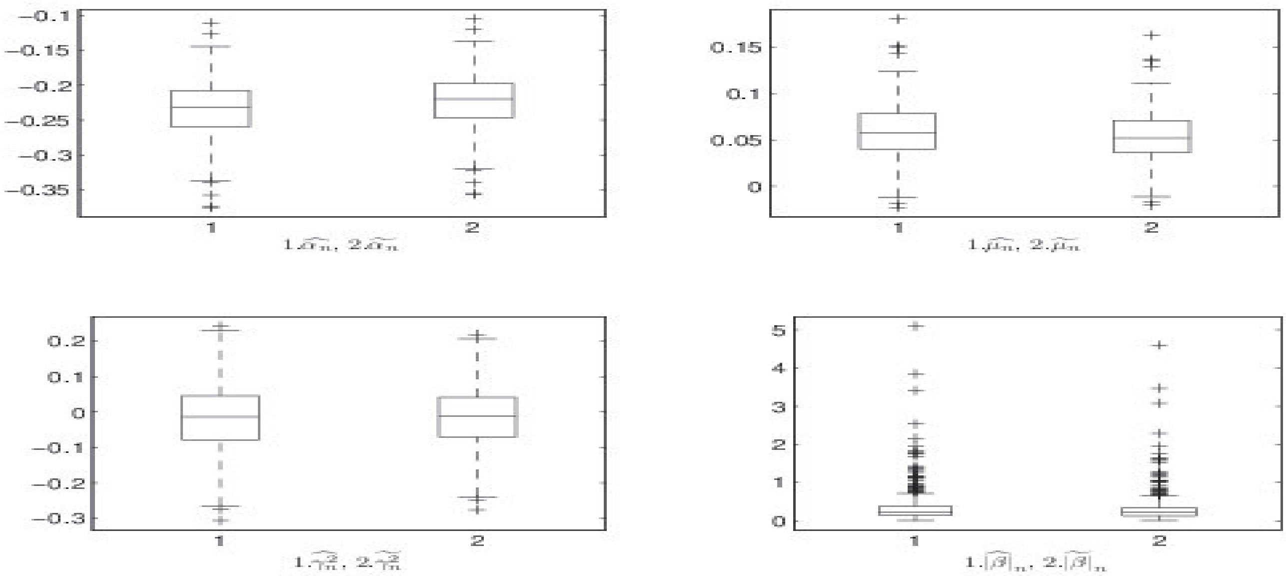


Fig2. The box plot summary of $\widehat{\underline{\theta}}_n$ (left box) and of $\widetilde{\underline{\theta}}_n$ (right box).

The plots of asymptotic density of each parameters in $\underline{\theta}$ according to two method are summarized in the following figure

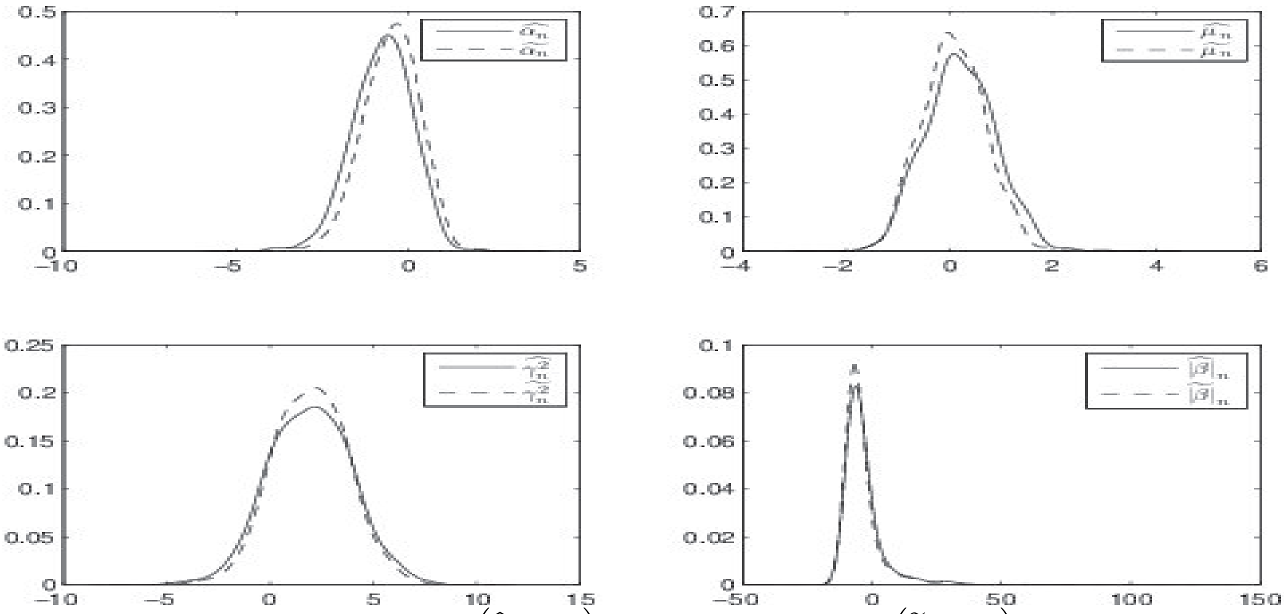


Fig3. The distribution of $\sqrt{n}\left(\underline{\widehat{\theta}}_n - \underline{\theta}_0\right)$ (solid line) and of $\sqrt{n}\left(\underline{\widetilde{\theta}}_n - \underline{\theta}_0\right)$ (dotted line).

A few comments are in order. First, regarding the results of table (2) it seems that in general the results of the first method is more performed than the second one. This is due undoubtedly to the narrow dependency of the lag d in autocorrelation functions involving in algorithm 4.3.2. In contrast to the asymptotic densities in figure Fig3 are in concordance with the theoretical results.

4.6 Application to exchange rate modeling

We apply our method to two foreign exchanges rate series of Algerian Dinar against U.S. Dollar noted $Y(t)$ and against the EURO noted $Z(t)$. This data consist of daily prices from January $3, 2000$ to September $29, 2011$. After removing the days when the market was closed (weekends, holidays,...), we provides 3056 observations for each series. Some descriptive statistics of such series are summarized in following table

The series	mean	Std. Dev	Median	Max	Min	Skewness	Kurtosis
$(Y(t))$	73.4511	4.2424	73.1261	81.2819	60.3453	-0.6005	3.7642
$(Z(t))$	88.6118	11.5755	91.0995	109.0699	67.2039	-0.5181	2.1330

Table3: Descriptive statistics of the series $(Y(t))_{t\geq 1}$ and $(Z(t))_{t\geq 1}$

The plot of series $(Y(t))_{t\geq 1}$ and $(Z(t))_{t\geq 1}$ are reported in Fig 4

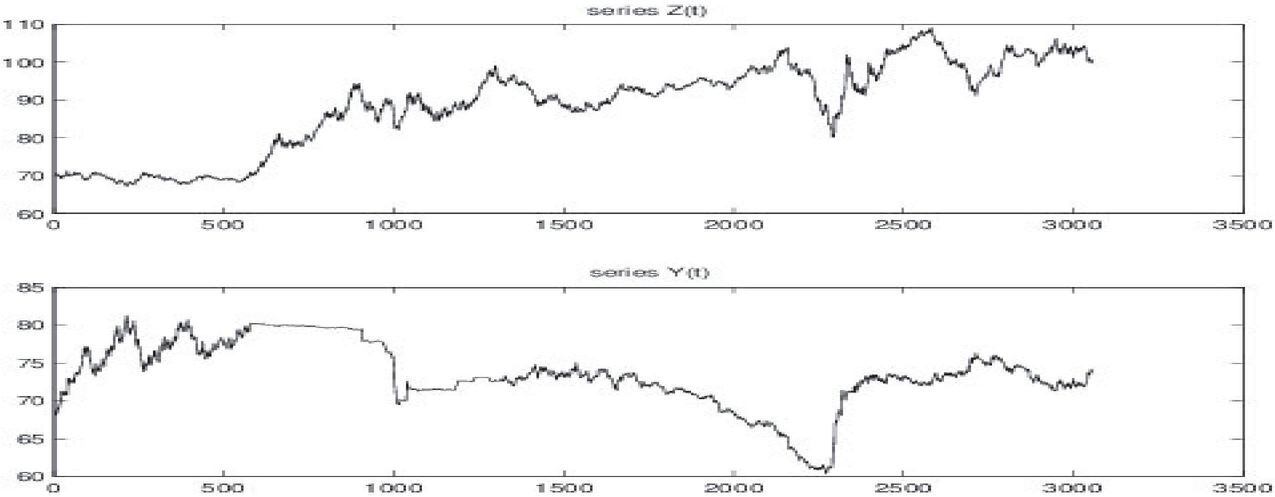


Fig4. The graphics of the series $(Y(t))$ and $(Z(t))$.

Using surrogate data test for nonlinearity, it can be showed that the series $(Y(t))_{t\geq 1}$ and $(Z(t))_{t\geq 1}$ are non linear, so we propose to mode them by an $COBL(1,1)$. The parameters corresponding to the fiited models gathered in following table

Parameters	α	μ	γ	β
$(Y_t)_{t\geq 0}$	0.0014	0.1013	0.0362	-2.8024
$(Z_t)_{t\geq 0}$	0.0008	0.0692	0.0256	-2.6982

Table4: Parameters of adjusted COBL $(1,1)$

The descriptive statistics of fitted trajectories according to the Table4 are summarized in Table5

The series	mean	Std. Dev	Median	Max	Min	Skewness	Kurtosis
$\left(\widetilde{Y}(t)\right)$	74.4587	1.2698	74.6736	76.3603	68.9355	-0.7795	3.2848
$\left(\widetilde{Z}(t)\right)$	91.5503	4.9783	92.6145	99.2591	70.0438	-1.0607	4.3296

Table 5: Descriptive statistics of the series $\left(\widetilde{Y}(t)\right)_{t\geq 1}$ and $\left(\widetilde{Z}(t)\right)_{t\geq 1}$

The results in Table 5 of fitted data according $COBL(1,1)$ model reveal a noticeable resemblance with the results of the brut series displayed in Table3 and hence the capability of $COBL(1,1)$ to model this data is then justified.

4.7 Conclusion

In this chapter, we have presented two methods of moment estimation for continuous-time bilinear process generated by some diffusion equation. The first method described by algorithm 4.3.1 is based on the second order moments of the process, its quadratic version and their associated incremented processes of order 1. In the second method, the estimator described by algorithm 4.3.2 is based on the autocorrelation functions of the discretized schemes. Contrary to the uncomputability of $QMLE$ and/or the inconsistency of direct inference based on a discretized version of the diffusion process, the main advantage of MM estimator lies in its consistency and its explicitly form. Finally, we investigated the empirical study of our estimators via monte Carlo simulation and an application to model the exchanges rate of the Algerian Dinar against the US-dollar and against the single European currency, in order to highlight the theoretical results. It is however interesting to extend the method for general models using the generalized method of moments (GMM). We leave this important issue for future researches.

4.8 Appendix

Proof of Proposition 4.2.4. Under **A2**, Equation (4.2.2) becomes

$$dq(t) = \left(\beta^2 + \left(2\alpha + \gamma^2\right) q(t)\right) dt + 2\left(\gamma q(t) + \beta X(t)\right) dW(t),$$

and under the condition **A1** we have $\frac{dE\left\{q(t)\right\}}{dt} = 0$ and from (4.2.2) we obtain

$$\begin{aligned} E\left\{q(t)\right\} &= m_q = -\frac{\beta^2}{2\alpha+\gamma^2}, \ \forall t \geq 0, \\ dE\left\{q(t)q(s)\right\} &= (2\alpha+\gamma^2)E\left\{q(t)q(s)\right\} dt + \beta^2 E\left\{q(s)\right\} dt \\ &= (2\alpha+\gamma^2)E\left\{q(t)q(s)\right\} dt + \beta^2 m_q dt, \ \forall t \geq s \geq 0, \end{aligned}$$

so, we get

$$\begin{aligned} E\left\{q(t)q(s)\right\} &= e^{(2\alpha+\gamma^2)(t-s)}\left\{E\left\{q^2(s)\right\} + \beta^2 m_q \int_s^t e^{(2\alpha+\gamma^2)(s-u)} du\right\} \\ &= e^{(2\alpha+\gamma^2)(t-s)}\left\{E\left\{q^2(s)\right\} + \beta^2 m_q \left(-\frac{1}{2\alpha+\gamma^2}\right)\left[e^{(2\alpha+\gamma^2)(s-t)} - 1\right]\right\} \\ &= e^{(2\alpha+\gamma^2)(t-s)}\left\{E\left\{q^2(s)\right\} + m_q \left(-\frac{\beta^2}{2\alpha+\gamma^2}\right)\left[e^{(2\alpha+\gamma^2)(s-t)} - 1\right]\right\} \\ &= e^{(2\alpha+\gamma^2)(t-s)}\left\{E\left\{q^2(s)\right\} + m_q^2\left[e^{(2\alpha+\gamma^2)(s-t)} - 1\right]\right\} \\ &= e^{(2\alpha+\gamma^2)(t-s)}\left\{E\left\{q^2\right\} - m_q^2 + m_q^2 e^{(2\alpha+\gamma^2)(s-t)}\right\} \\ &= e^{(2\alpha+\gamma^2)(t-s)}\left\{E\left\{q^2\right\} - m_q^2\right\} + m_q^2 \\ &= e^{(2\alpha+\gamma^2)(t-s)} K_q(0) + m_q^2. \end{aligned}$$

Since $m_q = -\frac{\beta^2}{2\alpha+\gamma^2}$ and $K_q(0) = Var\left\{q(t)\right\} = E\left\{q^2(t)\right\} - m_q^2$, then we have $E\left\{q(t)q(s)\right\} - m_q^2 = e^{(2\alpha+\gamma^2)(t-s)} K_q(0)$ and using the identity $Cov(q(t), q(s)) = E\left\{q(t)q(s)\right\} - E\left\{q(t)\right\} E\left\{q(s)\right\} = E\left\{q(t)q(s)\right\} - m_q^2$, we obtain

$$\begin{aligned} K_q(t,s) &= e^{(2\alpha+\gamma^2)(t-s)} K_q(0) \\ K_q(t,s) &= K_q(t-s) = e^{(2\alpha+\gamma^2)(t-s)} K_q(0), \ \forall t \geq s \end{aligned}$$

due to the second-order stationarity of $q(t)$. $\square$
Proof of Proposition 4.2.5. Due to the second-order stationary solution, we have $E\left\{X^{(r)}(t)\right\} = E\left\{X(t)-X(t-r)\right\} = E\left\{X(t)\right\} - E\left\{X(t-r)\right\} = 0$, so the variance and covariance function of the process $X^{(r)}(t)$ can be computed for all $h \geq r > 0$, as follows

$$
\begin{aligned}
K_X^{(r)}(0) &= 2\left(1-e^{\alpha r}\right) K_X(0),\\
K_X^{(r)}(h) &= E\left\{\left(X(t)-X(t-r)\right)\left(X(t+h)-X(t+h-r)\right)\right\}\\
&= E\left\{X(t)X(t+h)\right\} - E\left\{X(t)X(t+h-r)\right\} - E\left\{X(t-r)X(t+h)\right\} + E\left\{X(t-r)X(t+h-r)\right\}\\
&= 2K_X(0)e^{\alpha h} - K_X(0)e^{\alpha(h-r)} - K_X(0)e^{\alpha(h+r)}\\
&= \left(2-e^{-\alpha r}-e^{\alpha r}\right) e^{\alpha h} K_X(0),\ \forall h \geq r > 0,
\end{aligned}
$$

By the same manner we obtain the moments of the process $\left(q^{(r)}(t)\right)_{t\geq 0}$. Indeed, first, it is clear that $E\left\{q^{(r)}(t)\right\} = E\left\{q(t)-q(t-r)\right\} = E\left\{q(t)\right\} - E\left\{q(t-r)\right\} = 0$, and

$$
\begin{aligned}
K_q^{(r)}(0) &= E\left\{\left(q(t)-q(t-r)\right)^2\right\}\\
&= E\left\{q^2(t)\right\} - 2E\left\{q(t)q(t-r)\right\} + E\left\{q^2(t-r)\right\}\\
&= 2K_q(0) - 2K_q(0)e^{(2\alpha+\gamma^2)r}\\
&= 2\left(1-e^{(2\alpha+\gamma^2)r}\right) K_q(0)
\end{aligned}
$$

Moreover, we have for any $h \geq r > 0$,

$$
\begin{aligned}
K_q^{(r)}(h) &= E\left\{\left(q(t)-q(t-r)\right)\left(q(t+h)-q(t+h-r)\right)\right\}\\
&= E\left\{q(t)q(t+h)\right\} - E\left\{q(t)q(t+h-r)\right\} - E\left\{q(t-r)q(t+h)\right\} + E\left\{q(t-r)q(t+h-r)\right\}\\
&= K_q(0)\left(2-e^{-(2\alpha+\gamma^2)r}-e^{(2\alpha+\gamma^2)r}\right) e^{(2\alpha+\gamma^2)h},\ h \geq r > 0. \ \square
\end{aligned}
$$

Proof of Corollary 4.2.1. From the proposition 4.2.7 we get $\dfrac{\rho_X^{(r)}(\tau)}{\rho_X^{(r)}(1)} = \dfrac{R_X^{(r)}(\tau)}{R_X^{(r)}(1)} = \dfrac{K_X^{(r)}(\tau)}{K_X^{(r)}(1)} = e^{\alpha(\tau-1)}$, $\tau \geq 1$, so, $\rho_X^{(r)}(\tau) = e^{\alpha(\tau-1)}\rho_X^{(r)}(1)$. Hence, following Brockwell and Davis [16] page 112, we identify e^{α} as the root φ of $AR(1)$ part and the root ϕ of MA part may be identified as $\rho_X^{(r)}(1) = (1+\phi\varphi)(\phi+\varphi) / \left(1+2\phi\varphi+\phi^2\right)$. The same arguments for the process $\left(q_j^{(r)}\right)_{j\in\mathbb{N}}$

Proof of Lemma 4.3.1. Under the conditions of Lemma 4.3.1, it follows by the proposition 4.2.5 that $R_X^{(1)}(0) = 2\left(1-e^{\alpha}\right) K_X(0)$ and $R_q^{(1)}(0) = 2\left(1-e^{\delta}\right) K_q(0)$, so $\alpha = log\left(1-\dfrac{R_X^{(1)}(0)}{2K_X(0)}\right)$, $\delta = log\left(1-\dfrac{R_q^{(1)}(0)}{2K_q(0)}\right)$ where $\delta = 2\alpha+\gamma^2$. Moreover, since $m_X = -\dfrac{\mu}{\alpha}$ then β and μ can be deduced. $\square$

Proof of Lemma 4.3.3. Under the assumptions of the Theorem 4.3.1 we obtain from Proposition 4.2.5 and Remark 4.2.6 for any $\tau \geq 1$,

$$
\rho_X^{(1)}(\tau) = \frac{\left(2-e^{-\alpha}-e^{\alpha}\right)}{2\left(1-e^{\alpha}\right)} e^{\alpha\tau} \text{ and } \rho_q^{(1)}(\tau) = \frac{\left(2-e^{-(2\alpha+\gamma^2)}-e^{(2\alpha+\gamma^2)}\right)}{2\left(1-e^{(2\alpha+\gamma^2)}\right)} e^{(2\alpha+\gamma^2)\tau}
$$

Setting $k(x)=\dfrac{(2-e^{-x}-e^{x})}{2\,(1-e^{x})}$, so after some simple calculus the results follow. □

Proof of Proposition 4.4.2. We first show that the vector $\left(\widehat{K}^{\star}_X(0),\widehat{R}^{(1)}_X(0),\widehat{m}_X,\widehat{K}^{\star}_q(0),\widehat{R}^{(1)}_q(0)\right)'$ where $\widehat{K}^{\star}_X(0)=\frac{1}{n}\sum_{i=1}^{n}(X(i)-m_X)^2$ and $\widehat{K}^{\star}_q(0)=\frac{1}{n}\sum_{i=1}^{n}(q(i)-m_q)^2$ satisfies the proposition 4.4.2. For this purpose, denote $\underline{X}_i:=((X(i)-m_X)^2,X^{(1)}_i,X(i),(q(i)-m_q)^2,\ q^{(1)}_i)'$ and by Cramér-Wold device, we have to show that as $n\longrightarrow\infty$,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\underline{\lambda}'\underline{X}_i-\underline{\lambda}'\left(K_X(0),R^{(1)}_X(0),m_X,K_q(0),R^{(1)}_q(0)\right)'\right)\overset{\mathcal{D}}{\rightsquigarrow}\mathcal{N}\left(0,\lambda'\Sigma\lambda\right), \qquad (4.8.1)$$

for all vectors $\underline{\lambda}=(\lambda_1,...,\lambda_5)'\in\mathbb{R}^5$ such that $\underline{\lambda}'\Sigma\underline{\lambda}>0$. Since the strong mixing is preserved under linear transformations as well as its rate of convergence, then the sequences $\left(\underline{\lambda}'\underline{X}_i\right)_{i\in\mathbb{N}}$ is strongly mixing with exponentially decaying rate. So, under **A**3., the central limit theorem for strongly mixing processes is applicable (see Theorem 18.5.3, Ibragimov and Linnik [37]). Therefore, $n\longrightarrow\infty$,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\underline{\lambda}'\underline{X}_i-\underline{\lambda}'\left(K_X(0),R^{(1)}_X(0),m_X,K_q(0),R^{(1)}_q(0)\right)'\right)\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\sigma^2\right),$$

where

$$\sigma^2=Var\left\{\underline{\lambda}'\underline{X}_1\right\}+2\sum_{i=1}^{\infty}Cov\left(\underline{\lambda}'\underline{X}_1,\underline{\lambda}'\underline{X}_{i+1}\right). \qquad (4.8.2)$$

Evaluating (4.8.2) and rearranging with respect to $\underline{\lambda}$, we can shows that $\sigma^2=\underline{\lambda}'\Sigma\underline{\lambda}$. Moreover, since

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\underline{\lambda}'\underline{X}_i-\underline{\lambda}'\left(\widehat{K}_X(0),\widehat{R}^{(1)}_X(0),\widehat{m}_X,\widehat{K}_q(0),\widehat{R}^{(1)}_q(0)\right)'\right),$$

converges in probability to 0 as $n\longrightarrow\infty$ for every $\underline{\lambda}\in\mathbb{R}^5$ such that $\underline{\lambda}'\Sigma\underline{\lambda}>0$, (see the proof of Proposition 7.3.4, Brockwell and Davis [16]), it follows that $(\widehat{K}_X(0),\widehat{R}^{(1)}_X(0),\widehat{m}_X,\widehat{K}_q(0),\widehat{R}^{(1)}_q(0))'$ have the same asymptotic behavior as $\left(\widehat{K}^{\star}_X(0),\widehat{R}^{(1)}_X(0),\widehat{m}_X,\widehat{K}^{\star}_q(0),\widehat{R}^{(1)}_q(0)\right)'$. □

Proof of Theorem 4.4.3. Since the mapping $\mathcal{J}_0$ is continuous in $\underline{U}$, The strong consistency of $\widehat{\underline{V}}$ follows immediately from Theorem 4.4.1. Moreover, since $\widehat{\underline{U}}$ is asymptotically normal and $\mathcal{J}_0$ is differentiable at $\underline{U}_0$, then the asymptotic normality of $\widehat{\underline{V}}$ follows from proposition 4.4.2 and by application of delta method.□

Proof of Proposition 4.4.4. We will first focus on the asymptotic behavior of the vector $\left(\widehat{m}_X,\widehat{\underline{R}}^{(1)\prime}_X,\widehat{\underline{R}}^{(1)\prime}_q\right)'$. Denote

$$\underline{Y}_i:=\left(X(i),(X^{(1)}_i)^2,X^{(1)}_iX^{(1)}_{i+1},...,X^{(1)}_iX^{(1)}_{i+d},(q^{(1)}_i)^2,q^{(1)}_iq^{(1)}_{i+1},...,q^{(1)}_iq^{(1)}_{i+d}\right)'.$$

then, by the Cramér-Wold, we have to show that as $n\longrightarrow\infty$,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\underline{\lambda}'\underline{Y}_i-\underline{\lambda}'\left(m_X,\underline{R}^{(1)}_X,\underline{R}^{(1)}_q\right)\right)\overset{\mathcal{D}}{\rightsquigarrow}\mathcal{N}\left(0,\underline{\lambda}'\Lambda\underline{\lambda}\right), \qquad (4.8.3)$$

for all vectors $\underline{\lambda}= (\lambda_1, ..., \lambda_{2d+3})^{'} \in \mathbb{R}^{2d+3}$ such that $\underline{\lambda}^{'}\Lambda\underline{\lambda} > 0$. Similar arguments as in proof of proposition 4.4.2, we have as $n \longrightarrow \infty$,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\underline{\lambda}^{'}\underline{Y}_i - \underline{\lambda}^{'}\left(m_X, \underline{R}_X^{(1)}, \underline{R}_q^{(1)}\right)\right) \overset{\mathcal{D}}{\rightsquigarrow} \mathcal{N}\left(0, \nu^2\right),$$

with $\nu^2 = Var\left\{\underline{\lambda}^{'}\underline{Y}_1\right\} + 2\sum_{i=1}^{\infty} Cov\left(\underline{\lambda}^{'}\underline{Y}_1, \underline{\lambda}^{'}\underline{Y}_{i+1}\right)$. By evaluating the above expression and rearranging with respect to λ we obtain $\nu^2 = \underline{\lambda}^{'}\Lambda\ \underline{\lambda}$. □

Proof of Theorem 4.4.5. Since the mapping $\mathcal{J}$ is continuous in $\underline{U}$, The strong consistency of $\underline{\widetilde{V}}$ follows immediately from theorem (4.4.1). The fact that $\underline{\widetilde{U}}$ is asymptotically normal and $\mathcal{J}$ is differentiable at $\underline{U}_0$, then the asymptotic normality of $\underline{\widetilde{V}}$ follows from (4.4.3) and the application of delta method. □

Chapter 5

Frequency-domain estimation of continuous-time bilinear processes[5]

5. Ce chapitre est soumis dans le journal : Communication in Statistics- Theory and Methods.

Abstract

In this chapter, we study in frequency domain some probabilistic and statistical properties of continuous-time version of the well known bilinear processes driven by a standard Brownian motion. This class of processes which encompasses many commonly used processes in literature, were defined as a nonlinear stochastic differential equation which has raised considerable interest in the last few years. So, the $\mathbb{L}_2-$structure of the process is studied and its covariance function is given. These structures we lead to study the strong consistency and asymptotic normality of the Whittle estimate of the unknown parameters involved in the process. Finite sample properties are also considered through Monte Carlo experiments.

5.1 Introduction

Discrete-time series analysis has been well developed within the framework of linear and/or Gaussian models. Unfortunately these hypothesis lead to models that fail to capture certain phenomena commonly observed in practice such as limit cycles, self-excitation, asymmetric distribution, leptokurtosis and sudden jumping behavior. So, in recent times we have become more aware of the fact that there are many datasets that cannot be modeled as discrete-time linear models. Wegman et al. [69] provide a rich source of examples emanating from the oceanographic and meteorological sciences which are clearly non-linear. To model such series, one of the classes of non-linear models which has attracted considerably the attention of statistician and/or econometricians is the classes of discrete-time bilinear processes, introduced by Granger and Andersen [30]. The version of continuous-time of these processes have been widely studied and considered by several authors in time series analysis and in theory of stochastic differential equations. For instance, Le Breton and Musiela [45] and Bibi and Merahi [11] have considered the processes $(X(t))_{t\geq 0}$ generated by the following time-varying stochastic differential equation

(SDE)

$$\begin{aligned} dX(t) &= (\alpha(t)X(t) + \mu(t))\,dt + (\gamma(t)X(t) + \beta(t))\,dw(t), t \geq 0, X(0) = X_0 \\ &= \mu_t\,(X\,(t))\,dt + \sigma_t\,(X(t))\,dw(t), t \geq 0, X(0) = X_0 \end{aligned} \quad (5.1.1)$$

denoted hereafter $COBL\,(1,1)$ (called some time Black-Scholes models) in which $\mu_t\,(x) = \alpha(t)x + \mu(t)$ and $\sigma_t\,(x) = \gamma(t)x + \beta(t)$ which represents respectively the drift and the diffusion, $(w(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}$ defined on some basic filtered space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, P)$ with spectral representation $w(t) = \int_{\mathbb{R}} \frac{e^{it\lambda} - 1}{i\lambda} dZ(\lambda)$, where $Z(\lambda)$ is an orthogonal complex-valued stochastic measure on $\mathbb{R}$ with zero mean, $E\left\{|dZ(\lambda)|^2\right\} = dF(\lambda) = \frac{d\lambda}{2\pi}$ and uniquely determined by $Z([a,b[) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-i\lambda a} - e^{-i\lambda b}}{i\lambda} dw\,(\lambda)$, for all $-\infty < a < b < +\infty$. The SDE (5.1.1) is called time-invariant if there exists some constants α, μ, γ and β such that for all t, $\alpha(t) = \alpha$, $\mu(t) = \mu$, $\gamma(t) = \gamma$ and $\beta(t) = \beta$. The initial state $X(0)$ is a random variable defined on $(\Omega, \mathcal{A}, P)$ supposed to be not dependent on w such that $E\{X(0)\} = m_X(0)$ and $Var\{X(0)\} = K_X(0)$. The parameters $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are measurable deterministic functions and subject to the following assumption:

Assumption 2. *$\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are differentiable functions such that* $\forall T > 0$, $\int_0^T |\alpha(t)|\,dt < \infty$, $\int_0^T |\mu(t)|\,dt < \infty$, $\int_0^T |\gamma(t)|^2\,dt < \infty$ *and* $\int_0^T |\beta(t)|^2\;dt < \infty$.

The SDE (5.1.1) encompasses many commonly used models in the literature. Some specific examples among others are:

1. $COGARCH(1,1)$: This classes of processes is defined as a SDE by $dX\,(t) = \sigma\,(t)\,dB_1(t)$ with $d\sigma^2\,(t) = \left(\mu\,(t) - \alpha\,(t)\,\sigma^2\,(t)\right)dt + \gamma\,(t)\,\sigma^2\,(t)\,dB_2\,(t)$, $t > 0$ where B_1 and B_2 are independent Brownian motions and $\mu\,(t) > 0$, $\alpha\,(t) \geq 0$ and $\gamma\,(t) \geq 0$. So, the stochastic volatility equation can be regarded as a particular case of (5.1.1) by assuming constant the function $\beta(t) = 0$ for all t. (see Kluppelberg et al. [42] and the reference therein).

2. $CAR(1)$: This classes of SDE may be obtained by assuming $\gamma\,(t) = 0$ for all t. (see Brockwell [15] and the reference therein)

3. Gaussian Ornstein-Uhlenbeck (GOU) process: The GOU process is defined as $dX(t) = (\mu\,(t) - \alpha\,(t)\,X(t))\,dt + \beta\,(t)\,dw(t)$, with $\beta\,(t) > 0$ for all $t \geq 0$. So it can be obtained from (5.1.1) by assuming $\gamma(t) = 0$ for all t. (see Brockwell [15] and the reference therein).

4. Geometric Brownian motion (GBM): This class of processes is defined as a $\mathbb{R}-$valued solution process $(X(t))_{t\geq 0}$ of $dX(t) = \alpha\,(t)\,X(t)dt + \gamma\,(t)\,X(t)dw(t), t \geq 0$. So it can be obtained from (5.1.1) by assuming $\beta(t) = \mu\,(t) = 0$ for all t. (see Øksendal [7] and the reference therein).

The existence of solution process of SDE (5.1.1), was investigated by several authors, for instance among others, Le Breton and Musiela [45], Bibi and Merahi [11].

In this chapter, we shall investigate some probabilistic and statistical properties of second-order solution process of equation (5.1.1) which are also regular (or causal), i.e., $X(t)$ is $\sigma\,\{w(s), s \leq t\}$

$-$measurable. For this purpose, let $\mathbb{L}_r(F)$ be the real Hilbert space of complex valued functions $f_t(\lambda_{(r)})$ defined on $\mathbb{R}^r$ such that $f_t(-\lambda_{(r)}) = \overline{f_t(\lambda_{(r)})}$ with a inner product

$$\langle f_t, g_t \rangle_F = r! \int_{\mathbb{R}^r} Sym\left\{f_t(\lambda_{(r)})\right\} \overline{Sym\left\{g_s(\lambda_{(r)})\right\}} dF(\lambda_{(r)}),$$

where $\lambda_{(r)} = (\lambda_1, ..., \lambda_r) \in \mathbb{R}^r$, $Sym\left\{f_t(\lambda_{(r)})\right\} = \frac{1}{r!} \sum_{\pi \in \mathcal{P}} f_t\left(\lambda_{\pi(1),}, ..., \lambda_{\pi(r)}\right) \in \mathbb{L}_r(F)$ where $\mathcal{P}$ denotes the group of all permutations of the set $\{1, ..., r\}$ and $dF(\lambda_{(r)}) = \prod_{i=1}^{r} dF(\lambda_i)$. It is well known that if $(X(t))_{t \geq 0}$ is second-order and causal process (see Major [50] for further discussions) then it admits the so-called Wiener-Itô representation, i.e.,

$$X(t) = f_t(0) + \sum_{r \geq 1} \frac{1}{r!} \int_{\mathbb{R}^r} e^{it\underline{\lambda}_{(r)}} f_t(\lambda_{(r)}) dZ(\lambda_{(r)}), \tag{5.1.2}$$

where $\underline{\lambda}_{(r)} = \sum_{i=1}^{r} \lambda_i$ and the integrals are multiple Wiener-Itô stochastic integrals with respect to the stochastic measure $dZ(\lambda)$, $f_t(0) = E\{X(t)\}$, $dZ(\lambda_{(r)}) = \prod_{i=1}^{r} dZ(\lambda_i)$ and $f_t(\lambda_{(r)})$ are referred as the $r-th$ evolutionary transfer functions of $(X(t))_{t \geq 0}$, uniquely determined up to symmetrization and fulfill the condition

$$\sum_{r \geq 0} \frac{1}{r!} \int_{\mathbb{R}^r} \left|f_t(\lambda_{(r)})\right|^2 dF(\lambda_{(r)}) < \infty \text{ for all } t. \tag{5.1.3}$$

so $Var(X(t)) < +\infty$, and the symmetrization of $f_t(.)$ simplify some combinatorial reasoning. As a property of the representation (5.1.2) is that for any $f_t(\lambda_{(n)})$ and $f_s(\lambda_{(m)})$, we have

$$E\left\{\int_{\mathbb{R}^n} f_t(\lambda_{(n)}) dZ(\lambda_{(n)}) \overline{\int_{\mathbb{R}^m} f_s(\lambda_{(m)}) dZ(\lambda_{(m)})}\right\} = \delta_n^m n! \int_{\mathbb{R}^n} Sym\left\{f_t(\lambda_{(n)})\right\} \overline{Sym\left\{f_s(\lambda_{(n)})\right\}} dF(\lambda_{(n)})$$

where δ_n^m is the delta function. Another property linked with (5.1.2) is the diagram formula which state that

$$\begin{aligned} &\int_{\mathbb{R}} f_t(\lambda) dZ(\lambda) \int_{\mathbb{R}^n} g_s\left(\lambda_{(n)}\right) dZ(\lambda_{(n)}) \\ &= \int_{\mathbb{R}^{n+1}} g_s\left(\lambda_{(n)}\right) f_t\left(\lambda_{n+1}\right) dZ(\lambda_{(n+1)}) + \sum_{k=1}^{n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} g_s\left(\lambda_{(n)}\right) \overline{f_t\left(\lambda_k\right)} dF\left(\lambda_k\right) dZ(\lambda_{(n \backslash k)}) \end{aligned}$$

where $Z(d\lambda_{(n \backslash k)}) = Z\left(d\lambda_1\right) ... Z\left(d\lambda_{k-1}\right) . Z\left(d\lambda_{k+1}\right) ... Z\left(d\lambda_n\right)$.

The main aim of the chapter is to use the transfer functions approach associated with the solution process of (5.1.1) to establish some characterizations and second-order properties for a such process. So, in section 2 we summarize some of the basic probabilistic properties of $COBL(1,1)$ including the conditions ensuring the existence of the process $(X(t))_{t \geq 0}$ using its Wiener-Itô representation and $\mathbb{L}_2-$structure of SDE (5.1.1) based on the associated transfer functions. After a short remainder of the Whittle estimation procedure, we formulate and discuss in section 3 our main results on the asymptotic behavior of the Whittle estimator when the coefficients are constant.

5.2 Second-order properties of $COBL\,(1,1)$

The existence and uniqueness of the solution process of SDE (5.1.1) in time domain is ensured by the general results on SDE and under the Assumption 2, since the drift and the diffusion are Lipschitz with linear growth, i.e., $|\mu_t(x) - \mu_t(y)| \leq \sup_t |\alpha(t)|\,|x-y|$ and $|\sigma_t(x) - \sigma_t(y)| \leq \sup_t |\gamma(t)|\,|x-y|$, so the Itô solution is given by

$$X(t) = \Phi(t)\left\{X(0) + \int_0^t \Phi^{-1}(s)\left(\mu(s) - \gamma(s)\beta(s)\right)ds + \int_0^t \Phi^{-1}(s)\beta(s)\,dw(s)\right\},\ a.e., \quad (5.2.1)$$

where $\Phi(t) = \exp\left\{\int_0^t \left(\alpha(s) - \frac{1}{2}\gamma^2(s)\right)ds + \int_0^t \gamma(s)\,dw(s)\right\}$. In time-invariant case the solution reduces

$$X(t) = \Phi(t)\left\{X(0) + (\mu - \gamma\beta)\int_0^t \Phi^{-1}(s)ds + \beta\int_0^t \Phi^{-1}(s)dw(s)\right\}, \quad (5.2.2)$$

with $\Phi(t) = \exp\left\{\left(\alpha - \frac{1}{2}\gamma^2\right)t + \gamma w(t)\right\}$. The process (5.2.2) constitute a unique, stationary and ergodic solution to the time-invariant version of (5.1.1) if and only if the function $g(y) = \frac{1}{\sigma^2(y)}\exp\left\{2\int\limits_1^y \frac{\mu(x)}{\sigma^2(x)}dx\right\}$ is integrable on $[0,+\infty]$ (see Kutoyants [43], page 1). Moreover, (5.2.2) can be rewritten as

$$X(t) = e^{-\xi(t)}\left\{X(0) + \int_0^t e^{\xi(s)}d\eta(s)\right\}, t \geq 0 \quad (5.2.3)$$

where $-\xi(t) = \left(\alpha - \frac{1}{2}\gamma^2\right)t + \gamma W(t)$ and $\eta(t) = (\mu - \gamma\beta)\,t + \beta W(t)$, that is the solution process of celebrated generalized OU process defined by $dX(t) = -\xi(t)X(t)dt + d\eta(t), t \geq 0, X(0) = X_0$.

Remark 5.2.1. *Note that, the solution (5.2.1) is Markovian whenever $\beta(t) \neq 0$ for all t, otherwise the solution process is neither a Markov process nor a martingale.*

Remark 5.2.2. *When $\gamma(t) = 0$, $\alpha(t) < 0$ and $\beta(t) \neq 0$, this provides a second-order solution processes for GOU or for $CAR(1)$ equations. So, if we are interested in second-order non-Gaussian solution, it is necessary to assume that everywhere $\mu^2(t) + \beta^2(t) > 0$, $\gamma(t) \neq 0$ and not only $\alpha(t) < 0$ but $2\alpha(t) + \gamma^2(t) < 0$ as well.*

Remark 5.2.3. *It is worth noting that the condition $\gamma(t)\mu(t) \neq \alpha(t)\beta(t)$ for all t, must be hold, otherwise the equation (5.1.1) has only a degenerate solution,i.e., $X(t) = -\dfrac{\beta(t)}{\gamma(t)} = -\dfrac{\mu(t)}{\alpha(t)}$.*

The solution based on Wiener-Irô representation (5.1.2) is discussed along the rest of the section. For this purpose, we recalling the following two theorems due to Bibi and Merahi [11].

Theorem 5.2.4. *Assume that everywhere*

$$2\alpha(t) + \gamma^2(t) < 0, \quad (5.2.4)$$

then the process $(X(t))_{t\geq 0}$ generated by the SDE (5.1.1) has a regular second-order solution given by the Wiener-Itô representation (5.1.2). The evolutionary symmetrized transfer functions

$f_t(\lambda_{(r)}), (t,r) \in \mathbb{R}_+ \times \mathbb{N}$ *of this solution are given by the symmetrization of the following differential equations*

$$f_t^{(1)}(\lambda_{(r)}) = \begin{cases} \alpha(t) f_t(0) + \mu(t), & \text{if } r = 0 \\ \left(\alpha(t) - i\underline{\lambda}_{(r)}\right) f_t(\lambda_{(r)}) + r\left(\gamma(t) f_t(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta(t)\right), & \text{if } r \geq 1 \end{cases} \tag{5.2.5}$$

where the superscript $^{(j)}$ *denotes* $j-$*fold differentiation with respect to* t.

Proof. See Bibi and Merahi [11]. □

Remark 5.2.5. *The existence and uniqueness of the evolutionary symmetrized transfer functions* $f_t(\lambda_{(r)}), (t,r) \in \mathbb{R} \times \mathbb{N}$ *given by* (5.2.5) *is ensured by general results on linear ordinary differential equations (see, e.g., [40], ch. 1), i.e.,*

$$f_t(\lambda_{(r)}) = \begin{cases} \varphi_t\,(0) \left(f_0(0) + \int\limits_0^t \varphi_s^{-1}\,(0)\,\mu(s) ds\right) & \text{if } r = 0 \\ \varphi_t\left(\underline{\lambda}_{(r)}\right) \left(f_0(\lambda_{(r)}) + r\int\limits_0^t \varphi_s^{-1}\left(\underline{\lambda}_{(r)}\right)\left(\gamma(s) f_s(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta(s)\right) ds\right) & \text{if } r \geq 1 \end{cases} \tag{5.2.6}$$

where $\varphi_t\left(\underline{\lambda}_{(r)}\right) = \exp\left\{\int\limits_0^t \left(\alpha(s) - i\underline{\lambda}_{(r)}\right) ds\right\}$.

Corollary 5.2.1. *Assume that* $\alpha(t), \mu(t), \beta(t)$ *and* $\gamma(t)$ *are constants, then the transfer functions* $f(\lambda_{(r)})$ *for all* $r \in \mathbb{N}$ *reduces to* $f(\lambda_{(r)}) = -\frac{\mu}{\alpha}\delta_{\{r=0\}} + r\left(i\underline{\lambda}_{(r)} - \alpha\right)^{-1}\left(\gamma f(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta\right)$ *for any* $r \geq 0$, *or also* $f(\lambda_{(r)}) = \gamma^{r-1} r! \left(\beta - \frac{\mu}{\alpha}\gamma\right) \prod\limits_{j=1}^{r} \left(i\underline{\lambda}_{(j)} - \alpha\right)^{-1}$, *and hence the symmetrized version can be rewritten as*

$$Sym\left\{f(\lambda_{(r)})\right\} = (\mu\gamma - \alpha\beta)\,\gamma^{r-1} \int_0^{+\infty} \exp\{\alpha\lambda\} \prod\nolimits_{j=1}^{r} \frac{1 - \exp\{-i\lambda\lambda_j\}}{i\lambda_j} d\lambda.$$

Proof. Straightforward and hence omitted. □

In theorem 5.2.4and remark 5.2.5 a recursive formula is derived for the evolutionary transfer functions of regular solution of $COBL(1,1)$. Condition (5.2.4) give sufficient condition for that these transfer functions determine a solution process given by the Wiener-Itô representation (5.1.2) for equation (5.1.1). In this section we examine the second-order properties of such the solution process.

Theorem 5.2.6. *Under the condition* (5.2.4), *the mean, variance and covariance functions for* $COBL\,(1,1)$ *are given respectively by the expressions*

$$m_X(t) = \varphi_t(0)\left(f_0(0) + \int_0^t \varphi_s^{-1}(0)\mu(s) ds\right). \tag{5.2.7}$$

$$K_X(t) = \phi_t\,(0)\left(K_X(0) + \int_0^t \phi_s^{-1}\,(0)\,(\gamma(s) f_s(0) + \beta(s))^2\, ds\right). \tag{5.2.8}$$

$$K_X(t,s) = \varphi_t(0)\varphi_s^{-1}(0) K_X(s), t \geq s \geq 0. \tag{5.2.9}$$

Before proving the above theorem, we have to prove the following lemma

Lemma 5.2.7. *Consider the process $COBL(1,1)$ having a Wiener-Itô representation (5.1.2), then $dCov\,(Y(t),Y(t)) = Cov\,(dY(t),dY(t))$ where*

$$Y(t) = \sum_{r\geq 1}\frac{1}{r!}\int\limits_{\mathbb{R}^r}\left(f_0(\lambda_{(r)}) + r\int_0^t \varphi_s^{-1}\left(\underline{\lambda}_{(r)}\right)\left(\gamma(s)f_s(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta(s)\right)ds\right)dZ(\lambda_{(r)}). \tag{5.2.10}$$

Proof. By equation (5.2.10), we have

$$dY(t) = \sum_{r\geq 1}\frac{1}{r!}\int\limits_{\mathbb{R}^r} r\varphi_t^{-1}\left(\underline{\lambda}_{(r)}\right)\left(\gamma(t)f_t(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta(t)\right)dtdZ(\lambda_{(r)})$$

, so

$$\begin{aligned}
&Cov\,(dY(t),dY(t))\\
&= \int_{\mathbb{R}}|\varphi_t^{-1}\left(\lambda_1\right)|^2(\gamma(t)f_t(0)+\beta(t))^2dF(\lambda_1)dt + \sum\nolimits_{r\geq 2}\frac{1}{(r!)^2}\int_{\mathbb{R}^r} r^2|\varphi_s^{-1}\left(\underline{\lambda}_{(r)}\right)|^2|\gamma(t)f_t(\lambda_{(r-1)})|^2dF(\lambda_{(r)})dt\\
&= \varphi_t^{-2}(0)(\gamma(t)f_t(0)+\beta(t))^2dt + \varphi_t^{-2}(0)\gamma^2(t)\sum\nolimits_{r\geq 2}\frac{r^2}{(r!)^2}\int_{\mathbb{R}^r}|f_t(\lambda_{(r-1)})|^2dF(\lambda_{(r)})dt\\
&= \varphi_t^{-2}(0)(\gamma(t)f_t(0)+\beta(t))^2dt + \varphi_t^{-2}(0)\gamma^2(t)\sum\nolimits_{r\geq 2}\frac{r^2}{(r!)^2}\int_{\mathbb{R}^r}|f_t(\lambda_{(r-1)})|^2dF(\lambda_{(r)})dt\\
&= \varphi_t^{-2}(0)(\gamma(t)f_t(0)+\beta(t))^2dt + \varphi_t^{-2}(0)\gamma^2(t)\sum\nolimits_{r\geq 2}\frac{r^2}{(r!)^2}\int_{\mathbb{R}^r}|f_t(\lambda_{(r-1)})|^2dF(\lambda_{(r)})dt\\
&= \varphi_t^{-2}(0)(\gamma(t)f_t(0)+\beta(t))^2dt + \gamma^2(t)\varphi_t^{-2}(0)\sum\nolimits_{r\geq 2}\frac{1}{((r-1)!)^2}\int_{\mathbb{R}^{r-1}}|f_t(\lambda_{(r-1)})|^2dF(\lambda_{(r-1)})dt\int_{\mathbb{R}}dF(\lambda_r)\\
&= \varphi_t^{-2}(0)(\gamma(t)f_t(0)+\beta(t))^2dt + \gamma^2(t)\varphi_t^{-2}(0)\sum\nolimits_{r\geq 1}\frac{1}{(r!)^2}\int_{\mathbb{R}^r}|f_t(\lambda_{(r)})|^2dF(\lambda_{(r)})dt\\
&= \varphi_t^{-2}(0)(\gamma(t)f_t(0)+\beta(t))^2dt + \gamma^2(t)\varphi_t^{-2}(0)Cov(X(t),X(t))\\
&= \varphi_t^{-2}(0)(\gamma(t)f_t(0)+\beta(t))^2dt + \gamma^2(t)Cov(Y(t),Y(t))dt
\end{aligned}$$

The expression of $dCov\,(Y(t),Y(t))$ may be achieved upon observation that $Y(t)$ can be rewritten as follow :

$$\begin{aligned}
&Y(t)\\
&= \int_{\mathbb{R}}\left\{f_0(\lambda) + \int_0^t \varphi_s^{-1}\left(\lambda\right)\left(\gamma(s)f_s(0)+\beta(s)\right)ds\right\}dZ(\lambda)\\
&+ \sum\nolimits_{r\geq 2}\frac{1}{r!}\int\limits_{\mathbb{R}^r}\left(f_0(\lambda_{(r)}) + r\int_0^t \varphi_s^{-1}\left(\underline{\lambda}_{(r)}\right)\gamma(s)f_s(\lambda_{(r-1)})ds\right)dZ(\lambda_{(r)})\\
&= \int_0^t\left\{\int_{\mathbb{R}}\varphi_s^{-1}\left(\lambda\right)dZ(\lambda)\right\}\left(\gamma(s)f_s(0)+\beta(s)\right)ds + \sum\nolimits_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r}f_0(\lambda_{(r)})dZ(\lambda_{(r)})\\
&+ \int_0^t\left\{\sum\nolimits_{r\geq 2}\frac{1}{(r-1)!}\int_{\mathbb{R}^r}\varphi_s^{-1}\left(\underline{\lambda}_{(r)}\right)f_s(\lambda_{(r-1)})dZ(\lambda_{(r)})\right\}\gamma(s)ds.
\end{aligned}$$

Then we have

$Cov\,(Y(t),Y(t))$

$$= \sum_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r}|f_0(\lambda_{(r)})|^2 dZ(\lambda_{(r)}) + \int_0^t \varphi_s^{-2}\,(0)\,(\gamma(s)f_s(0)+\beta(s))^2\,ds$$
$$+\int_0^t \gamma^2(s)\varphi_s^{-2}\,(0)\left\{\sum_{r\geq 1}\frac{1}{(r!)^2}\int_{\mathbb{R}^r}|f_s(\lambda_{(r)})|^2 dF(\lambda_{(r)})\right\}ds$$
$$= \sum_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r}|f_0(\lambda_{(r)})|^2 dZ(\lambda_{(r)}) + \int_0^t \varphi_s^{-2}\,(0)\,(\gamma(s)f_s(0)+\beta(s))^2\,ds + \int_0^t \gamma^2(s)\varphi_s^{-2}\,(0)\,Cov\,(X(s),X(s))\,ds.$$

And hence

$$dCov\,(Y(t),Y(t)) = \varphi_t^{-2}\,(0)\,(\gamma(t)f_t(0)+\beta(t))^2\,dt + +\gamma^2(t)\varphi_t^{-2}\,(0)\,Cov\,(X(t),X(t))\,dt$$
$$= \varphi_t^{-2}\,(0)\,(\gamma(t)f_t(0)+\beta(t))^2\,dt + \gamma^2(t)Cov\,(Y(t),Y(t))\,dt.$$

□

Proof of Theorem 5.2.6. Since $E\{X(t)\} = f_t(0)$, then the expression of $m_X(t)$ in (5.2.7) follows immediately from (5.2.6). To show (5.2.8) and (5.2.9), let $X(t) - f_t(0) = \varphi_t\,(0)\,Y(t)$ where the process $(Y(t))_{t\geq 0}$ is given by (5.2.10). Then $K_X(t) = \varphi_t^2(0)Cov\,(Y(t),Y(t))$ and

$$dK_X(t) = 2\alpha(t)\varphi_t^2(0)Cov\,(Y(t),Y(t))\,dt + \varphi_t^2(0)dCov\,(Y(t),Y(t))$$

Moreover, by lemma 5.2.7 we obtain

$$\varphi_t^2(0)Cov\,(dY(t),dY(t)) = |\beta(t)+\gamma(t)f_t(0)|^2\,dt + \gamma^2(t)K_X(t)dt$$

and thus

$$dK_X(t) = \left[2\alpha(t)+\gamma^2(t)\right]K_X(t)dt + [\gamma(t)f_t(0)+\beta(t)]^2\,dt,$$

its solution is given by

$$K_X(t) = \phi_t\,(0)\left(K_X(0) + \int_0^t \phi_s^{-1}\,(0)\,[\gamma(s)f_s(0)+\beta(s)]^2\,ds\right),$$

Since $dK_X(t,s) = \alpha(t)\varphi_t(0)\varphi_s(0)E\{Y(t)Y(s)\}\,dt$, $t\geq s$ and $K_X(t,s) = \varphi_t(0)\varphi_s(0)E\{Y(t)Y(s)\}$ then (5.2.8) follows □

Corollary 5.2.2. *In time-invariant case and under the condition (5.2.4), we have the following results* $m_X(0) = -\dfrac{\mu}{\alpha}$, $K_X(0) = \dfrac{(\alpha\beta-\mu\gamma)^2}{\alpha^2\,|2\alpha+\gamma^2|}$, $K_X(h) = K_X(0)e^{\alpha|h|}$, $h\in\mathbb{R}$ *and the corresponding spectral density is*

$$f\,(\lambda) = \frac{-\alpha K_X(0)}{\pi\,(\alpha^2+\lambda^2)} \tag{5.2.11}$$

.

Remark 5.2.8. *Le Breton and Musiela [45] have showed that there exists a wide-sense Brownian motion process* $(w^*(t))_{t\geq 0}$ *uncorrelated with* $X(0)$ *such that* $(X(t))_{t\geq 0}$ *admits the linear representation*

$$dX(t) = (\alpha(t)X(t)+\mu(t))\,dt + \xi\,(t)\,dw^*(t) \tag{5.2.12}$$

where $\xi^2\,(t) = \gamma^2\,(t)\,K_X(t) + (\gamma\,(t)\,m_X(t)+\beta\,(t))^2$.

5.3 Whittle estimator

5.3.1 An overview

In this section we consider the time-invariant version which depends on four parameters μ, α, β and γ gathered in vector $\underline{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)' = (\mu, \alpha, \beta, \gamma)'$, its true value denoted by $\underline{\theta}_0$ is unknown and therefore it must be estimated. For the purpose estimation, we assume observed the time-invariant $COBL(1,1)$ process at discrete time $\underline{X}_N = \{X(t_1), ..., X(t_N)\}$ with $t_1 < t_2 < ... < t_N$. Because the theory of statistical inference for an alternative class of continuous time processes is now well documented and comprehensively developed by several authors (see for instance Kutoyants [43] and the reference therein). The traditional approaches of estimating the parameter $\underline{\theta}_0$ based on $(\underline{X}_N)_{N\geq 1}$ are first, a time-domain approach (including among others, GMM, contrast function, least squares, $QML, ...$) and second, frequency domain approach. A time-domain approach is to use the Kalman-filter to calculate the conditional log-likelihood $l_{\underline{\theta}}(X(t_i)|X(t_{i-1}))$ and solve $\max_{\theta} \sum_{i=1}^{N} l_{\underline{\theta}}(X(t_i)|X(t_{i-1}))$. For a frequency domain approach, a sample estimate $\widehat{f}_{\underline{N}}(\lambda)$ of spectral density is needed, and then by minimizing some objective function, e.g., $\min_{\underline{\theta}} \int_{-\infty}^{+\infty} \left\{ \log f_{\underline{\theta}}(\lambda) + \frac{\widehat{f}_N(\lambda)}{f_{\underline{\theta}}(\lambda)} \right\} d\lambda$ an estimator is obtained. Solving this optimization problem yields the so-called Whittle estimator. However, for the case of equally spaced observations, i.e., $t_i = i\Delta$, $i \in \mathbb{N}$ and for some sampling interval $\Delta = \Delta(N) > 0$ the FFT (or periodogram) is an efficient tool of getting an asymptotically unbiased estimate of the spectral density function $f_{\underline{\theta}}(\lambda)$ of a strictly stationary process whenever $\Delta \to 0$ (see Florens [27] for further discussion), contrary to irregularly spaced observations the problem is not so simple and/or $\Delta \nrightarrow 0$ leads to inconsistency. Therefore, in the sequel, we shall assume that the time-invariant $COBL(1,1)$ process is centred and observed at equally spaced-time and we associate with the unique solution process of SDE (5.1.1) the process $(X(t\Delta), t \in \mathbb{N})$ with Δ is small enough which inherits the Markovian structure of the former, as well as stationarity and ergodicity. Hence, the periodogram of $\underline{X}_N = (X(t\Delta), 1 \leq t \leq N)'$ is defined as $\widehat{f}_N^{\Delta}(\lambda) = \widehat{f}_N(\lambda) = |I_N(\lambda)|^2$, where $I_N(\lambda) = I_N^{\Delta}(\lambda) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} X(t\Delta) e^{i\lambda t}$, and for computational purpose, the periodogram is evaluated at some Fourier frequencies $\lambda_j = \frac{2\pi j}{N}, j \in \left\{ -\frac{[N-1]}{2}, ..., \frac{[N]}{2} \right\}$. To obtain an approximate quasi-likelihood function, we consider that the discrete observation is obtained from a discretization of the time-invariant version of (5.1.1), as for example the Euler schema is given by.

$$X((t+1)\Delta) = X(t\Delta) + \mu(X(\Delta t))\Delta + \sigma(X(t\Delta))(w((t+1)\Delta) - w(t\Delta))$$

while the exact discretization of (5.1.1) is given by

$$X((t+1)\Delta) = X(t\Delta) + (1 - \alpha((t+1)\Delta)) X(t\Delta) + \exp\{-\xi((t+1)\Delta)\} e((t+1)\Delta) \quad (5.3.1)$$

where (see expression (5.2.3)) $\alpha((t+1)\Delta) = e^{-(\xi((n+1)\Delta) - \xi(n\Delta))}$ and $e((t+1)\Delta) = \int_{t\Delta}^{(t+1)\Delta} e^{-\xi(u)} d\eta(u)$. It is worth noting that when $\gamma = 0$ (OU process) the diffusion function however is independent of $X(t)$ and $e(t)$ becomes a Gaussian process with zero mean and variance equal to

$\frac{1}{2|\alpha|}\left(1-e^{-2|\alpha|\Delta}\right)$ and independent of $X(t)$ and therefore the likelihood obtained from the above discretization schemes are the same up to a reparametrization. The quasi-likelihood function of the parameter vector $\underline{\theta}_0$ based on the observation vector $\underline{X}_N$ is given by

$$L_N\left(\underline{\theta}\right)=\left(2\pi\right)^{-N/2}\left(\det\left(R_{\Delta}\left(\underline{\theta}\right)\right)\right)^{-1/2}\exp\left\{-\frac{1}{2}\underline{X}'_N R_{\Delta}^{-1}\left(\underline{\theta}\right)\underline{X}_N\right\}, \tag{5.3.2}$$

where $R_{\Delta}\left(\underline{\theta}\right)$ is an $N\times N$ Toplitz matrix with $(j,k)-th$ entries is $R_{\Delta}(j,k)=R_{\Delta}\left(|k-j|\right)=Cov\left\{X\left(j\Delta\right),X\left(k\Delta\right)\right\},\ j,k=1,...,N$, its empirical estimate is $\widehat{R}_N\left(h\right)=\frac{1}{N}\sum\limits_{k=1}^{N-|h|}X\left(k\Delta\right)X\left(\left(k+h\right)\Delta\right)$. So by taking logarithm in (5.3.2) we obtain up to a constant, the log quasi-likelihood function

$$l_N\left(\underline{\theta}\right)=-\frac{1}{2}\log\det\left(R_{\Delta}\left(\underline{\theta}\right)\right)-\frac{1}{2}\underline{X}'_N R_{\Delta}^{-1}\left(\underline{\theta}\right)\underline{X}_N \tag{5.3.3}$$

Since the sequence $\left(I_N\left(\lambda_j\right)\right)_{1\leq j\leq N}$ is approximately non correlated with covariance matrix *diag* $\left(f_{\underline{\theta}}\left(\lambda_j\right),_{1\leq j\leq N}\right)$, thus if U denotes the unitary matrix operator that transforms $\left\{X\left(\Delta\right),...,X\left(N\Delta\right)\right\}$ to $\left\{I_N\left(\lambda_1\right),...,I_N\left(\lambda_N\right)\right\}$ then $UR_{\Delta}\left(\underline{\theta}\right)U^*$ is approximately a diagonal matrix with $f_{\underline{\theta}}\left(\lambda_j\right),j=1,...,N$ on the diagonal. Hence an approximation for $l_N\left(\underline{\theta}\right)$ is now $-\frac{1}{2}\sum\limits_{j=1}^{N}\left\{\log f_{\underline{\theta}}\left(\lambda_j\right)+f_{\underline{\theta}}^{-1}\left(\lambda_j\right)\widehat{f}_N\left(\lambda_j\right)\right\}$ for N large enough. Therefore, Whittle estimator of $\underline{\theta}_0$ is defined as $\widehat{\underline{\theta}}_N=Arg\min\limits_{\underline{\theta}\in\Theta}\widehat{Q}_N\left(\underline{\theta}\right)$ where $\widehat{Q}_N\left(\underline{\theta}\right)$ is the objective function defined by

$$\widehat{Q}_N\left(\underline{\theta}\right)=\int_{-\pi}^{\pi}\left(\log f_{\underline{\theta}}\left(\lambda\right)+f_{\underline{\theta}}^{-1}\left(\lambda\right)\widehat{f}_N\left(\lambda\right)\right)d\lambda, \tag{5.3.4}$$

which is similar to the Kullbach-Leibler criterion.

Remark 5.3.1. *An approximation of the objective function* $\widehat{Q}_N\left(\underline{\theta}\right)$ *may be given by replacing the integral* $\int\limits_{-\pi}^{\pi}$ *in* (5.3.4) *by a Riemann sum evaluated at the Fourier frequencies i.e.,*

$$\widetilde{Q}_N\left(\underline{\theta}\right)=\frac{1}{N}\sum_{j=1}^{N}\left(\log f_{\underline{\theta}}\left(\lambda_j\right)+f_{\underline{\theta}}^{-1}\left(\lambda_j\right)\widehat{f}_N\left(\lambda_j\right)\right).$$

The Whittle estimator is one of the standard estimator for $ARMA$ processes which is extremely flexible under various modifications of $ARMA$ models. For instance, the Whittle estimator also works with infinite variance of innovation process and hence may be extended to long memory models. The applicability of Whittle estimator criterion $\widehat{Q}_N\left(\underline{\theta}\right)$ to non linear processes was considered by several authors. Dzhaparidze and Yaglon [26] have suggested the following conditions for discrete, strictly stationary, non-Gaussian and mixing processes $\left(X\left(t\right)\right)_t$

(a) : $\left(X\left(t\right)\right)_t$ has finite absolute moments of all order, i.e., $E\left\{\left|X^k\left(t\right)\right|\right\}<+\infty,\ k=1,2,..$

(b) : $\sum\limits_{t_1\geq 0}...\sum\limits_{t_{k-1}\geq 0}\left|C\left(t_1,...,t_{k-1}\right)\right|<+\infty, k=2,3,..$

where $C\left(t_1,...,t_{k-1}\right)$ denotes the $kth-$order cumulant of the process. Brillinger [12], has replaced the condition (**b**) by

$(\mathbf{c}) : \sum_{t_1 \geq 0} ... \sum_{t_{k-1} \geq 0} |t_j C(t_1, ..., t_{k-1})| < +\infty,\ k = 2, 3, ..$

The condition $(\mathbf{c})$ is a mixing condition which measures the asymptotic independence between the values of the process, which is much easier to verify than others conditions that measures the mixing (for instance $\beta-$mixing coefficient). Note here, that under the condition (5.2.4) and using the representation (5.2.12) it can be shown that the SDE (5.1.1) is exponentially ergodic and holds the exponentially $\beta-$mixing properties. Now we are ready to formulate the main results on the asymptotic behavior of the Whittle estimator for $COBL(1,1)$. We start with the consistency.

5.3.2 Consistency

The weak consistency of the Whittle estimator is generally considered in the literature. For an in-depth detailed mathematical framework on the subject we refer the reader to Dzhaparidze and Yaglon [26] and the references therein. To study the strong consistency of $\widehat{\underline{\theta}}_N$, we consider the following regularities assumptions

A0. $\underline{\theta}_0 \in \Theta$ and Θ is a compact subset of $\mathbb{R}^4$.

A1. The function $f_{\underline{\vartheta}}(\lambda)$ is strictly positive

A2. $\inf_{\underline{\theta} \in \Theta} \int_{-\pi}^{+\pi} f_{\underline{\theta}_0}(\lambda) f_{\underline{\theta}}^{-1}(\lambda)\, d\lambda = 1$

The compactness of Θ in Assumption **A0.**, is often imposed in order that several results from real analysis can be used. A sufficient condition for that **A1**. holds is that the condition (5.2.4) hold true, (see Subba Rao and Terdik [62]), while **A2.** in imposed for identification purpose.

Theorem 5.3.2. *Under Assumptions* **A0.**, $-$**A2.**, *almost surely* $\widehat{\underline{\theta}}_N \to \underline{\theta}_0$ *as* $N \to \infty$.

To prove the Theorem 5.3.2, we introduce the criterion $Q(\underline{\theta}) = \int_{-\pi}^{\pi} \left(\log f_{\underline{\theta}}(\lambda) + f_{\underline{\theta}}^{-1}(\lambda) f_{\underline{\theta}_0}(\lambda) \right) d\lambda$ and the perturbed spectral density of $f_{\underline{\theta}}^{-1}(\lambda)$ defined by $\varphi_{\underline{\theta},\delta}(\lambda) = \left(f_{\underline{\theta}}(\lambda) + \delta \right)^{-1}$ for some $\delta > 0$. Also for any integrable function $g_{\underline{\theta}}(\lambda)$ on $[-\pi, \pi]$, we associated its Fourier coefficient $\widehat{g}_t(\underline{\theta}) = \int_{-\pi}^{\pi} g_{\underline{\theta}}(\lambda) e^{it\lambda} d\lambda$ and the corresponding trunked Cesaro sum $g_M(\underline{\theta}, \lambda) = \frac{1}{M} \sum_{u=-M}^{M} \left(1 - \frac{|u|}{M} \right) \widehat{g}_u(\underline{\theta}) e^{-iu\lambda}$. So, if $g_{\underline{\theta}}(\lambda) = g_{\underline{\theta}}(-\lambda)$, then $\lim_{M \to +\infty} \sup_{\lambda} \left| g_{\underline{\theta}}(\lambda) - g_M(\underline{\theta}, \lambda) \right| = 0$, moreover, if $g_{\underline{\theta}}(\lambda)$ is continuous uniformly in λ, then $\lim_{M \to +\infty} \sup_{\lambda, \underline{\theta}} \left| g_{\underline{\theta}}(\lambda) - g_M(\underline{\theta}, \lambda) \right| = 0$. First, we establish the following lemma

Lemma 5.3.3. *almost surely* $\lim_{N \to \infty} \sup_{\underline{\theta}_0 \in \Theta} \left| \widehat{Q}_N(\underline{\theta}) - Q(\underline{\theta}) \right| = 0$.

Proof. It is sufficient to prove that $\lim_{N \to \infty} \sup_{\underline{\theta}_0 \in \Theta} \int_{-\pi}^{\pi} f_{\underline{\theta}}^{-1}(\lambda) \widehat{f}_N(\lambda)\, d\lambda = \int_{-\pi}^{\pi} f_{\underline{\theta}}^{-1}(\lambda) f_{\underline{\theta}_0}(\lambda)\, d\lambda$. Indeed, let $f_M^{-1}(\underline{\theta}, \mu)$, $\mu \in [-\pi, \pi]$ be the Cesaro sum associated with the Fourier coefficients of $f_{\underline{\theta}}^{-1}(\lambda)$.

Then, it is not difficult to see that for N, M large enough, and by the ergodic theorem, there exist $\epsilon > 0$ such that, almost surely

$$\left| \int_{-\pi}^{\pi} f_{\underline{\theta}}^{-1}(\lambda) \widehat{f}_N(\lambda) \, d\lambda - \int_{-\pi}^{\pi} f_M^{-1}(\underline{\theta}, \lambda) \widehat{f}_N(\lambda) \, d\lambda \right| \leq \epsilon R_\Delta(0),$$

and as $N \to +\infty$

$$\int_{-\pi}^{\pi} f_M^{-1}(\underline{\theta}, \mu) \widehat{f}_N(\lambda) \, d\lambda = \int_{-\pi}^{\pi} \left\{ \sum_{u=-M}^{M} \left(1 - \frac{|u|}{M}\right) \widehat{f}_u^{-1}(\underline{\theta}) e^{-i\lambda u} \widehat{f}_N(\lambda) \right\} d\lambda \to \sum_{u=-M}^{M} \left(1 - \frac{|u|}{M}\right) \widehat{f}_u^{-1}(\underline{\theta}) R_\Delta(u)$$

$$= \sum_{u=-M}^{M} \left(1 - \frac{|u|}{M}\right) \widehat{f}_u^{-1}(\underline{\theta}) \int_{-\pi}^{\pi} f_{\underline{\theta}_0}(\lambda) e^{-i\lambda u} d\lambda = \int_{-\pi}^{\pi} f_M^{-1}(\underline{\theta}, \mu) f_{\underline{\theta}_0}(\lambda) \, d\lambda \to \int_{-\pi}^{\pi} f_{\underline{\theta}}^{-1}(\lambda) f_{\underline{\theta}_0}(\lambda) \, d\lambda, \text{ as } M \to +\infty$$

and the result follows as $\epsilon \to 0$. □ □

Proof of Theorem 5.4.4. Suppose $\widehat{\underline{\theta}}_N$ is not strongly consistent for $\underline{\theta}_0$, then by the compactness of Θ, there is a subsequence $\widehat{\underline{\theta}}_{N(n)}$ converging to some $\underline{\vartheta} \in \Theta$ and $\underline{\vartheta} \neq \underline{\theta}_0$. Thus

$$\begin{aligned} \lim \inf_{n\to\infty} \widehat{Q}_{N(n)}\left(\widehat{\underline{\theta}}_{N(n)}\right) &\geq \sup_{\delta>0} \left\{ \lim \inf_{n\to\infty} \left\{ \int_{-\pi}^{\pi} \left(\log f_{\widehat{\underline{\theta}}_{N(n)}}(\lambda) + \varphi_{N(n),\delta}(\lambda) \widehat{f}_N(\lambda) \right) d\lambda \right\} \right\}, \\ &= \int_{-\pi}^{\pi} \log f_{\underline{\vartheta}}(\lambda) \, d\lambda + \sup_{\delta>0} \left\{ \int_{-\pi}^{\pi} f_{\underline{\theta}_0}(\lambda) \varphi_{\vartheta,\delta}(\lambda) \, d\lambda \right\} \\ &\to \int_{-\pi}^{\pi} \log f_{\underline{\vartheta}}(\lambda) \, d\lambda + \int_{-\pi}^{\pi} f_{\underline{\theta}_0}(\lambda) f_{\underline{\vartheta}}^{-1}(\lambda) \, d\lambda \text{ } a.s. \text{ as } \delta \to 0 \end{aligned}$$

so by **A2.,** we have $\int_{-\pi}^{\pi} \log f_{\underline{\vartheta}}(\lambda) \, d\lambda + \int_{-\pi}^{\pi} f_{\underline{\theta}_0}(\lambda) f_{\underline{\vartheta}}^{-1}(\lambda) \, d\lambda > \int_{-\pi}^{\pi} \log f_{\underline{\vartheta}}(\lambda) \, d\lambda$ for $\underline{\vartheta} \neq \underline{\theta}_0$, thus $\lim \inf_{n\to\infty} \widehat{Q}_{N(n)}\left(\widehat{\underline{\theta}}_{N(n)}\right) > \int_{-\pi}^{\pi} \log f_{\underline{\vartheta}}(\lambda) \, d\lambda$. But $\widehat{Q}_{N(n)}\left(\widehat{\underline{\theta}}_{N(n)}\right) \leq \widehat{Q}_{N(n)}(\underline{\theta})$ for any $\underline{\theta} \in \Theta$ and therefore

$$\lim \sup_{n\to\infty} \widehat{Q}_{N(n)}\left(\widehat{\underline{\theta}}_{N(n)}\right) \leq \inf_{\underline{\theta}\in\Theta} \lim \sup_{n\to\infty} \widehat{Q}_{N(n)}\left(\widehat{\underline{\theta}}_{N(n)}\right) = \inf_{\underline{\theta}\in\Theta} \widehat{Q}(\underline{\theta}) = \int_{-\pi}^{\pi} \log f_{\underline{\vartheta}}(\lambda) \, d\lambda.$$

Hence, the contradiction $\lim \sup_{n\to\infty} \widehat{Q}_{N(n)}\left(\widehat{\underline{\theta}}_{N(n)}\right) \leq \int_{-\pi}^{\pi} \log f_{\underline{\theta}_0}(\lambda) \, d\lambda < \lim \inf_{n\to\infty} \widehat{Q}_{N(n)}\left(\widehat{\underline{\theta}}_{N(n)}\right)$, $a.s.$□

5.3.3 Asymptotic normality

To show the asymptotic normality of $\widehat{\underline{\theta}}_N$,we define the $k-$th order cumulant spectral density (when exists) associated with the discretized version $(X(\Delta t))_{t\in\mathbb{Z}}$, i.e.,

$$f_{\underline{\theta}}(\lambda_1,...,\lambda_{k-1}) = \frac{1}{(2\pi)^{k-1}} \sum_{s_1,...,s_{k-1}\in\mathbb{Z}} C_\Delta(s_1,....,s_{k-1}) \exp\left\{-i\sum_{j=1}^{k-1}\lambda_j s_j\right\},$$

where $\sum_{j=1}^{k}\lambda_j = 0\,[\mathrm{mod}\ 2\pi]$ and consider the following additional assumptions

A3. $\underline{\theta}_0 \in \mathring{\Theta}$, with $\mathring{\Theta}$ denotes the interior of Θ.

A4. $(X(t))_{t\geq 0}$ is strictly stationary with $E\left\{X^4(t)\right\} < +\infty$

A5. the functions $f_{\underline{\vartheta}}(\lambda)$ and of $f_{\underline{\vartheta}}^{-1}(\lambda)$ are squared integrable on $[-\pi,\pi]$, bounded and continuous of all $\left(\lambda,\underline{\theta}'\right) \in [-\pi,\pi]\times\Theta$.

A6. The functions $\underline{\theta} \to \int_{-\pi}^{\pi} \log f_{\underline{\theta}}(\lambda)\,d\lambda$ and $\underline{\theta} \to \int_{-\pi}^{\pi} f_{\underline{\theta}_0}(\lambda)\, f_{\underline{\theta}}^{-1}(\lambda)\,d\lambda$ are twice continuously differentiable under the integral sign with respect to $\underline{\theta}$.

The principal well known results that we use for establish the asymptotic normality are summarized in the following Lemma due to Hannan [35]

Lemma 5.3.4. *Under the assumption* **A5**, **A6** *we have*

1. $R_N(u) = \sqrt{N}\left(\widehat{R}_N(u) - R_\Delta(u)\right)$ *has an asymptotic Gaussian joint distribution with* 0 *mean and joint asymptotic covariance*

$$\lim_{N\to\infty} NCov\left(R_N(u), R_N(v)\right) = \int_{-\pi}^{\pi} f_{\underline{\theta}}^2(\lambda)\, e^{i\lambda(u-v)} d\lambda + \int_{-\pi}^{\pi} f_{\underline{\theta}}^2(\lambda)\, e^{i\lambda(u+v)} d\lambda$$
$$+ \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{\underline{\theta}}(\lambda,-w,w)\, e^{i(\lambda u + wv)} d\lambda dw.$$

2. *For any continuous function* ϕ *let* $H(\phi) = \int_{-\pi}^{\pi} \phi(\lambda) f_{\underline{\theta}}(\lambda)\,d\lambda$ *and* $\widehat{H}_N(\phi) = \int_{-\pi}^{\pi} \phi(\lambda)\widehat{f}_N(\lambda)\,d\lambda$

   Then $\widehat{H}_N(\phi) \to H(\phi)$, *a.s. as* $N\to\infty$, *moreover* $\lim_{N\to+\infty} NCov\left(\widehat{H}_N(\phi_1), \widehat{H}(\phi_2)\right) = \Sigma(\underline{\theta}_0)$ *where*

$$\Sigma(\underline{\theta}) = 2\pi\int_{-\pi}^{\pi} \phi_1(\lambda)\,\overline{\phi}_2(\lambda)\, f_{\underline{\theta}}^2(\lambda)\,d\lambda + 2\pi\int_{-\pi}^{\pi} \phi_1(\lambda)\,\overline{\phi}_2(-\lambda)\, f_{\underline{\theta}}^2(\lambda)\,d\lambda$$
$$+ 2\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi_1(\lambda)\,\overline{\phi}_2(-w)\, f_{\underline{\theta}}(\lambda,-w,w)\,d\lambda dw.$$

The second main result of the chapter is the following

Theorem 5.3.5. *Under the conditions* ***A0.–A6.***, $\sqrt{N}\left(\widehat{\underline{\theta}}_N - \underline{\theta}_0\right)$ *is asymptotically*

$$\mathcal{N}\left(\underline{O}, J^{-1}\left(\underline{\theta}_0\right) V(\underline{\theta}_0) J^{-1}\left(\underline{\theta}_0\right)\right)$$

distributed where $J\left(\underline{\theta}_0\right)$ *and* $V(\underline{\theta}_0)$ *are* $4 \times 4-$*matrices with* $(i,j)-$*entries are given by*

$$J^{(i,j)}\left(\underline{\theta}_0\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{\underline{\theta}_0}^{-2}(\lambda)\frac{\partial}{\partial\theta_i} f_{\underline{\theta}}(\lambda)\frac{\partial}{\partial\theta_j} f_{\underline{\theta}}(\lambda)\, d\lambda,$$

$$V^{(i,j)}(\underline{\theta}_0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{\underline{\theta}_0}^{2}(\lambda)\frac{\partial}{\partial\theta_j} f_{\underline{\theta}}^{-1}(\lambda)\frac{\partial}{\partial\theta_i} f_{\underline{\theta}}^{-1}(\lambda)\, d\lambda + \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial\theta_j} f_{\underline{\theta}}(\lambda_1)\frac{\partial}{\partial\theta_i} f_{\underline{\theta}}(\lambda_2)\, f_{\underline{\theta}_0}(-\lambda_1,\lambda_2,-\lambda_2)\, d\lambda_1 d\lambda_2.$$

The proof of theorem 5.3.5 rest classically on a Taylor series expansion of the score vector around $\underline{\theta}_0$, we have

$$\underline{O} = \sqrt{N}\frac{\partial\widehat{Q}_N\left(\widehat{\underline{\theta}}_N\right)}{\partial\underline{\theta}} = \sqrt{N}\frac{\partial\widehat{Q}_N\left(\underline{\theta}_0\right)}{\partial\underline{\theta}} + \frac{\partial^2\widehat{Q}_N\left(\widetilde{\underline{\theta}}\right)}{\partial\underline{\theta}\partial\underline{\theta}'}\sqrt{N}\left(\widehat{\underline{\theta}}_N - \underline{\theta}_0\right) + o(1),$$

where $\widetilde{\underline{\theta}}$ is on the line segment joining $\underline{\theta}_0$ and $\widehat{\underline{\theta}}_N$ and $o(1)$ represents a random variable which tends to zero almost surely and $N \to +\infty$. First, under the condition **A6.** the first and second partial derivative of the objective function (5.3.4) are

$$\frac{\partial\widehat{Q}_N\left(\underline{\theta}\right)}{\partial\theta_j} = \int_{-\pi}^{\pi} f_{\underline{\theta}}^{-1}(\lambda)\frac{\partial}{\partial\theta_j} f_{\underline{\theta}}(\lambda)\, d\lambda - \int_{-\pi}^{\pi} f_{\underline{\theta}}^{-2}(\lambda)\frac{\partial}{\partial\theta_j} f_{\underline{\theta}}(\lambda)\,\widehat{f}_N(\lambda)\, d\lambda \tag{5.3.5}$$

$$= \int_{-\pi}^{\pi} F_{\underline{\theta}}^{(j)}(\lambda)\left(\widehat{f}_N(\lambda) - f_{\underline{\theta}}(\lambda)\right) d\lambda, \text{ with } F_{\underline{\theta}}^{(j)}(\lambda) = f_{\underline{\theta}}^{-2}(\lambda)\frac{\partial}{\partial\theta_j} f_{\underline{\theta}}(\lambda).$$

$$\frac{\partial\widehat{Q}_N\left(\underline{\theta}\right)}{\partial\theta_i\partial\theta_j} = \int_{-\pi}^{\pi} f_{\underline{\theta}}^{-1}(\lambda)\frac{\partial^2}{\partial\theta_i\partial\theta_j} f_{\underline{\theta}}(\lambda)\, d\lambda - \int_{-\pi}^{\pi} f_{\underline{\theta}}^{-2}(\lambda)\frac{\partial}{\partial\theta_i} f_{\underline{\theta}}(\lambda)\frac{\partial}{\partial\theta_j} f_{\underline{\theta}}(\lambda)\, d\lambda + \int_{-\pi}^{\pi}\frac{\partial^2}{\partial\theta_i\partial\theta_j} f_{\underline{\theta}}^{-1}(\lambda)\,\widehat{f}_N(\lambda)\, d\lambda.$$

We will show some intermediate results summarized in the following lemma and hence theorem 5.3.5 will straightforwardly follow.

Lemma 5.3.6. *Under the conditions* ***A0.,-A6.*** *we have*

1. $\lim_{N\to\infty} Var\left\{\sqrt{N}\int_{-\pi}^{\pi}\left(F_{j,M}(\underline{\theta},\lambda)\left(\widehat{f}_N(\lambda) - E\left\{\widehat{f}_N(\lambda)\right\}\right)\right)d\lambda\right\} = 0$ *where* $F_{j,M}\left(\underline{\theta},\lambda\right) = F_M^{(j)}\left(\underline{\theta},\lambda\right)$ $-F_{\underline{\theta}}^{(j)}(\lambda)$ *with* $F_M^{(j)}\left(\underline{\theta},\lambda\right)$ *is the Cesaro's sum associated with* $F_{\underline{\theta}}^{(j)}(\lambda)$.

2. $T_N = \sqrt{N}\left(\frac{\partial\widehat{Q}_N\left(\underline{\theta}\right)}{\partial\underline{\theta}} - E\left\{\frac{\partial\widehat{Q}_N\left(\underline{\theta}\right)}{\partial\underline{\theta}}\right\}\right) \rightsquigarrow \mathcal{N}\left(\underline{O}, V(\underline{\theta}_0)\right)$ *with* $V(\underline{\theta}_0)$ *is positive definite matrix and*

$$\sqrt{N}E\left\{\frac{\partial\widehat{Q}_N\left(\underline{\theta}\right)}{\partial\underline{\theta}}\right\} \to \underline{O}$$

3. *almost surely* $\frac{\partial^2\widehat{Q}_N\left(\underline{\theta}\right)}{\partial\underline{\theta}\partial\underline{\theta}'} \to J\left(\underline{\theta}_0\right)$ *as* $N \to +\infty$ *and* $J\left(\underline{\theta}_0\right)$ *is positive definite matrix.*

Proof. To prove **1**, we first observe that since $F_{\underline{\theta}}^{(j)}(\lambda)$ is continuous for all λ, then we have for all $\varepsilon > 0$, there is an $M > 0$ such that $\max_{a,b} \sup_{\lambda} |F_{j,M}(\underline{\theta}_0, \lambda)| < \varepsilon$. Moreover, by assertion 2 in Lemma 5.3.4 we have as $N \to +\infty$

$$Var\left\{\sqrt{N}\int_{-\pi}^{\pi} F_{j,M}(\underline{\theta},\lambda)\left(\widehat{f}_N(\lambda) - E\left\{\widehat{f}_N(\lambda)\right\}\right)d\lambda\right\}$$
$$\leq \int_{-\pi}^{\pi} |F_{j,M}(\underline{\theta},\lambda)|^2 f^2(\lambda)\,d\lambda + \int_{-\pi}^{\pi} F_{j,M}(\underline{\theta},\lambda)\,\overline{F}_{j,M}(\underline{\theta},-\lambda)\,f^2(\lambda)\,d\lambda$$
$$+\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_{j,M}(\underline{\theta},\lambda)\,\overline{F}_{j,M}(\underline{\theta},-\mu)\,f(\lambda,\mu,-\mu)\,d\lambda d\mu$$

and the result follow since $F_{j,M}(\underline{\theta},\lambda) \to 0$ as $M \to +\infty$. The assertion **2.**, may be proved by expressing T_N as function of $R_N(.)$ (or some approximations). Indeed, from (5.3.5) and by **A5.**, we have

$$\sqrt{N}\frac{\partial \widehat{Q}_N(\underline{\theta})}{\partial \underline{\theta}} = \sqrt{N}\int_{-\pi}^{\pi}\left(F_{\underline{\theta}}^{(j)}(\lambda)\left(\widehat{f}_N(\lambda) - E\left\{\widehat{f}_N(\lambda)\right\}\right)\right)d\lambda + \sqrt{N}\int_{-\pi}^{\pi}\left(F_{\underline{\theta}}^{(j)}(\lambda)\left(E\left\{\widehat{f}_N(\lambda)\right\} - f_{\underline{\theta}}(\lambda)\right)\right)d\lambda$$
$$= \sqrt{N}\int_{-\pi}^{\pi}\left(F_{\underline{\theta}}^{(j)}(\lambda)\left(\widehat{f}_N(\lambda) - E\left\{\widehat{f}_N(\lambda)\right\}\right)\right)d\lambda + O(N^{1/2-\alpha})$$
$$= \sqrt{N}\int_{-\pi}^{\pi}\left(F_{M}^{(j)}(\underline{\theta},\lambda)\left(\widehat{f}_N(\lambda) - E\left\{\widehat{f}_N(\lambda)\right\}\right)\right)d\lambda + O(N^{1/2-\alpha})$$
$$+\sqrt{N}\int_{-\pi}^{\pi}\left(\left(F_{\underline{\theta}}^{(j)}(\lambda) - F_{M}^{(j)}(\underline{\theta},\lambda)\right)\left(\widehat{f}_N(\lambda) - E\left\{\widehat{f}_N(\lambda)\right\}\right)\right)d\lambda.$$

So the second point of this assertion follows. Since the last term of the above equality tends to 0 as $N \to +\infty$ by assertion **1**. By Bernstein's lemma the asymptotic distribution of $\sqrt{N}\frac{\partial \widehat{Q}_N(\underline{\theta})}{\partial \underline{\theta}}$ is the same as of $\sqrt{N}\int_{-\pi}^{\pi} F_{M}^{(j)}(\underline{\theta},\lambda)\left(\widehat{f}_N(\lambda) - E\left\{\widehat{f}_N(\lambda)\right\}\right)d\lambda$. On the other hand,

$$\sqrt{N}\int_{-\pi}^{\pi}\left(F_{M}^{(j)}(\underline{\theta},\lambda)\left(\widehat{f}_N(\lambda) - E\left\{\widehat{f}_N(\lambda)\right\}\right)\right)d\lambda$$
$$= \sum\nolimits_{|u|\leq M-1}\left(1 - \frac{|u|}{M}\right)\sqrt{N}\left\{\widehat{R}_N(-u) - \left(1 - \frac{|u|}{M}\right)R_\Delta(-u)\right\}\widehat{F}^{(j)}(u)$$
$$= \sum\nolimits_{|u|\leq M-1}\left(1 - \frac{|u|}{M}\right)\widehat{F}^{(j)}(u)\,R_N(u) + o(1)$$

The result follow by assertion 1 in Lemma 5.3.4. The last assertion follow from the convergence of $\int_{-\pi}^{\pi}\frac{\partial^2}{\partial\theta_i\partial\theta_j}f_{\underline{\theta}}^{-1}(\lambda)\,\widehat{f}_N(\lambda)\,d\lambda$ to $\int_{-\pi}^{\pi}\frac{\partial^2}{\partial\theta_i\partial\theta_j}f_{\underline{\theta}}^{-1}(\lambda)\,f_{\underline{\theta}}(\lambda)\,d\lambda$ (see the second assertion of lemma 5.3.4, with $h_{\underline{\theta}}(\lambda) = \frac{\partial^2}{\partial\theta_i\partial\theta_j}f_{\underline{\theta}}^{-1}(\lambda)$) □ □

5.4 Monte Carlo experiments

We provide in this section some simulations results for the Whittle estimator and their asymptotic behavior given in the above section for estimating the unknown vector $\underline{\theta} = (\alpha,\mu,\gamma,\beta)$ involved in the model. For this purpose, we simulated 500 independent trajectories from a second-order stationary series according to $COBL(1,1)$ of length $n \in \{1000, 1500, 2000\}$ with standard Brownian motion. The results of simulation experiments for estimating the vector $\underline{\theta}_0$ are reported in tables below in which we have indicated in the line "Mean of" correspond to the average of the parameters estimates over the 500 repetitions. In order to show the performance of such method, we have reported (results between bracket) the root-mean square errors ($RMSE$) of each estimates. Note that the choice of parameters values must be satisfied the condition (5.2.4) .

5.4.1 GOU

The first design of our experiment consists to estimate the parameter of the Gaussian Ornstein-Uhlenbeck (GOU) process, i.e.,

$$dX(t) = (\mu - \alpha X(t))\, dt + \beta dw(t) \tag{5.4.1}$$

in which $\alpha > 0$ and $\beta^2 > 0$. Moreover, from equation (5.3.1), we obtain

$$X((t+1)\Delta) - \frac{\mu}{\alpha} = e^{-\alpha\Delta}\left(X(t\Delta) - \frac{\mu}{\alpha}\right) + \beta\sqrt{\frac{1-e^{-2\alpha\Delta}}{2\alpha}}e((t+1)\Delta)$$

where e is a Gaussian white noise independent of X. We deduce that $E\{X\} = \frac{\mu}{\alpha}$, $Var\{X\} = \frac{\beta^2}{2\alpha}\left(1+e^{-\alpha\Delta}\right)$, $Cov(X((t+1)\Delta), X(t\Delta)) = e^{-\alpha\Delta}Var\{X\}$ and hence $f_{\underline{\theta}}(\lambda) = \frac{\beta^2\left(1-e^{-2\alpha\Delta}\right)}{2\alpha\left|1-e^{-i\lambda-\alpha\Delta}\right|^2}$ that is independent of μ and thus may be considered as a nuisance parameter. So the vector $\underline{\theta}$ that we must estimate by Whittle method is $\underline{\theta} = (\alpha, \beta)'$ The results of simulation of such model are reported in the following table

length	1000	1500	2000
mean of $\widehat{\underline{\theta}}_n = \begin{pmatrix}\widehat{\alpha}_n \\ \widehat{\beta}_n\end{pmatrix}$	$\begin{pmatrix}2.037\,(0.0370) \\ 1.494\,(0.0053)\end{pmatrix}$	$\begin{pmatrix}2.028\,(0.0204) \\ 1.494\,(0.0055)\end{pmatrix}$	$\begin{pmatrix}2.018\,(0.085) \\ 1.496\,(0.003)\end{pmatrix}$
design(1): $\alpha = 2$ and $\beta = 1.5$			
mean of $\widehat{\underline{\theta}}_n = \begin{pmatrix}\widehat{\alpha}_n \\ \widehat{\beta}_n\end{pmatrix}$	$\begin{pmatrix}1.516\,(0.0610) \\ 0.499\,(0.0010)\end{pmatrix}$	$\begin{pmatrix}1.512\,(0.0104) \\ 0.498\,(0.0012)\end{pmatrix}$	$\begin{pmatrix}1.508\,(0.0081) \\ 0.499\,(0.0003)\end{pmatrix}$
design(2): $\alpha = 1.5$ and $\beta = 0.5$			
mean of $\widehat{\underline{\theta}}_n = \begin{pmatrix}\widehat{\alpha}_n \\ \widehat{\beta}_n\end{pmatrix}$	$\begin{pmatrix}1.008\,(0.0070) \\ 0.998\,(0.0018)\end{pmatrix}$	$\begin{pmatrix}1.006\,(0.0061) \\ 0.998\,(0.0020)\end{pmatrix}$	$\begin{pmatrix}1.004\,(0.0030) \\ 0.998\,(0.0014)\end{pmatrix}$
design(3): $\alpha = 1.0$ and $\beta = 1.0$			
mean of $\widehat{\underline{\theta}}_n = \begin{pmatrix}\widehat{\alpha}_n \\ \widehat{\beta}_n\end{pmatrix}$	$\begin{pmatrix}0.756\,(0.0065) \\ 0.997\,(0.0021)\end{pmatrix}$	$\begin{pmatrix}0.754\,(0.0045) \\ 0.998\,(0.0020)\end{pmatrix}$	$\begin{pmatrix}0.753\,(0.0029) \\ 0.998\,(0.0015)\end{pmatrix}$
design(4): $\alpha = 0.75$ and $\beta = 1.0$			

Table(1): The results of simulation by the Whittle estimator of GOU(1)

On the other hand, the asymptotic distribution of estimated density are shown in Figure 1 followed by the box plot summary of the statistical properties of each estimates.

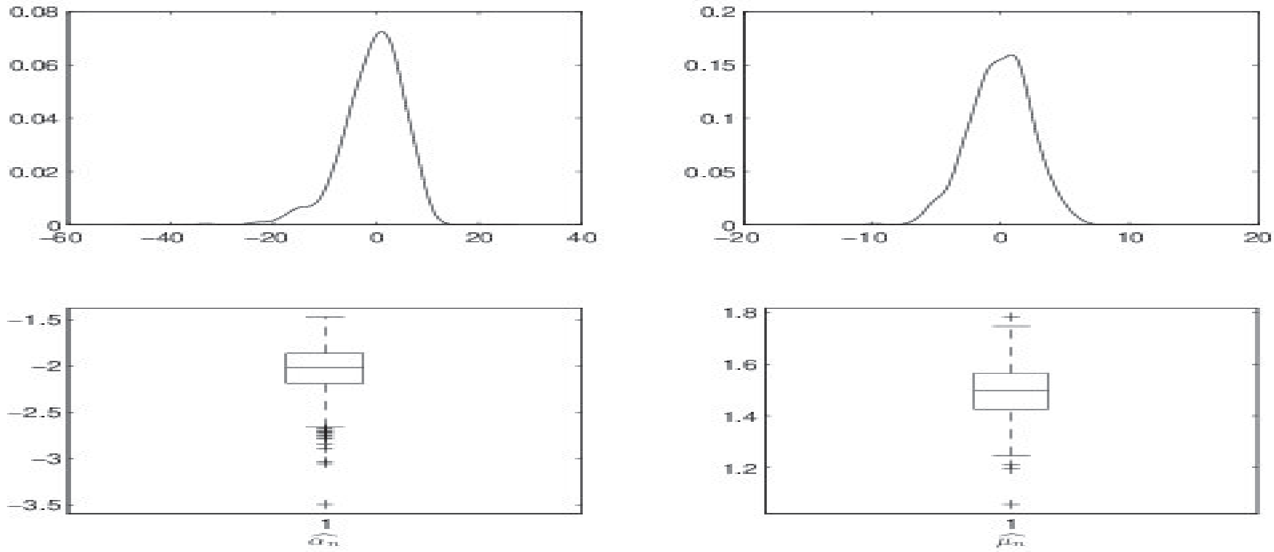


Fig1. Top panal: The asymptotic distribution of $\sqrt{n}\left(\underline{\widehat{\theta}}_n - \underline{\theta}_0\right)$. Bottom panal: box plot summary for GOU(1)

5.4.2 COBL(1,1)

In the second design, we consider the $COBL(1,1)$ generated by the following SDE

$$dX(t) = (\alpha X(t) + \mu)\, dt + (\gamma X(t) + \beta)\, dw(t), t \geq 0, X(0) = X_0 \tag{5.4.2}$$

in which, it can be assumed that $\beta = 0$, otherwise the transformation $Y(t) = \dfrac{\mu}{\gamma\mu - \alpha\beta}(\gamma X(t) + \beta)$ can be fulfilled in (5.4.2). So the vector $\underline{\theta}$ that we must estimate is thus $\underline{\theta} = (\alpha,\ \gamma, \mu)'$. The $CARMA$ representation (5.2.12) becomes $dX(t) = (\alpha X(t) + \mu)\, dt + \xi dw^{\star}(t),\ \xi^2 = \gamma^2\left(K_X(0) + m_X^2\right)$. The Euler discretization yields

$$X(t+h) - X(t) = (\alpha X(t) + \mu)\, h + \xi(w^{\star}(t+h) - w^{\star}(t))$$

while the exact discretization is given by

$$X(t+\Delta) - X(t) = -\frac{\mu}{\alpha}\left(1 - e^{\alpha\Delta}\right) - \left(1 - e^{\alpha\Delta}\right) X(t) + \zeta e^{\alpha t}\int_t^{t+\Delta} e^{-\alpha s} dw^{\star}(s)$$

so we obtain

$$X((t+1)\Delta) + \frac{\mu}{\alpha} = e^{\alpha\Delta}\left(X(t\Delta) + \frac{\mu}{\alpha}\right) + \zeta\sqrt{\frac{1 - e^{2\alpha\Delta}}{-2\alpha}} e((t+1)\Delta) \tag{5.4.3}$$

where e is a Gaussian white noise independent of X. Equation (5.4.3) means that the exact discretization of $COBL(1,1)$ is an $AR(1)$ model with coefficient $e^{\alpha\Delta} > 0$, so we have $E\{X\} = -\frac{\mu}{\alpha}$, $Var\{X\} = \frac{\zeta^2}{-2\alpha}$, $Cov(X((t+1)\Delta), X(t\Delta)) = e^{\alpha\Delta} Var\{X\}$ and hence $f_{\underline{\theta}}(\lambda) = \frac{\zeta^2\left(e^{2\alpha\Delta}-1\right)}{2\alpha\left|1-e^{-i\lambda+\alpha\Delta}\right|^2}$.
The results of simulation of $COBL(1,1)$ are reported in the following table

length		1000	1500	2000
mean of $\widehat{\underline{\theta}}_n =$	$\begin{pmatrix} \widehat{\alpha}_n \\ \widehat{\gamma}_n \\ \widehat{\mu}_n \end{pmatrix}$	$\begin{pmatrix} -2.601\,(0.111) \\ 1.561\,(0.301) \\ 1.081\,(0.008) \end{pmatrix}$	$\begin{pmatrix} -2.572\,(0.071) \\ 1.541\,(0.210) \\ 1.013\,(0.002) \end{pmatrix}$	$\begin{pmatrix} -2.543\,(0.049) \\ 1.506\,(0.281) \\ 1.011\,(0.001) \end{pmatrix}$
design(1): $\alpha = -2.5$, $\gamma = 1.5$ and $\mu = 1$				
mean of $\widehat{\underline{\theta}}_n =$	$\begin{pmatrix} \widehat{\alpha}_n \\ \widehat{\gamma}_n \\ \widehat{\mu}_n \end{pmatrix}$	$\begin{pmatrix} -0.143\,(0.125) \\ 0.623\,(0.127) \\ 0.475\,(0.011) \end{pmatrix}$	$\begin{pmatrix} -0.146\,(0.075) \\ 0.542\,(0.023) \\ 0.481\,(0.071) \end{pmatrix}$	$\begin{pmatrix} -0.152\,(0.055) \\ 0.502\,(0.012) \\ 0.048\,(0.011) \end{pmatrix}$
design(2): $\alpha = -1.5$, $\gamma = 0.5$ and $\mu = 0.5$				
mean of $\widehat{\underline{\theta}}_n =$	$\begin{pmatrix} \widehat{\alpha}_n \\ \widehat{\gamma}_n \\ \widehat{\mu}_n \end{pmatrix}$	$\begin{pmatrix} -0.985\,(0.143) \\ 1.015\,(0.073) \\ 0.915\,(0.212) \end{pmatrix}$	$\begin{pmatrix} -0.947\,(0.107) \\ 1.005\,(0.041) \\ 0.965\,(0.017) \end{pmatrix}$	$\begin{pmatrix} -0.978\,(0.021) \\ 0.975\,(0.012) \\ 0.981\,(0.051) \end{pmatrix}$
design(3): $\alpha = -1.0$, $\gamma = 1.0$ and $\mu = 1.0$				
mean of $\widehat{\underline{\theta}}_n =$	$\begin{pmatrix} \widehat{\alpha}_n \\ \widehat{\gamma}_n \\ \widehat{\mu}_n \end{pmatrix}$	$\begin{pmatrix} -1.352\,(0.151) \\ 0.952\,(0.021) \\ 0.743\,(0.024) \end{pmatrix}$	$\begin{pmatrix} -1.219\,(0.041) \\ 0.972\,(0.071) \\ 0.747\,(0.012) \end{pmatrix}$	$\begin{pmatrix} -1.230\,(0.060) \\ 0.975\,(0.017) \\ 0.746\,(0.025) \end{pmatrix}$
design(4): $\alpha = -1.25$, $\gamma = 1.0$ and $\mu = 0.75$				

Table(2): The results of simulation by the Whittle estimator of COBL(1,1)

The plots of asymptotic density and their box plots of each parameters in $\underline{\theta}$ according to first design are summarized in the following figure

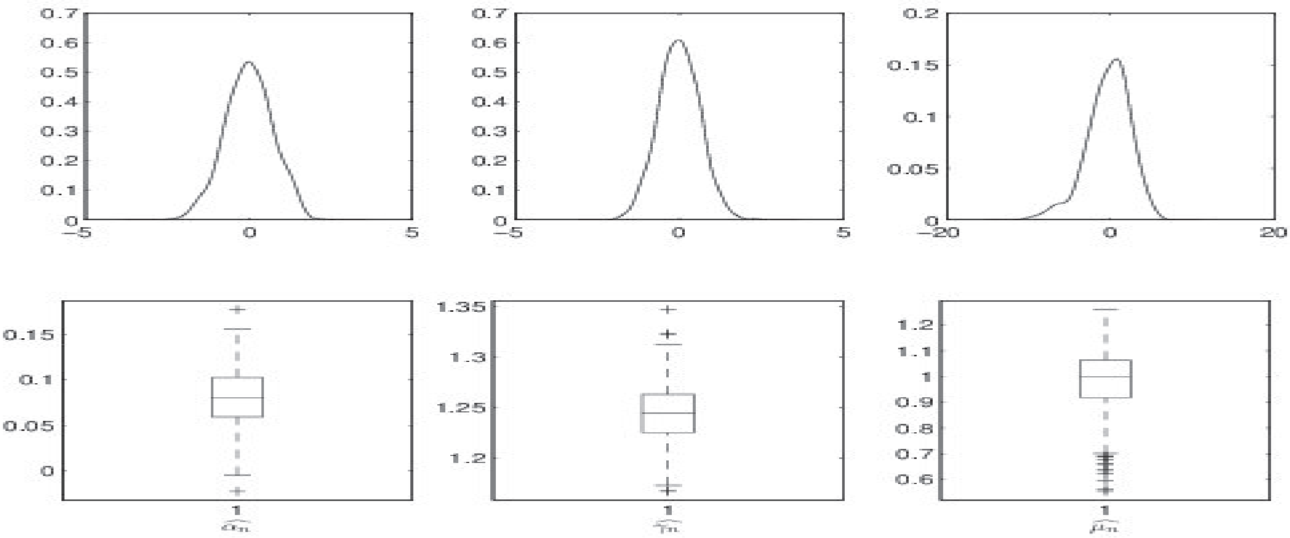


Fig2. Top panal: The asymptotic distribution of $\sqrt{n}\left(\underline{\widehat{\theta}}_n - \underline{\theta}_0\right)$. Bottom panal: box plot summary for GOU(1)

5.5 Concluding remarks

In this chapter, we have presented the Whittle's estimator for continuous-time bilinear ($COBL$) process generated by some diffusion equation. The method described in section 5.3 is based on the $CARMA$ representation of the $COBL$ process. So, we have analyzed the asymptotic properties of such method. We have showed that this method is consistency and asymptotically normal under some regularity condition. Finally, we investigated the empirical study of our estimators via monte Carlo simulation in order to highlight the theoretical results. It is however interesting to extend the method for general models, we leave this important issue for future researches.

Chapter 6

GMM estimation of continuous-time bilinear processes[6]

6. Ce chapitre est soumis dans le journal : Electronic Journal of Applied Statistical Analysis.

Abstract

This chapter examines the moments properties in frequency domain of the class of first order continuous-time bilinear processes ($COBL(1,1)$) with time-varying coefficients. So, we used the associated transfer functions to study the structure of second-order of the process and its powers. In time-invariant case, an expression of the moments of any order are showed and the continuous-time AR (CAR) representation of $COBL(1,1)$ is given as well as some moments properties of special cases. Based on these results we are able to examine the statistical properties such that we develop an estimation method of the process via the so-called generalized method of moments (GMM) illustrated by a Monte Carlo study and applied to modelling two foreign exchange rates of Algerian Dinar against $U.S.$-Dollar (USD/DZD) and against the single European currency Euro (EUR/DZD).

6.1 Introduction

One of the major difficulties that arises in the statistical analysis of linear and/or non linear stochastic differential equations (SDE) and thus poses a challenge to statisticians and econometricians for some time, is certainly their identifications. So, in financial application, estimation methods have usually carried by some discretization schema and hence various techniques are adapted. Indeed, Kallsen and Muhle-Karbe [39], and Haug et al. [32] have proposed an asymptotic inference of moments method (MM) for discretized continuous $GARCH$ process, Bibi and Merahi [10] have proposed a MM for estimating the parameters of continuous-time bilinear processes, Chan et al. [18] investigated an empirical comparison of generalized method of moment (GMM) of several discretized diffusions processes. Broze et al. [17] studies the effect of discretization schema on the consistency of the direct inference based on likelihood, Hyndman [34] and Guyon and Souchet [31] extended the so-called Yule-Walker estimator for a discretized version of an $CAR(p)$. The Levinson-Durbin-type algorithms for continuous-time autoregressive models was studied by Pham and Le Breton [56]. For in deep lecture we advised interested readers to see Kessler [41] end the reference therein and to monographs by Rao [58] and Kutoyants [43].

In this chapter we consider the class of continuous-time bilinear processes $(X(t))_{t\in\mathbb{R}_+}$ ($COBL$ for short) generated by the following time-varying stochastic differential equation (SDE)

$$\begin{aligned} dX(t) &= (\alpha(t)X(t)+\mu(t))\,dt + (\gamma(t)X(t)+\beta(t))\,dW(t),\ X(0)=X_0, \\ &= \mu_t\,(X(t))\,dt + \sigma_t\,(X(t))\,dW(t) \end{aligned} \tag{6.1.1}$$

where $\mu_t\,(x) = \alpha(t)x+\mu(t)$ and $\sigma_t\,(x) = \gamma(t)x+\beta(t)$ which represents the drift and the diffusion functions, $(W(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}$ defined on some basic filtered space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}\,, P)$ with spectral representation $W(t) = \int_{\mathbb{R}} \frac{e^{it\lambda}-1}{i\lambda} dZ(\lambda)$, where $Z(\lambda)$ is an orthogonal complex-valued stochastic measure on $\mathbb{R}$ with zero mean, $E\left\{|dZ(\lambda)|^2\right\} = dF(\lambda) = \frac{d\lambda}{2\pi}$ and uniquely determined by $Z([a,b[) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-i\lambda a}-e^{-i\lambda b}}{i\lambda} dW(\lambda)$, for all $-\infty < a < b < +\infty$, the initial state $X(0)$ is a random variable, defined on $(\Omega, \mathcal{A}, P)$, independent of W such that $E\{X(0)\} = m_1(0)$ and $Var\{X(0)\} = R_1(0)$. Special cases of this process are the Brownian motion with drift ($\alpha(t) = 0$ and $\gamma(t) = 0$), the Gaussian Ornstein-Uhlenbeck (GOU) process ($\gamma(t) = 0$) and the volatility of the $COGARCH(1,1)$ process defined by $dX(t) = \sigma\,(t)\,dW_1(t)$ where $d\sigma^2\,(t) = \left(\mu\,(t) - \alpha\,(t)\,\sigma^2\,(t)\right) dt + \gamma\,(t)\,\sigma^2\,(t)\,dW_2(t)$ in which $\mu\,(t) > 0$,$\alpha(t), \gamma\,(t) \geq 0$ for all $t \geq 0$ and $W_1(t)$ an $W_2(t)$ are independent Bm independent of $(X(0), \sigma\,(0))$. The SDE (6.1.1) is called time-invariant if there exists some constants α, μ, γ and β such that for all t, $\alpha(t) = \alpha$, $\mu(t) = \mu$, $\gamma(t) = \gamma$ and $\beta(t) = \beta$. The main aim here is focused firstly on the conditions ensuring the existence of the processes $(X(t))_{t\in\mathbb{R}_+}$ and its powers $(X^k(t))_{t\in\mathbb{R}_+}$, $k \geq 2$, using the transfer functions associated with the model. Secondly, we extend the generalized method of moments (GMM) for a discretized time-invariant version of SDE (6.1.1) and hence an estimates of the parameters involving in the model is ready for study their asymptotic properties. To ensure the existence and uniqueness of the solution process $(X(t))_{t\geq 0}$ of equation (6.1.1) we assume that the parameters $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are measurable deterministic functions and subject to the following assumption:

Assumption 3. *$\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are differentiable functions such that $\forall T > 0$,*

$$\int_0^T |\alpha(t)|\,dt < \infty,\ \int_0^T |\mu(t)|\,dt < \infty,\ \int_0^T |\gamma(t)|^2\,dt < \infty \ \ \textit{and} \ \int_0^T |\beta(t)|^2\ dt < \infty.$$

The remainder of this chapter is structured as follows. Section 2 outlines the Wienet-Itô spectral representation for general non linear SDE, in particular the recursive evolutionary transfer functions of SDE (6.1.1) are given and hence the associated spectral representation of $(X(t))_{t\geq 0}$ and its powers is showed. Section 3, investigated the moments properties of $(X(t))_{t\geq 0}$ and its powers and an explicit formula for time-invariant version are derived. Section 4, is dedicated for the estimate of time invariant SDE (6.1.1) via generalized moments method (GMM), so its consistency and asymptotic normality are studied. Numerical illustrations are given in Section 5

6.2 Wiener-Itô representation

The existence and uniqueness of the solution process of SDE (6.1.1) in time domain is ensured by the general results on SDE and under the Assumption 3. Moreover, since the drift and the diffusion are Lipschitz with linear growth, i.e., $|\mu_t(x) - \mu_t(y)| \le \sup_t |\alpha(t)| |x - y|$ and $|\sigma_t(x) - \sigma_t(y)| \le \sup_t |\gamma(t)| |x - y|$, then the Itô solution is given by (see Le Breton and Musiela [45] and Bibi and Merahi [11].)

$$X(t) = \Phi(t) \left\{ X(0) + \int_0^t \Phi^{-1}(s) (\mu(s) - \gamma(s) \beta(s)) \, ds + \int_0^t \Phi^{-1}(s) \beta(s) \, dW(s) \right\}, \ a.e., \quad (6.2.1)$$

where the process $(\Phi(t))_{t\ge0}$ is given by $\Phi(t) = \exp\left\{ \int_0^t \left(\alpha(s) - \frac{1}{2}\gamma^2(s)\right) ds + \int_0^t \gamma(s) \, dW(s) \right\}$ its mean function is $\Psi(t) = \exp\left\{ \int_0^t \alpha(s) ds \right\}$. In time-invariant case $(\Phi(t))_{t\ge0}$ reduces to $\Phi(t) = \exp\{-\xi(t)\}$ where $-\xi(t) = \left(\alpha - \frac{1}{2}\gamma^2\right) t + \gamma W(t)$ and then the solution process (6.2.1) reduces to

$$X(t) = e^{-\xi(t)} \left\{ X(0) + \int_0^t e^{\xi(s)} d\eta(s) \right\}, t \ge 0. \quad (6.2.2)$$

with $\eta(t) = (\mu - \gamma\beta) t + \beta W(t)$, that is the solution process of the celebrated generalized OU . So, by Itô formula, we obtain $dX(t) = -\xi(t) X(t) dt + d\eta(t), t \ge 0, X(0) = X_0$. Hence the above equation can be considered as a random coefficient time-continuous autoregressive ($CRCA$) representation of SDE (6.1.1). The autoregressive coefficient is obviously the random variable $-\xi(t)$.

In the first part of this chapter, we shall investigate in frequency domain, some probabilistic and statistical properties of second-order solution process of equation (6.1.1) which are also regular (or causal), i.e., $X(t)$ is $\sigma\{W(s), s \le t\}$ −measurable, such solution were given by Iglói and Terdik [38] for time-invariant version of SDE (6.1.1). For this purpose, let $\Im = \Im(W) := \sigma(W(t), t \ge 0)$ (resp. $\Im_t := \sigma(W(s), s \le t)$) be the σ-algebra generated by $(W(t))_{t\ge0}$ (resp. generated by $W(s)$ up to time t) and let $\mathbb{L}_2(\Im) = \mathbb{L}_2(\mathbb{C}, \Im, P)$ be the Hilbert space of nonlinear $\mathbb{L}_2$−functional of $(W(t))_{t\ge0}$. It is well known that any second-order regular process $(X(t))_{t\ge0}$ (i.e., $X(t)$ is $\Im_t$−measurable) admits the so-called Wiener-Itô orthogonal (or also chaotic) representation (see for instance Major [50]), i.e.,

$$X(t) = g_t(0) + \sum_{r\ge1} \frac{1}{r!} \int_{\mathbb{R}^r} g_t(\lambda_{(r)}) e^{it\underline{\lambda}_{(r)}} dZ(\lambda_{(r)}), \quad (6.2.3)$$

wherein $g_t(0) = E\{X(t)\}$, $\lambda_{(r)} = (\lambda_1, ..., \lambda_r) \in \mathbb{R}^r$, $\underline{\lambda}_{(r)} = \sum_{i=1}^r \lambda_i$ with $\lambda_{(0)} = \underline{\lambda}_{(0)} = 0$, and the integrals in (6.2.3) are the multiple Wiener-Itô stochastic integrals with respect to the stochastic measure $dZ\left(\lambda_{(r)}\right) = \prod_{i=1}^r dZ(\lambda_i)$ and $\left(g_t(\lambda_{(r)})\right)_{r\ge0}$ are referred as the $r - th$ evolutionary transfer functions uniquely determined up to symmetrization and $g_t(\lambda_{(r)}) \in \mathbb{L}_2(G) = \mathbb{L}_2(\mathbb{C}^n, B_{\mathbb{C}^n}, G)$ for all $t \ge 0$, i.e., $\sum_{r\ge0} \frac{1}{r!} \|g_t\|^2 < \infty$ for all t, where $\|g_t\|^2 = \int_{\mathbb{R}^r} \left|g_t(\lambda_{(r)})\right|^2 dG(\lambda_{(r)})$ with $dG(\lambda_{(r)}) =$

$\frac{1}{(2\pi)^r}d\lambda_{(r)}$ and $d\lambda_{(r)} = \prod_{i=1}^{r} d\lambda_i$. As a property of the representation (6.2.3) is that for any $f_t(\lambda_{(n)})$ and $f_s(\lambda_{(m)})$, we have

$$E\left\{\int_{\mathbb{R}^n} f_t(\lambda_{(n)})dZ(\lambda_{(n)})\overline{\int_{\mathbb{R}^m} f_s(\lambda_{(m)})dZ(\lambda_{(m)})}\right\} = \delta_n^m n! \int_{\mathbb{R}^n} Sym\left\{f_t(\lambda_{(n)})\right\}\overline{Sym\left\{f_s(\lambda_{(n)})\right\}}dF(\lambda_{(n)}) \tag{6.2.4}$$

where δ_n^m is the delta function and $Sym\left\{f_t(\lambda_{(n)})\right\} = \frac{1}{n!}\sum_{\pi\in\Pi(n)} f_t\left(\lambda_{\pi(n)}\right)$ where $\Pi(n)$ denotes the group of all permutation of the set $\{1,...,n\}$. Another property linked with (6.2.3) is the diagram formula which state that

$$\int_{\mathbb{R}} f_t(\lambda)dZ(\lambda)\int_{\mathbb{R}^n} g_s\left(\lambda_{(n)}\right)dZ(\lambda_{(n)}) \tag{6.2.5}$$
$$= \int_{\mathbb{R}^{n+1}} g_s\left(\lambda_{(n)}\right) f_t\left(\lambda_{n+1}\right) dZ(\lambda_{(n+1)}) + \sum_{k=1}^{n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} g_s\left(\lambda_{(n)}\right)\overline{f_t\left(\lambda_k\right)}dF\left(\lambda_k\right)dZ(\lambda_{(n\backslash k)})$$

where $dZ(\lambda_{(n\backslash k)}) = \prod_{i=1,i\neq k}^{n} dZ\left(\lambda_i\right)$. The following theorem due to Bibi and Merahi [11], give a recursive evolutionary transfer functions associated to the second-order regular solution of SDE (6.1.1).

Theorem 6.2.1. *Assume that everywhere*

$$2\alpha\left(t\right)+\gamma^2\left(t\right)<0, \tag{6.2.6}$$

then the process $(X(t))_{t\geq 0}$ generated by the SDE (6.1.1) has a regular second-order solution given by the series (6.2.3) where the evolutionary symmetrized transfer functions of this solution are given by the symmetrization of the solution of the following first-order differential equation

$$g_t^{[1](1)}(\lambda_{(r)}) = \begin{cases} \alpha(t)g_t^{[1]}(0)+\mu(t), & \text{if } r=0 \\ \left(\alpha(t)-i\underline{\lambda}_{(r)}\right)g_t^{[1]}(\lambda_{(r)})+r\left(\gamma(t)g_t^{[1]}(\lambda_{(r-1)})+\delta_{\{r=1\}}\beta(t)\right), & \text{if } r\geq 1 \end{cases} \tag{6.2.7}$$

where $g_t^{[1]}(0) = E\left\{X(t)\right\}$ and the superscript $^{(j)}$ denotes $j-$fold differentiation with respect to t.

Remark 6.2.2. *The existence and uniqueness of the evolutionary transfer functions $g_t^{[1]}(\lambda_{(r)}), (t,r) \in \mathbb{R}\times\mathbb{N}$ of (6.2.7) are ensured by general results on linear ordinary differential equations, so,*

$$g_t^{[1]}(\lambda_{(r)}) = \begin{cases} \varphi_1\left(t\right)\left(g_0^{[1]}(0)+\int_0^t\varphi_1^{-1}\left(s\right)\mu(s)ds\right) & \text{if } r=0 \\ \varphi_{1,t}\left(\underline{\lambda}_{(r)}\right)\left(g_0^{[1]}(\lambda_{(r)})+r\int_0^t\varphi_{1,s}^{-1}\left(\underline{\lambda}_{(r)}\right)\left(\gamma(s)g_s^{[1]}(\lambda_{(r-1)})+\delta_{\{r=1\}}\beta(s)\right)ds\right) & \text{if } r\geq 1 \end{cases} \tag{6.2.8}$$

where $\varphi_{1,t}\left(\underline{\lambda}_{(r)}\right) = \exp\left\{\int_0^t\left(\alpha(s)-i\underline{\lambda}_{(r)}\right)ds\right\}$ and $\varphi_1\left(t\right) = \varphi_{1,t}\left(0\right)$.

In time-invariant case we shall assume through the paper that

$$\alpha,\mu,\gamma,\beta\in\mathbb{R},\ \gamma\neq 0, \alpha\beta\neq\mu\gamma, 2\alpha+\gamma^2<0. \tag{6.2.9}$$

The condition $\alpha\beta\neq\mu\gamma$ is imposed otherwise the time-invariant version of (6.1.1) has only a degenerated solution given by $X(t) = -\frac{\beta}{\gamma} = -\frac{\mu}{\alpha}$.

Example 6.2.3. *In time-invariant version and under the condition 6.2.9, the transfer functions* $g^{[1]}(\lambda_{(r)})$ *for all* $r \in \mathbb{N}$ *are given by*

$$g^{[1]}(\lambda_{(r)})) = \begin{cases} -\frac{\mu}{\alpha}, r = 0 \\ \left(i\underline{\lambda}_{(r)} - \alpha\right)^{-1}\left(r\gamma g^{[1]}(\lambda_{(r-1)}) + \delta_{\{r=1\}}\beta\right), r \geq 1 \end{cases}$$

or equivalently $g^{[1]}(\lambda_{(r)}) = \gamma^{r-1} r! \left(\beta - \frac{\mu}{\alpha}\gamma\right) \prod_{j=1}^{r} \left(i\underline{\lambda}_{(j)} - \alpha\right)^{-1}$ *and the symmetrized version can be rewritten as*

$$Sym\left\{g^{[1]}(\lambda_{(r)})\right\} = (\mu\gamma - \alpha\beta)\,\gamma^{r-1} \int_0^{+\infty} \exp\{\alpha\lambda\} \prod_{j=1}^{r} \frac{1 - \exp\{-i\lambda\lambda_j\}}{i\lambda_j} d\lambda.$$

and hence

$$m_1 = -\frac{\mu}{\alpha},\ R_1(\tau) = Cov(X(t), X(t+\tau)) = R_1(0)e^{\alpha|\tau|},\ \textit{where } R_1(0) = \frac{|\alpha\beta - \mu\gamma|^2}{\alpha^2\,|2\alpha + \gamma^2|}. \tag{6.2.10}$$

Hence, the second-order properties for time-invariant versions of the nested models can be easily deduced.

6.2.1 Wiener-Itô representation for $\left(X^k(t)\right)_{t\geq 0}$

In this subsection, we examine the structure of the process $\left(X^k(t)\right)_{t\geq 0}$, $\forall k \geq 2$ in which the condition

$$2\alpha(t) + (2k-1)\gamma^2(t) < 0,\ a.e.,\ \text{for all } t \geq 0 \tag{6.2.11}$$

is imposed. The following lemma give Wiener-Itô representation of $\left(X^k(t)\right)_{t\geq 0}$.

Lemma 6.2.4. *Suppose that the solution process of SDE (6.1.1) is regular. Then under the condition (6.2.11), the process* $\left(X^k(t)\right)_{t\geq 0}$ *is regular and has a Wiener-Itô representation, i.e.,*

$$X^k(t) = g_t^{[k]}(0) + \sum_{r\geq 1} \frac{1}{r!} \int_{\mathbb{R}^r} e^{it\underline{\lambda}_{(r)}} g_t^{[k]}(\lambda_{(r)}) dZ(\lambda_{(r)}),$$

where the transfer functions $g_t^{[k]}(\lambda_{(r)})$, $r \geq 0$ *satisfying the following first-order differential equation*

$$\begin{aligned} &g_t^{[k](1)}(\lambda_{(r)}) \\ &= \begin{cases} k\left(\alpha(t) + \frac{1}{2}\gamma^2(t)(k-1)\right) g_t^{[k]}(0) + k\left(\gamma(t)\beta(t)(k-1) + \mu(t)\right) g_t^{[k-1]}(0) \\ +\frac{1}{2}\beta^2(t)k(k-1)g_t^{[k-2]}(0),\ \ \textit{if } r = 0 \\ \\ \left(k\left(\alpha(t) + \frac{1}{2}\gamma^2(t)(k-1)\right) - i\underline{\lambda}_{(r)}\right) g_t^{[k]}(\lambda_{(r)}) + k\left(\gamma(t)\beta(t)(k-1) + \mu(t)\right) g_t^{[k-1]}(\lambda_{(r)})) \\ +\frac{1}{2}\beta^2(t)k(k-1)g_t^{[k-2]}(\lambda_{(r)}) + kr\left(\gamma(t)g_t^{[k]}(\lambda_{(r-1)}) + \beta(t)g_t^{[k-1]}(\lambda_{(r-1)})\right), r \geq 1 \end{cases} \end{aligned} \tag{6.2.12}$$

Proof. The proof follows upon the observation that by applying the Itô's formulae for $f(x) = x^k$ for any integer $k \geq 2$, then the process $\left(X^k(t)\right)_{t\geq 0}$ satisfying the following stochastic differential equation

$$dX^k(t) = \left(k(\alpha(t) + \frac{1}{2}\gamma^2(t)(k-1))X^k(t) + k\left(\mu(t) + \gamma(t)\beta(t)(k-1)\right)X^{k-1}(t) + \frac{1}{2}\beta^2(t)k(k-1)X^{k-2}(t)\right)dt$$
$$+ k\left(\gamma(t)X^k(t) + \beta(t)X^{k-1}(t)\right)dW(t), \ a.e.,$$

So, using the diagram formula (6.2.5) the result follows. □

Remark 6.2.5. *The existence and uniqueness of the evolutionary symmetrized transfer functions $g_t^{[2]}(\lambda_{(r)}), (t,r) \in \mathbb{R}\times\mathbb{N}$ given by (6.2.12) is ensured by general results on linear ordinary differential equations (see, e.g., [40], chap. 1) so, the evolutionary transfer functions $g_t^{[k]}$ are given recursively by*

$$g_t^{[k]}(\lambda_{(r)}) = \begin{cases} \varphi_{k,t}(0)\left(g_0^{[k]}(0) + \int_0^t \varphi_{k,s}^{-1}(0)\,\mu_s^{[k]}(0)ds\right) & \text{if } r = 0 \\ \varphi_{k,t}\left(\underline{\lambda}_{(r)}\right)\left(g_0^{[k]}(\lambda_{(r)}) + \int_0^t \varphi_{k,s}^{-1}\left(\underline{\lambda}_{(r)}\right)\mu_s^{[k]}(\lambda_{(r)})ds\right) & \text{if } r \geq 1 \end{cases} \tag{6.2.13}$$

in which $\varphi_{k,t}\left(\underline{\lambda}_{(r)}\right) = \exp\left\{\int_0^t \left(k\left(\alpha(s) + \frac{1}{2}\gamma^2(s)(k-1)\right) - i\underline{\lambda}_{(r)}\right)ds\right\}$, $g_t^{[k]}(0) = m_k(t) = E\left\{X^k(t)\right\}, t \geq 0$, *and*

$$\mu_t^{[k]}(\lambda_{(r)}) = \begin{cases} 2\left(\gamma(t)\beta(t) + \mu(t)\right)g_t^{[1]}\left(\lambda_{(r)}\right) + \beta^2(t)\delta_{\{r=0\}} + 2r\left(\gamma(t)g_t^{[2]}(\lambda_{(r-1)}) + \beta(t)g_t^{[1]}(\lambda_{(r-1)})\right), k = 2 \\ k\left((k-1)\gamma(t)\beta(t) + \mu(t)\right)g_t^{[k-1]}\left(\lambda_{(r)}\right) + \frac{1}{2}k(k-1)\beta^2(t)g_t^{[k-2]}\left(\lambda_{(r)}\right) \\ +kr\left(\gamma(t)g_t^{[k]}(\lambda_{(r-1)}) + \beta(t)g_t^{[k-1]}(\lambda_{(r-1)})\right), k \geq 3 \end{cases}$$

6.3 Moments properties of $\left(X^k(t)\right)_{t\geq 0}$

Since (6.1.1) is non linear with deterministic coefficients, the solution process (6.2.3) is non Gaussian in general, its first and second moment however are insufficient and hence the resort to higher order moments for the identification purpose is therefore necessary. In this section, we examine the moments properties of the process $\left(X^k(t)\right)_{t\geq 0}, \forall k \geq 2$.

Theorem 6.3.1. *Let $(X(t))_{t\geq 0}$ be the solution process of SDE (6.1.1), then under the condition (6.2.11), the mean $m_k(t)$ variance $R_k(t)$ and covariance functions $R_k(t,s)$ of $\left(X^k(t)\right)_{t\geq 0}$, $k \geq 2$*

are given respectively for all $t \geq s \geq 0$ *by*

$$m_k(t) = \varphi_k(t)\varphi_k^{-1}(s)\{m_k(s) \tag{6.3.1}$$
$$+ k\int_s^t \varphi_k(s)\varphi_k^{-1}(u)\left((\mu(u) + (k-1)\gamma(u)\beta(u))\, m_{k-1}(u) + \frac{1}{2}\beta^2(u)(k-1)m_{k-2}(u)\right)du\}$$

$$R_k(t) = \phi_k(t)\phi_k^{-1}(s)R_k(s) + \int_s^t \phi_k(t)\phi_k^{-1}(u)[\gamma^2(u)m_k^2(u) + 2k\left(\mu(u) + (2k-1)\gamma(u)\beta(u)\right)m_{2k-1}(u) \tag{6.3.2}$$
$$- 2k\left(\mu(u) + (k-1)\gamma(u)\beta(u)\right)m_k(u)m_{k-1}(u) + k(2k-1)\beta^2(u)m_{2k-2}(u) - k(k-1)\beta^2(u)m_k(u)m_{k-2}(u)]du,$$

$$R_k(t,s) = \varphi_k(t)\varphi_k^{-1}(s)\{R_k(s) + k\int_s^t \varphi_k(s)\varphi_k^{-1}(u)((\mu(u) + \gamma(u)\beta(u)(k-1))\, Cov\left(X^{k-1}(u), X^k(s)\right) \tag{6.3.3}$$
$$+ \frac{1}{2}\beta^2(u)(k-1)Cov\left((X^{k-2}(u), X^k(s))\right)du\}$$

where $\varphi_k(t) = \exp\left\{k\int_0^t \left(\alpha(u) + \frac{1}{2}\gamma^2(u)(k-1)\right)du\right\}$ *and* $\phi_k(t) = \exp\left\{k\int_0^t \left(2\alpha(u) + (2k-1)\gamma^2(u)\right)du\right\}$.

Proof. The fact that $g_t^{[k]}(0) = m_k(t)$, then from (6.2.12) we can obtain the following ordinary differential equation

$$m_k^{(1)}(t) = k\left(\alpha\left(t\right) + \frac{1}{2}\gamma^2(t)(k-1)\right)m_k(t) + k\left(\gamma(t)\beta(t)(k-1) + \mu(t)\right)m_{k-1}(t) + \frac{1}{2}\beta^2(t)k(k-1)m_{k-2}(t), \tag{6.3.4}$$

and the expression (6.3.1) is obtained by solving the above differential equation. To prove (6.3.2) we have $Var\left\{X^k(t)\right\} = R_k(t) = E\left\{X^{2k}(t)\right\} - \left(E\left\{X^k(t)\right\}\right)^2$ with $E\left\{X^k(t)\right\} = m_k(t) = g_t^{[k]}(0)$ and $E\left\{X^{2k}(t)\right\} = m_{2k}(t) = g_t^{[2k]}(0), \forall t \geq 0$ which implies $R_k(t) = g_t^{[2k]}(0) - \left(g_t^{[k]}(0)\right)^2$, then by differentiating with respect to t and from the formula (6.2.12) in which we substitute respectively $\dfrac{dg_t^{[k]}(0)}{dt}$, $\dfrac{dg_t^{[2k]}(0)}{dt}$ we find

$$\begin{aligned}
\frac{dR_k(t)}{dt} &= \frac{dg_t^{[2k]}(0)}{dt} - 2g_t^{[k]}(0)\frac{dg_t^{[k]}(0)}{dt}\\
&= k\left(2\alpha(t) + \gamma^2(t)(2k-1)\right)\left(g_t^{[2k]}(0) - \left(g_t^{[k]}(0)\right)^2\right) + 2k\left(\gamma(t)\beta(t)(2k-1) + \mu(t)\right)g_t^{[2k-1]}(0)\\
&- 2k\left(\gamma(t)\beta(t)(k-1) + \mu(t)\right)g_t^{[k]}(0)g_t^{[k-1]}(0) + \beta^2(t)k(2k-1)g_t^{[2k-2]}(0) - \beta^2(t)k(k-1)g_t^{[k]}(0)g_t^{[k-2]}(0)\\
&= k\left(2\alpha(t) + \gamma^2(t)(2k-1)\right)R_k(t) + 2k\left(\gamma(t)\beta(t)(2k-1) + \mu(t)\right)m_{2k-1}(t)\\
&- 2k\left(\gamma(t)\beta(t)(k-1) + \mu(t)\right)m_k(t)m_{k-1}(t) + \beta^2(t)k(2k-1)m_{2k-2}(t) - \beta^2(t)k(k-1)m_k(t)m_{k-2}(t).
\end{aligned}$$

Therefore the expression (6.3.2) is ensured by applying the general results on linear ordinary differential equations. It remains to prove (6.3.3), then we have for all $t \geq s$

$$R_k(t,s) = Cov(X^k(t), X^k(s)) = \sum_{r\geq 1}\frac{1}{(r!)^2}E\left\{\int_{\mathbb{R}^r} g_t^{[k]}\left(\lambda_{(r)}\right)e^{it\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})\overline{\int_{\mathbb{R}^r} g_s^{[k]}\left(\lambda_{(r)}\right)e^{is\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})}\right\}.$$

By differentiating with respect to t and the use of the formula (6.2.12) we obtain

$$\frac{dR_k(t,s)}{dt}$$
$$= \sum_{r\geq 1}\frac{1}{(r!)^2}E\left\{\int_{\mathbb{R}^r}\frac{d\left(g_t^{[k]}\left(\lambda_{(r)}\right)e^{it\underline{\lambda}_{(r)}}\right)}{dt}dZ(\lambda_{(r)})\overline{\int_{\mathbb{R}^r}g_s^{[k]}\left(\lambda_{(r)}\right)e^{is\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})}\right\}$$
$$= \sum_{r\geq 1}\frac{1}{(r!)^2}E\left\{\int_{\mathbb{R}^r}\left(k\left(\alpha(t)+\frac{1}{2}(k-1)\gamma^2(t)\right)g_t^{[k]}\left(\lambda_{(r)}\right)+\mu_t^{[k]}\left(\lambda_{(r)}\right)\right)e^{it\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})\overline{\int_{\mathbb{R}^r}g_s^{[k]}\left(\lambda_{(r)}\right)e^{is\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})}\right\},$$

Now apply the property of orthogonality (6.2.4) to get

$$\frac{dR_k(t,s)}{dt} =$$
$$= k\left(\alpha(t)+\frac{1}{2}(k-1)\gamma^2(t)\right)\sum_{r\geq 1}\frac{1}{(r!)^2}E\left\{\int_{\mathbb{R}^r}\left(g_t^{[k]}\left(\lambda_{(r)}\right)\right)e^{it\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})\overline{\int_{\mathbb{R}^r}g_s^{[k]}\left(\lambda_{(r)}\right)e^{is\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})}\right\}$$
$$+ k\left((k-1)\gamma(t)\beta(t)+\mu(t)\right)\sum_{r\geq 1}\frac{1}{(r!)^2}E\left\{\int_{\mathbb{R}^r}\left(g_t^{[k-1]}\left(\lambda_{(r)}\right)\right)e^{it\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})\overline{\int_{\mathbb{R}^r}g_s^{[k]}\left(\lambda_{(r)}\right)e^{is\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})}\right\}$$
$$+ \frac{1}{2}k(k-1)\beta^2(t)\sum_{r\geq 1}\frac{1}{(r!)^2}E\left\{\int_{\mathbb{R}^r}\left(g_t^{[k-2]}\left(\lambda_{(r)}\right)\right)e^{it\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})\overline{\int_{\mathbb{R}^r}g_s^{[k]}\left(\lambda_{(r)}\right)e^{is\underline{\lambda}_{(r)}}dZ(\lambda_{(r)})}\right\},$$

which implies

$$\frac{dR_k(t,s)}{dt}$$
$$= k\left(\alpha(t)+\frac{1}{2}(k-1)\gamma^2(t)\right)R_k(t,s) + k\left((k-1)\gamma(t)\beta(t)+\mu(t)\right)Cov(X^{k-1}(t),X^k(s))$$
$$+ \frac{1}{2}k(k-1)\beta^2(t)Cov(X^{k-2}(t),X^k(s)), t\geq s,$$

and the expression (6.3.2) is now obtained by solving the above ordinary differential equation.
□ □

In time-invariant case, we have

Theorem 6.3.2. *Consider the time-invariant of SDE (6.1.1), then under the condition (6.2.11) the moments up to the order k of the process solution are given by the following expressions*

1. *If $\beta \neq 0$, we have*

$$m_1 = -\frac{\mu}{\alpha},$$
$$m_2 = \frac{2(\gamma\beta+\mu)\mu-\alpha\beta^2}{\alpha(2\alpha+\gamma^2)},$$
$$m_3 = \frac{(2\gamma\beta+\mu)\left(2(\gamma\beta+\mu)\mu-\alpha\beta^2\right)-\mu\beta^2(2\alpha+\gamma^2)}{\alpha(2\alpha+\gamma^2)(\alpha+\gamma^2)}$$
$$m_4 = -\frac{2(3\gamma\beta+\mu)}{(2\alpha+3\gamma^2)}m_3 - \frac{3\beta^2}{(2\alpha+3\gamma^2)}m_2$$

2. *If* $\beta = 0$*, then* $m_k = (-1)^k \prod_{j=1}^{k} \dfrac{\mu}{\alpha + \frac{1}{2}(j-1)\gamma^2}$ *for all* $k \geq 0$.

Proof. 1. If $\beta \neq 0$, then since $g_t^{[k]}(0) = m_k(t)$, for $k \geq 1$, thus from (6.2.7) we obtain $m_1^{(1)}(t) = \alpha(t)m_1(t) + \mu(t)$. In time-invariant case and under the condition (6.2.11), the process $X(t)$ is second order stationary, its moments are independent of t, so $m_1(t) = m_1$ which implies $\alpha m_1 + \mu = 0$ and $m_1 = -\dfrac{\mu}{\alpha}$. For the same raison and from the expression (6.2.12) we can obtain a difference equation for all $k \geq 2$ as follow

$$\left(\alpha + \frac{1}{2}(k-1)\gamma^2\right) m_k + \left((k-1)\gamma\beta + \mu\right) m_{k-1} + \frac{1}{2}(k-1)\beta^2 m_{k-2} = 0, \tag{6.3.5}$$

hence, $m_2 = -\dfrac{2(\mu + \gamma\beta)}{2\alpha + \gamma^2} m_1 - \dfrac{\beta^2}{2\alpha + \gamma^2}$. The expressions for m_3 and m_4 maybe obtained from (6.3.4).

2. If $\beta = 0$, then in time-invariant case we obtain the difference equation (6.3.5) becomes as $\left(\alpha + \frac{1}{2}(k-1)\gamma^2\right) m_k + \mu m_{k-1} = 0$ which implies

$$m_k = -\frac{\mu}{\left(\alpha + \frac{1}{2}(k-1)\gamma^2\right)} m_{k-1}, \forall k \geq 1$$

with $m_0 = 1$ and hence $m_k = (-1)^k \prod_{j=1}^{k} \dfrac{\mu}{\alpha + \frac{1}{2}(j-1)\gamma^2}, \forall k \geq 0$ and the proof of the theorem is complete.□

□

Example 6.3.3. *Table(1) illustrated some finite-order moment for the GOU process defined by* $dX(t) = (\mu - \alpha X(t))\,dt + \beta dW(t)$ *with* $\alpha > 0$ *and* $\beta \neq 0$

m_1	m_2	m_3	m_4	*Kurtosis*	*Skewness*
$\dfrac{\mu}{\alpha}$	$\dfrac{2\mu^2 + \alpha\beta^2}{2\alpha^2}$	$\dfrac{\mu\left(2\mu^2 + 3\alpha\beta^2\right)}{2\alpha^3}$	$\dfrac{4\mu^4 + 10\alpha\beta^2\mu^2 + 3\alpha^2\beta^4}{4\alpha^4}$	$-\dfrac{12\mu^2}{\alpha\beta^2}$	$-\left(\dfrac{2}{\alpha}\right)^{\frac{3}{2}}\left(\dfrac{\mu}{\beta}\right)^3$

Table(1): First finite-order moment of GOU process

Remark 6.3.4. *By equations (6.2.10) and Table (1) the parameters* μ, α, β *and* γ *can be expressed as function of the finite moment of the process. Indeed,*

1. *For GOU process, we have*

$$\alpha = -\log\left(\left|\frac{R_1(1)}{R_1(0)}\right|\right), \mu = m_1\alpha \text{ and } \beta^2 = -\frac{12\alpha m_1^2}{Kurtosis\,(X)}$$

2. *For* $COBL(1,1)$ *and when* $\beta = 0$ *we obtain*

$$\alpha = \log\left(\left|\frac{R_1(1)}{R_1(0)}\right|\right), \mu = -m_1\alpha \text{ and } \gamma^2 = \frac{-2\alpha Var(X)}{Var(X) + m_1^2}$$

These relationship can be used for estimating the process by the moment method (MM).

6.4 GMM estimation

In what follows, we focus on estimating of the unknown parameters of time-invariant version. For this purpose we shall assume that $\beta = 0$ in SDE (6.1.1) i.e.,

$$dX(t) = (\alpha X(t) + \mu)\,dt + \gamma X(t)\;dW(t) \tag{6.4.1}$$

this assumption can be fulfilled by the transformation $Y(t) = \dfrac{\mu}{(\gamma\mu - \alpha\beta)}(\beta + \gamma X(t))$. So, the parameters of interest are gathered in the vector $\underline{\theta} = (\mu, \alpha, \gamma)' \in \mathbb{R}^3$, its true values denoted by $\underline{\theta}_0 = (\mu_0, \alpha_0, \gamma_0)'$ belonging to an Euclidean compact permissible parameter subspace Θ of $\mathbb{R}^3$. In statistical literature of continuous-time models, several technique of estimation were proposed (interested readers are advised to see the monographs by Bergstrom [6], Rao [58] and Kutoyants [43] and the references therein). However, in recent years, a number of diffusion processes which have a similar second-order properties to that of a $CARMA$ processes have been estimated via some discretization schema and hence adaptive methods related to discrete-time linear models are however applied. So, for the SDE (6.4.1), the Euler-Maruyama scheme yields

$$X(t+\Delta) = X(t) + \int_t^{t+\Delta} (\alpha X(s) + \mu)\,ds + \gamma \int_t^{t+\Delta} X(s)\;dW(s)$$

where Δ is some small enough constant sampling interval, hence an approximation of discrete-time version of SDE (6.4.1) is given by

$$X(t+1) = X(t) + (\alpha X(t) + \mu)\,\Delta + \eta\,(t+1) \tag{6.4.2}$$

in which $(\eta\,(t+1))_{t\geq 0}$ is a some white noise with $E\{\eta\,(t+1)\,|I_t\} = 0$ and $Var\{\eta\,(t+1)\,|I_t\} = \gamma^2 X^2(t)\Delta$ and I_t denotes the information available up a time t, and hence (6.4.2) can be viewed as an $AR(1)$ model with heteroskedasticity. This finding leads us to estimate the vector $\underline{\theta}_0$ of the process in discrete time using the Generalized Method of Moments (GMM) due to Hansen [33]. For this purpose, we use the orthogonality conditions given by the vector $E_{\underline{\theta}_0}\left\{\underline{g}_t\,(\underline{\theta}_0)\right\} = O$ equation where

$$\underline{g}_t\,(\underline{\theta}) = \begin{pmatrix} \eta\,(t+1) \\ \eta^2\,(t+1) - \gamma^2 X^2(t)\Delta \\ \left(\eta^2\,(t+1) - \gamma^2 X^2(t)\Delta\right) X(t) \end{pmatrix}.$$

A GMM estimator of $\underline{\theta}_0$ is defined as any measurable solution $\widehat{\underline{\theta}}_n$ of

$$\widehat{\underline{\theta}}_n = Arg \min_{\underline{\theta}\in\Theta} \left\{\widehat{Q}_n = \widehat{\underline{g}}'_n\,(\underline{\theta})\,W_n \widehat{\underline{g}}_n\,(\underline{\theta})\right\},$$

where $\widehat{\underline{g}}_n\,(\underline{\theta}) = \frac{1}{n}\sum_{t=1}^{n} \underline{g}_t\,(\underline{\theta})$ and W_n is a sequence of positive definite weighting matrices. Under the condition (6.2.11) for each $\underline{\theta} \in \Theta$, the process $\left(\underline{g}_t\,(\underline{\theta})\right)_{t\in\mathbb{Z}}$ is stationary, ergodic and such that $\left\|E_{\underline{\theta}_0}\left\{\underline{g}_t\,(\underline{\theta})\right\}\right\| < +\infty$ for any $\underline{\theta} \in \Theta$ and hence, almost surely $\widehat{\underline{g}}_n\,(\underline{\theta}) \to E_{\underline{\theta}_0}\left\{\underline{g}_0\,(\underline{\theta})\right\}$ as $n \to +\infty$. To analyze the large sample properties of the proposed estimator, it is necessary to impose the following regularity conditions on the process $(X(t))_{t\in\mathbb{Z}}$, on the matrix W_n and on the parameter space Θ.

A1. The sequence of matrices (W_n) converges in probability to a non random positive definite matrix W.

A2. The matrix $E_{\underline{\theta}_0}\left\{\dfrac{\partial \underline{g}'_t(\underline{\theta}_0)}{\partial \underline{\theta}}\right\} W E_{\underline{\theta}_0}\left\{\dfrac{\partial \underline{g}_t(\underline{\theta}_0)}{\partial \underline{\theta}}\right\}$ is a finite non singular matrix of constants.

A3. The parameter $\underline{\theta}_0$ is in the interior of Θ.

We are now in a position to state the following results.

Theorem 6.4.1. *Beside the assumption* (6.2.11)*, under the conditions* ***A****1*–***A****3, $\widetilde{\underline{\theta}}_n$ converges in probability to $\underline{\theta}_0$.*

Proof. From the first-order conditions (organized as column vector) for the minimization of $\widehat{Q}_n(\underline{\theta})$ we have

$$\frac{\partial \widehat{\underline{g}}'_n(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n \widehat{\underline{g}}_n(\widehat{\underline{\theta}}_n) = \underline{O}. \tag{6.4.3}$$

Taking the first-order Taylor-series expansion of the score vector $\widehat{g}_n(\underline{\theta})$ around $\underline{\theta}_0$, we obtain $\widehat{g}_n\left(\widehat{\underline{\theta}}_n\right) = \widehat{\underline{g}}_n(\underline{\theta}_0) - \dfrac{\partial \widehat{\underline{g}}_n(\underline{\theta}_*)}{\partial \underline{\theta}}\left(\widehat{\underline{\theta}}_n - \underline{\theta}_0\right)$ where $\underline{\theta}_*$ is an intermediate point on the line segment joining $\widehat{\underline{\theta}}_n$ and $\underline{\theta}_0$. Substituting for $\widehat{\underline{g}}_n\left(\widehat{\underline{\theta}}_n\right)$ into (6.4.3) yields $\dfrac{\partial \widehat{\underline{g}}'_n(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n \left\{\widehat{\underline{g}}_n(\underline{\theta}_0) - \dfrac{\partial \widehat{\underline{g}}_n(\underline{\theta}_*)}{\partial \underline{\theta}}\left(\widehat{\underline{\theta}}_n - \underline{\theta}_0\right)\right\}$ $= \underline{O}$. Rearranging the above expression gives almost surely

$$\widetilde{\underline{\theta}}_n - \underline{\theta}_0 = \left\{\frac{\partial \widehat{\underline{g}}'_n(\widetilde{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n \frac{\partial \widehat{\underline{g}}_n(\underline{\theta}_*)}{\partial \underline{\theta}}\right\}^{-1} \frac{\partial \widehat{\underline{g}}'_n(\underline{\theta}_n)}{\partial \underline{\theta}} W_n \widehat{\underline{g}}_n(\underline{\theta}_0).$$

Since the process $(X(t))_{t\geq 0}$, is an ergodic process, then under the conditions $A1.-A3.$, we have

$$p\lim_{n\to\infty} \frac{\partial \widehat{\underline{g}}_n(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n = B = E_{\underline{\theta}_0}\left\{\frac{\partial g(\underline{\theta}_0)}{\partial \underline{\theta}}\right\} W,$$

$$p\lim_{n\to\infty} \frac{\partial \widehat{\underline{g}}'_n(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n \frac{\partial \widehat{\underline{g}}_n(\underline{\theta}_*)}{\partial \underline{\theta}} = A = E_{\underline{\theta}_0}\left\{\frac{\partial \underline{g}'_t(\underline{\theta}_0)}{\partial \underline{\theta}}\right\} W E_{\underline{\theta}_0}\left\{\frac{\partial \underline{g}_t(\underline{\theta}_0)}{\partial \underline{\theta}}\right\}$$

Hence from Slutsky's and the dominated convergence theorem, it follows that

$$p\lim_{n\to\infty} \left\{\frac{\partial \widehat{\underline{g}}'_n(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n \frac{\partial \widehat{\underline{g}}_n(\underline{\theta}_*)}{\partial \underline{\theta}}\right\}^{-1} \frac{\partial \widehat{\underline{g}}'_n(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n = A^{-1}B'$$

is finite, and since $p\lim_{n\to\infty} \widehat{\underline{g}}_n(\underline{\theta}_0) = \underline{O}$, the weak consistency of $\widetilde{\underline{\theta}}_n$ follows. ■ □

Theorem 6.4.2. *Under the conditions of theorem* 6.4.1*, we have* $\sqrt{n}\left(\widehat{\underline{\theta}}_n - \underline{\theta}_0\right) \rightsquigarrow \mathcal{N}(\underline{O}, \Sigma(\underline{\theta}_0))$ *where*

$$\Sigma(\underline{\theta}_0) = A^{-1}E_{\underline{\theta}_0}\left\{\frac{\partial \underline{g}'(\underline{\theta}_0)}{\partial \underline{\theta}}\right\} W\Sigma_{as} W E_{\underline{\theta}_0}\left\{\frac{\partial \underline{g}(\underline{\theta}_0)}{\partial \underline{\theta}}\right\} A'^{-1}$$

with $\Sigma_{as} = \lim\limits_{n\to+\in} Var\left\{\sqrt{n}\widehat{\underline{g}}_n(\underline{\theta})\right\}$.

Proof. The proof rests classically on a Taylor-series expansion of the score vector $\widehat{\underline{g}}_n(\underline{\theta})$ around $\underline{\theta}_0$. Thus, by the same argument used in Theorem 6.4.1, we have

$$\left\{\frac{\partial\widehat{\underline{g}}'(\widehat{\underline{\theta}}_n)}{\partial\underline{\theta}}W_n\frac{\partial\widehat{\underline{g}}(\widehat{\underline{\theta}}_*)}{\partial\underline{\theta}}\right\}\left(\widehat{\underline{\theta}}_n-\underline{\theta}_0\right)=\frac{\partial\widehat{\underline{g}}'(\widehat{\underline{\theta}}_n)}{\partial\underline{\theta}}W_n\widehat{\underline{g}}_n(\underline{\theta}_0). \tag{6.4.4}$$

On the other hand, for any $\underline{\lambda}\in\mathbb{R}^3$, the sequence $\left\{\underline{\lambda}'\widehat{\underline{g}}_n(\underline{\theta}),I_t\right\}_t$ is a square integrable martingale difference. The central limit theorem of Billingsley [13] and the Wold-Cramer device show that $'\widehat{\underline{g}}_n(\underline{\theta})\rightsquigarrow N(\underline{O},\Sigma_{as})$. Moreover; by (6.4.4) we have the following limits

$$A=p\lim_{n\to\infty}\frac{\partial\widehat{\underline{g}}'(\widehat{\underline{\theta}}_n)}{\partial\underline{\theta}}W_n\frac{\partial\widehat{\underline{g}}(\widehat{\underline{\theta}}_*)}{\partial\underline{\theta}},B=p\lim_{n\to\infty}\frac{\partial\widehat{\underline{g}}'(\widehat{\underline{\theta}}_n)}{\partial\underline{\theta}}W_n$$

so the result simply follows from Slutsky's theorem. ■ □

We now discuss the optimal choice of the weighting matrix W which matters for asymptotic efficiency. It is clear that the asymptotic variance of $\widehat{\underline{\theta}}_n$ depends on W_n via W. When appropriately choosing W, it is possible to minimize the asymptotic variance of $\widehat{\underline{\theta}}_n$. Then the minimum variance that can be achieved is when $W=\Sigma_{as}^{-1}$. In this particular case, the asymptotic variance of $\widehat{\underline{\theta}}_n$ is $\left\{E_{\underline{\theta}_0}\left\{\frac{\partial g'_{\underline{\theta}_0}}{\partial\underline{\theta}}\right\}\Sigma_{as}^{-1}E_{\underline{\theta}_0}\left\{\frac{\partial g_{\underline{\theta}_0}}{\partial\underline{\theta}}\right\}\right\}^{-1}$ and $nQ_n(\underline{\theta})$ has an asymptotic chi-square distribution with appropriate degrees of freedom. One can note that this choice is only sufficient for efficiency. Hence, estimating the matrix Σ_{as} by a consistent estimator $\widehat{\Sigma}_{asn}$ is crucial since: i) it is the optimal weighting matrix of GMM; ii) it is a part of the construction of $\widehat{\underline{\theta}}_n$ and its asymptotic variance (needed to construct confidence intervals and to make statistical tests based on $\widehat{\underline{\theta}}_n$). In practice, the Newey-West estimator can be used $\widehat{V}_n=\widehat{\Omega}_n(0)+2\sum_{j=1}^{q}K\left(\frac{j}{q}\right)\widehat{\Omega}_n(j)$ where $\widehat{\Omega}_n(j)=n^{-1}\sum_{t=K+1}^{n}\underline{W}_t\underline{W}'_{t-j}$ with $\underline{W}_t-\frac{1}{n}\sum_{t=K+1}^{n}g_t\left(\widehat{\underline{\theta}}_n\right)$. The truncated lag q needs to go to infinity at some appropriate rate with respect to the sample, and the kernel weight $K(.)$ is assumed to belong to K where $K=\{k:R\to[-1,1]\mid k(0)=1,\ k(x)=k(-x),\forall x\in\mathbb{R},\ \int|k(x)|dx<\infty$, and k is continuous but at some countable points$\}$. Examples of such kernel weights are given in table(2) below

Names	Expressions	Names	Expressions
Truncated	$k_T(x)=\begin{cases}1 & if\ \mid x\mid\leq 1,\\ 0 & otherwise,\end{cases}$	Parzen	$k_P(x)=\begin{cases}1-6x^2+6\lvert x\rvert^3 & if\ \lvert x\rvert\leq 1/2,\\ 2(1-\lvert x\rvert)^3 & if\ 1/2<\lvert x\rvert\leq 1,\\ 0 & otherwise\end{cases}$
Bartlett	$k_B(x)=\begin{cases}1-\lvert x\rvert & if\ \lvert x\rvert\leq 1,\\ 0 & otherwise,\end{cases}$	Tukey-Hanning	$k_H(x)=\begin{cases}(1+\cos\pi x)/2 & if\ \lvert x\rvert\leq 1,\\ 0 & otherwise,\end{cases}$

Table(2): Example of kernel weights

It can be shown that Bartlett and Parzen kernels all product positive semi-definite estimates of V while this is not necessarily the case for truncated and Tukey-Hanning kernels.

6.5 Empirical evidence

6.5.1 Simulation study

In this subsection we give some numerical illustration of the use of the GMM method described in previous section in order to underline its interest in statistical inference of continuous-time models. We simulated $n = 500$ independent trajectories via some specifications of $COBL\,(1,1)$ model with length $N \in \{1000, 2000\}$driven by a standard Bm distribution and vector of parameters $\underline{\theta}$ described in the bottom of each table below. The vector $\underline{\theta}$ is chosen to satisfied the second order stationarity and existence of moments up to fourth order. For the purpose of illustration, we consider the following models

$$\text{Model}\,(1) : dX(t) = (\alpha X(t) + \mu)\,dt + \beta dW(t)$$
$$\text{Model}\,(2) : dX(t) = (\alpha X(t) + \mu)\,dt + \beta X(t)dW(t)$$

their vector of parameters $\underline{\theta}= (\mu, \alpha, \beta)'$ is estimated by the GMM algorithm noted $\widehat{\underline{\theta}}_g$ and as a parameter of configuration we estimate $\underline{\theta}$ by the moment method noted $\widehat{\underline{\theta}}_M$. Both methods have been executed under the $MATLAB/8$ using "$fminsearch.m$" as a minimizer function. In Tables below, the column "Mean" correspond to the average of the parameters estimates over the $n = 500$ simulations. In order to show the performance of $(G)\,MM$, we have reported in each table the root mean squared error $(RMSE)$ (results between brackets)

Model(1)

The results of estimating the Model(1) are summarized in the following table

$\widehat{\underline{\theta}}$	$N = 1000$		$N = 2000$		$N = 1000$		$N = 2000$	
	$\underline{\underline{Mean}}$		$\underline{\underline{Mean}}$		$\underline{\underline{Mean}}$		$\underline{\underline{Mean}}$	
	GMM	MM	GMM	MM	GMM	MM	GMM	MM
$\widehat{\mu}$	0.2556 (0.0352)	0.2538 (0.0402)	0.2516 (0.0183)	0.2569 (0.0271)	2.0475 (0.0374)	2.0042 (0.0271)	2.0031 (0.0344)	2.0200 (0.0331)
$\widehat{\alpha}$	-1.5338 (0.0287)	-1.5391 (0.0302)	-1.5218 (0.0325)	-1.5310 (0.0210)	-0.5304 (0.0862)	-0.5151 (0.0672)	-0.5003 (0.0802)	-0.5031 (0.0770)
$\widehat{\beta}$	0.7462 (0.0101)	0.7493 (0.0213)	0.7450 (0.1022)	0.7492 (0.0151)	-1.4903 (0.0441)	-1.5359 (0.0451)	-1.4947 (0.0345)	-1.4947 (0.0345)
Design(1): $\underline{\theta} = (0.25, -1.5, 0.75)'$					Design(2): $\underline{\theta} = (2.0, -0.5, -1.5)'$			
Table(3): $(G)\,MM$ estimation of Model(1)								

The plots of asymptotic density of each component of $\underline{\widehat{\theta}}$ according to two methods are summarized in the following figure

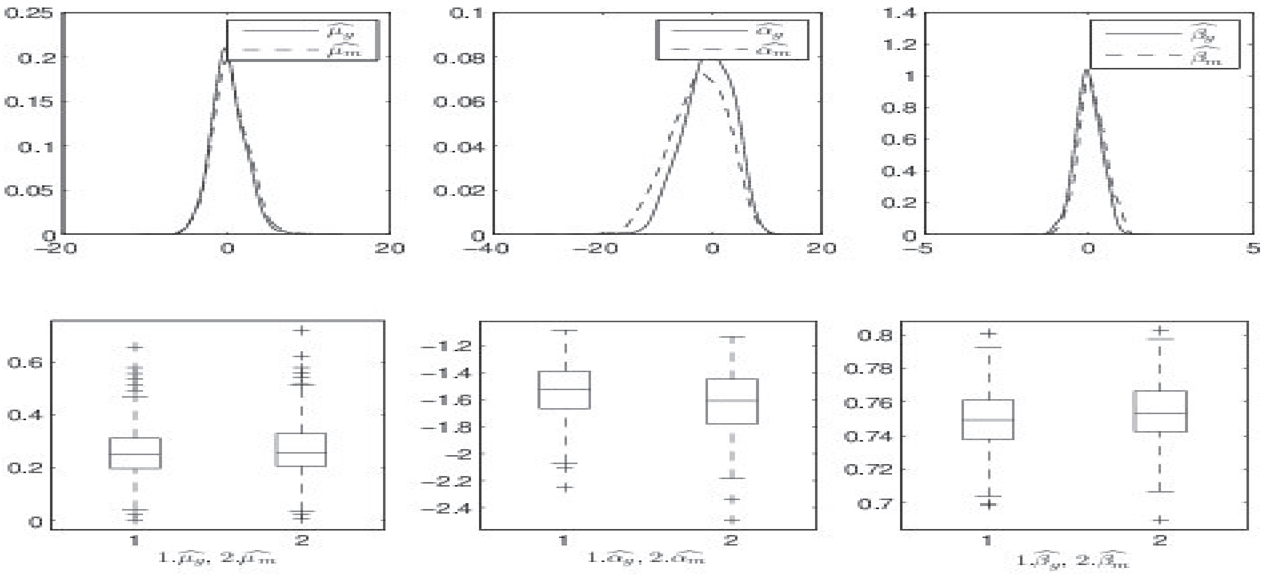


Fig1. Top panels: the asymptotic distribution of $\sqrt{n}(\widehat{\theta}_G(i) - \theta(i))$ (resp. $\sqrt{n}(\widehat{\theta}_M(i) - \theta(i))$). Bottom panels: Box plot summary of $\widehat{\theta}_G(i)$ (resp, $\widehat{\theta}_M(i)$) $i = 1, ..., 3$, according to the first design of table(3)

Model(2)

For the second model, we illustrated the results of its estimation in the following table

	$N = 1000$		$N = 2000$		$N = 1000$		$N = 2000$	
$\underline{\widehat{\theta}}$	$\underline{\underline{Mean}}$		$\underline{\underline{Mean}}$		$\underline{\underline{Mean}}$		$\underline{\underline{Mean}}$	
	GMM	MM	GMM	MM	GMM	MM	GMM	MM
$\widehat{\mu}$	0.2520 (0.0192)	0.2532 (0.0187)	0.2501 (0.0172)	0.2506 (0.0162)	0.5026 (0.0171)	0.5060 (0.0191)	0.5038 (0.0143)	0.5021 (0.0201)
$\widehat{\alpha}$	−1.4879 (0.0307)	−1.5352 (0.0452)	−1.5080 (0.0217)	−1.5064 (0.0251)	−1.5724 (0.0161)	−1.5307 (0.0142)	−1.5058 (0.0201)	−1.5072 (0.0162)
$\widehat{\beta}$	0.7537 (0.0121)	0.7449 (0.0157)	0.7449 (0.0157)	0.7512 (0.0609)	−0.50139 (0.0201)	−0.4946 (0.0211)	−0.4982 (0.0125)	−0.4920 (0.0302)
Design(1): $\underline{\theta} = (0.25, -1.5, 0.75)'$					Design(2): $\underline{\theta} = (0.5, -1.5, -0.5)'$			
Table(4): $(G)\,MM$ estimation of Model(2)								

The plots of asymptotic density of each parameters in $\underline{\widehat{\theta}}$ according to two methods are summarized in the following figure

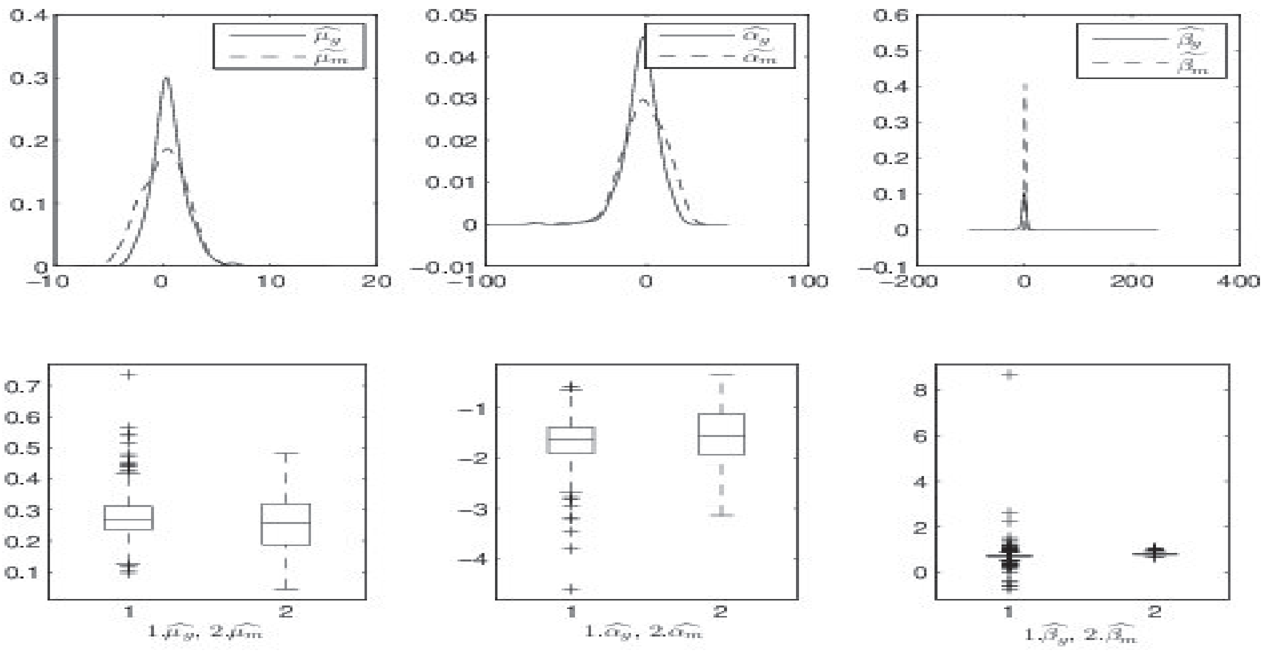


Fig2. Top panels: the asymptotic distribution of $\sqrt{n}(\widehat{\theta}_G(i) - \theta(i))$ (resp. $\sqrt{n}(\widehat{\theta}_M(i) - \theta(i))$). Bottom panels: Box plot summary of $\widehat{\theta}_G(i)$ (resp, $\widehat{\theta}_M(i)$) $i = 1, ..., 3$, according to design of table(4)

Now, a few comments are in order. Inspection of Table(3) reveals that the results of GMM and of MM methods are reasonably close on each other and also for their $RMSE$ with sone non significant deviation. These observations maybe seen regarding the plots of asymptotic distributions of their estimates and their elementary statistics summarized in box plots which represents a strong similarities,. This finding is however violated in Table(4). Indeed, In spite of its well estimate of the true values of unknown parameters, there are some difference regarding the plots presented in Fig2. It is clear that the asymptotic variances of $\widehat{\mu}_g$ and of $\widehat{\alpha}_g$ are smaller than of $\widehat{\mu}_m$ and of $\widehat{\alpha}_m$, contrary to that of $\widehat{\beta}$ Moreover, it can be seen from the their box plots, that the elementary statistics of two methods represents a significant dissimilarities.

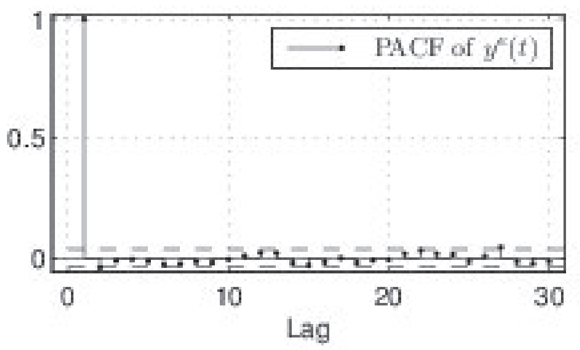
6.5.2 Real data analysis

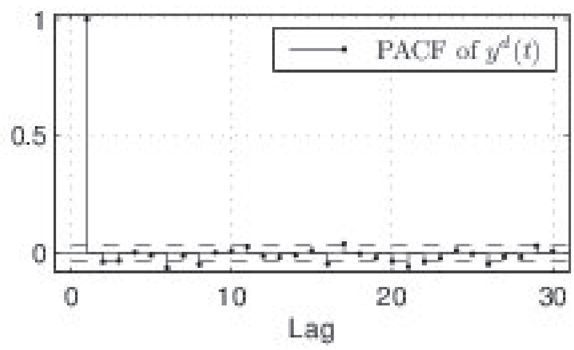
In this subsection, the proposed method is now investigated to real financial time series. So we apply the method to two foreign exchange rates of Algerian Dinar against $U.S.$-Dollar (USD/DZD) and against the single European currency Euro (EUR/DZD), noted respectively $\left(y^d(t)\right)$ and $(y^e(t))$ from January $3, 2000$ to September $29, 2011$. After removing the days when the market was closed (weekends, holidays,...), we provides 3055 observations of each series supposed to be uniformly distributed on 611 weeks. Table(5) below

Series	means	Std.Dev	Median	Skewness	Kurtosis	J. Bera
$y^e(t)$	88.61	11.57	91.09	-0.51	2.13	232.46
$y^d(t)$	73.45	4.24	73.12	-0.60	3.76	258.00

Table(5): Descriptive statistics of the series $(y^e(t))_{t\geq 1}$, $\left(y^d(t)\right)_{t\geq 1}$

provides descriptive statistics of such series, as a first finding, it is seen from the Jarque-Bera normality test that the series $y^d(t)$ and $y^e(t)$ are not normally distributed, this excludes its modelling by a GOU model. Moreover the sample partial autocorrelation function figure $Fig3a$





$Fig3a$. The PACF of $(y^e(t))$ and $\left(y^d(t)\right)$

of the prices series $(y^e(t))$ and $\left(y^d(t)\right)$ indicate that a discrete bilinear model with appropriates coefficients would well describe the series $\left(y^d(t)\right)$ and $(y^e(t))$. For it, the $(G)\,MM$ estimates followed by their $RMSE$ (results between bracket) of the series of prices $(y^e(t))_{t\geq 1}$ and $\left(y^d(t)\right)_{t\geq 1}$ noted hereafter $(\widehat{y}^e(t))_{t\geq 1}$ and $\left(\widehat{y}^d(t)\right)_{t\geq 1}$ via model(2) are given in Table (6)

	MM			GMM		
$\underline{\theta}$	μ	α	β	μ	α	β
$\widehat{y}^e_g(t)$	17.2911 (0.0307)	-0.1952 (0.0101)	0.0807 (0.0812)	17.4418 (0.0725)	-0.2125 (0.0613)	0.0771 (0.0621)
$\widehat{y}^d_g(t)$	25.3216 (0.0501)	-0.3447 (0.0817)	0.0479 (0.1002)	24.1221 (0.1320)	-1.0447 (0.1522)	0.0378 (0.0204)

Table(6): The $(G)\,MM$ estimates of $(y^e(t))_{t\geq 1}$, $\left(y^d(t)\right)_{t\geq 1}$

The graphics of original series $y^e(t)$ and $y^d(t)$ stacked on their estimates series $\widehat{y}^e(t)$ and $\widehat{y}^d(t)$ are shown the following figures below. For the first series

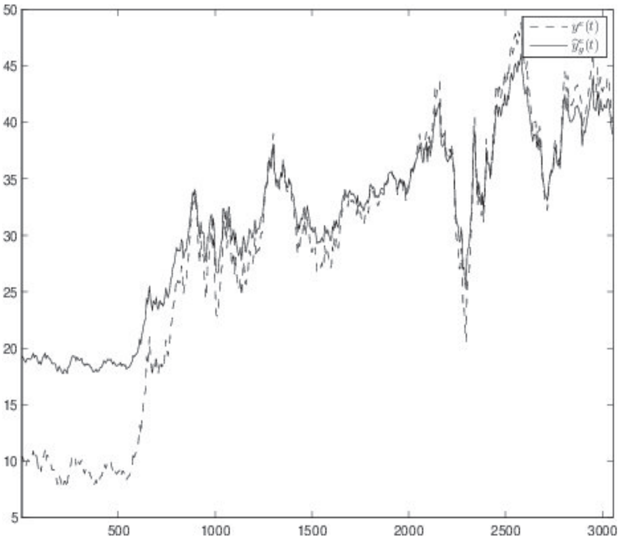


Fig4. Trajectory of model(2) fitted to the price of EUR/DZD via GMM method

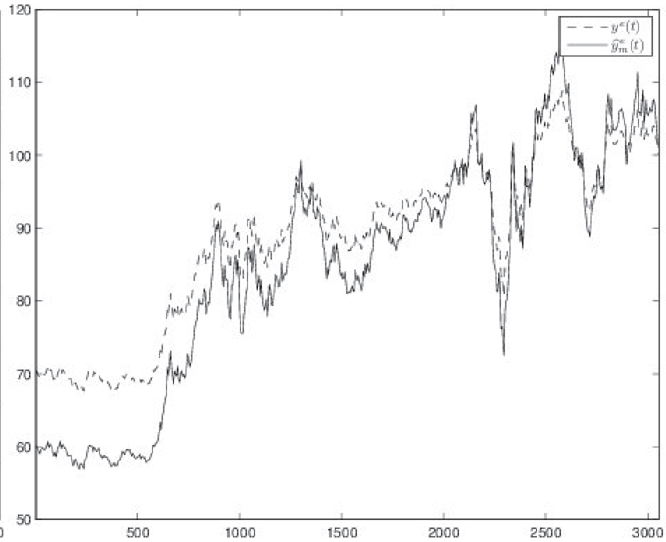


Fig5. Trajectory of model(2) fitted to the price of EUR/DZD via MM method

For the second one

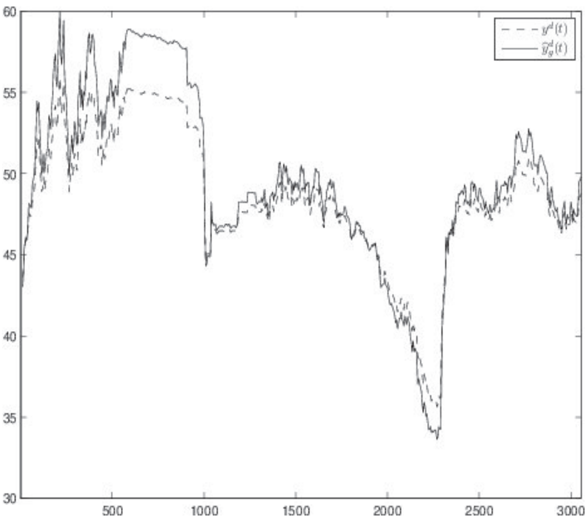


Fig5. Trajectory of model(2) fitted to the price of USD/DZD via GMM method

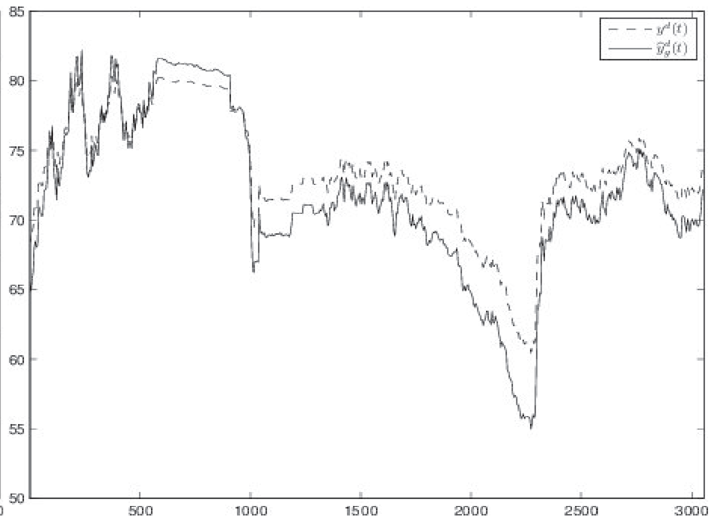


Fig6. Trajectory of model(2) fitted to the price of USD/DZD via MM method

It is clear, regarding the figures Fig3, Fig4, Fig5 and Fig6 that the plots of of original series $y^e(t)$ and $y^d(t)$ display a very similar pattern with respect to their estimates $\widehat{y}^e(t)$ and $\widehat{y}^d(t)$ via methods of GMM and MM.

6.6 Summary

In this chapter, the estimation of the unknown parameters in $COBL$ process has been presented by two alternative approaches. The chapter has shown that the GMM and MM estimating procedures are provided an efficient approach for estimating a discretized version of such models This methods are highlighted by a Monte Carlo study and an application to the foreign exchange rates of Algerian Dinar against $U.S.$-Dollar (USD/DZD) and against the single European currency Euro (EUR/DZD). The results of simulation and/or of the application shows the interest of the proposal methods whether their asymptotic properties or in modelling the real data.

Chapter 7

Yule-Walker type estimator of first-order bilinear differential model for stochastic processes[7]

7. Ce chapitre est soumis dans le journal : Statistical Method and Applications.

Abstract

This chapter, studies in time domain, some class of diffusion processes generated by a first order continuous-time bilinear stochastic processes ($COBL(1,1)$) with time-varying coefficients. So, we used the Itô formula approach for examining the $\mathbb{L}_2-$structure of the process and its powers. In time-invariant case, an expression of the moments of any order are given and the linear representation of such process is given as well as moments properties of some specifications. Based on these results we are able to examine the statistical properties as well as develop an estimation method of the process via the so-called Yule-Walker (YW) type algorithm which relate with the unknown coefficients of CAR representation. The method is illustrated by a Monte Carlo study and applied to modelling the electricity consumption sampled each 15mn in Algeria.

7.1 Introduction

In this chapter we consider the class of diffusion processes $(X(t))_{t\geq 0}$ generated by the following time-varying stochastic differential equation (SDE)

$$\begin{aligned} dX(t) &= (\alpha(t)X(t) + \mu(t))\, dt + (\gamma(t)X(t) + \beta(t))\, dw(t),\ X(0) = X_0, \\ &= \mu_t(X(t))\, dt + \sigma_t(X(t))\, dw(t), t \geq 0, X(0) = X_0 \end{aligned} \quad (7.1.1)$$

noted hereafter $COBL(1,1)$ in which $\mu_t(x) = \alpha(t)x + \mu(t)$ and $\sigma_t(x) = \gamma(t)x + \beta(t)$ which represents respectively the drift and the diffusion functions, $(w(t))_{t\geq 0}$ is a standard real Brownian motion (Bm) defined on some filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, P)$ and the initial state $X(0)$ is a random variable, defined on $(\Omega, \mathcal{A}, P)$, independent of w such that $E\{X(0)\} = m_1(0)$ and $Var\{X(0)\} = R_1(0)$. Special cases of this process are the Brownian motion with drift

$(\alpha(t) = 0$ and $\gamma(t) = 0$), the Gaussian Ornstein-Uhlenbeck (GOU) process $(\gamma(t) = 0$) and the volatility of the $COGARCH(1,1)$ process defined by $dX(t) = \sigma(t)\,dw_1(t)$ where $d\sigma^2(t) = \left(\mu(t) - \alpha(t)\sigma^2(t)\right)dt + \gamma(t)\sigma^2(t)\,dw_2(t)$ in which $\mu(t) > 0$, $\alpha(t), \gamma(t) \geq 0$ for all $t \geq 0$ and $w_1(t)$ an $w_2(t)$ are independent Bm independent of $(X(0), \sigma(0))$ (see for instance Kluppelberg et al. [42]). The SDE (7.1.1) is called time-invariant if there exists some constants α, μ, γ and β such that for all t, $\alpha(t) = \alpha$, $\mu(t) = \mu$, $\gamma(t) = \gamma$ and $\beta(t) = \beta$. To ensure the existence and uniqueness of the solution process $(X(t))_{t\geq 0}$ of equation (7.1.1) we shall assume that the parameters $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are measurable deterministic functions and subject to the following assumption

Assumption 4. *$\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are differentiable functions such that for any $T > 0$*

$$\int_0^T |\alpha(t)|\,dt < \infty, \int_0^T |\mu(t)|\,dt < \infty, \int_0^T |\gamma(t)|^2\,dt < \infty \text{ and } \int_0^T |\beta(t)|^2\ dt < \infty.$$

The existence and uniqueness of the Itô solution process $(X(t))_{t\geq 0}$ of equation (7.1.1) in time domain is however ensured by the general results on SDE and under the Assumption 4 since the drift and the diffusion functions are Lipschitz with linear growth, i.e., $|\mu_t(x) - \mu_t(y)| \leq \sup_t |\alpha(t)|\,|x-y|$ and $|\sigma_t(x) - \sigma_t(y)| \leq \sup_t |\gamma(t)|\,|x-y|$ so, a such solution is given by

$$X(t) = \Phi(t)\left\{X(0) + \int_0^t \Phi^{-1}(s)\left(\mu(s) - \gamma(s)\beta(s)\right)ds + \int_0^t \Phi^{-1}(s)\beta(s)\,dw(s)\right\}, \ a.e., \quad (7.1.2)$$

where the process $\Phi(t)$ is defined by $\Phi(t) = \exp\left\{\int_0^t \left(\alpha(s) - \frac{1}{2}\gamma^2(s)\right)ds + \int_0^t \gamma(s)\,dw(s)\right\}$. The solution (7.1.2) is however Markovian when $\beta(t) \neq 0$ for all t , otherwise it is neither a Markov process nor a martingale. A wide literature is available now on the probabilistic properties and on statistical inference and their asymptotic properties of SDE (7.1.1) driven by a Brownian and/or fractional Brownian motion (see for instance Kutoyants [43] and the references therein). Without doubt, the discretization schema remain the cornerstone of such developments. So, in recent years, a number of methods already developed in linear case are explored for SDE (7.1.1). Indeed, Kallsen and Muhle-Karbe [39], and Haug et al. [32] have proposed an asymptotic inference of moments method (MM) for discretized continuous $GARCH$ process, Bibi and Merahi [10] have explored a MM for estimating the parameters of continuous-time bilinear processes, Chan et al. [18] investigated an empirical comparison of generalized method of moment (GMM) of several discretized diffusions processes. Broze et al. [17] studies the effect of discretization schema on the consistency of the direct inference based on likelihood, Hyndman [34] and Guyon and Souchet [61] extended the so-called Yule-Walker estimator for a discretized version of an $CAR(p)$. The Levinson-Durbin-type algorithms for continuous-time autoregressive models was studied by Pham and Le Breton [56]. Nevertheless, some procedures or methods developed in linear cases are not yet explored (or not adapted) for the SDE (7.1.1) to our best knowledge.

The main aim of the chapter is twofold, the first one is dedicated to study the second-order properties of the processes $(X(t))_{t\geq 0}$ and its powers using the Itô approach. The second aim is to explore the Yulker-Walter (YW) algorithm for estimating the time-invariant version of (7.1.1). So, In the next section we give the conditions ensuring of stability of the SDE (7.1.1) based on the Itô formula, then we give an expression of its moments of higher order in time-invariant case. After a general description of Yulker-Walter (YW) method for $CAR(p)$ process, presented in section 3, we extend the approach for a discretized time-invariant version of SDE (7.1.1). In order to show the quality of YW algorithm for $COBL(1,1)$ model we present in section 4 a Monte Carlo simulations. Our results are also exploited for modelling of mean electricity consumption sampled each 15mn in Algeria. We give a summary of the chapter in Section 5.

7.2 Second-order properties of $(X(t))_{t\geq 0}$ and its powers

In this section we give conditions that assure the existence of finite moments of the process $(X(t))_{t\geq 0}$ and of its powers up to any fixed order k, and we show how they can be calculated using an iterative procedure for time-invariant specification.

7.2.1 Moments properties of $(X(t))_{t\geq 0}$

Assume that everywhere

$$2\alpha(t) + \gamma^2(t) < 0. \tag{7.2.1}$$

The moments properties of the process $(X(t))_{t\geq 0}$ is described in the following theorem.

Theorem 7.2.1. *Under the condition (7.2.1), the mean function $m_1(t) = E\{X(t)\}$, the variance $R_1(t)$ and the covariance function $R_1(t,s) = E\{(X(t) - m_1(t))(X(s) - m_1(s))\}, t \geq s$ for the process $(X(t))_{t\geq 0}$ generated by the SDE (7.1.1) are given respectively by*

$$m_1(t) = \varphi_1(t)\left(m_1(0) + \int_0^t \varphi_1^{-1}(s)\mu(s)ds\right). \tag{7.2.2}$$

$$R_1(t) = \phi_1(t)\left(R_1(0) + \int_0^t \phi_1^{-1}(s)\left(\gamma(s)m_1(s) + \beta(s)\right)^2 ds\right), \tag{7.2.3}$$

$$R_1(t,s) = \varphi_1(t)\varphi_1^{-1}(s)R_1(s), t \geq s \geq 0,. \tag{7.2.4}$$

where for any positive integer k,

$$\phi_k(t) = \exp\left\{k\int_0^t \left(2\alpha(u) + (2k-1)\gamma^2(u)\right)du\right\} \text{ and } \varphi_k(t) = \exp\left\{\int_0^t \left(\alpha(t)k + \frac{1}{2}\gamma^2(t)k(k-1)\right)du\right\}$$

so $\varphi_1(t)$ becomes the mean function of the process $(\Phi(t))_{t\geq 0}$ in (7.1.2).

Proof. The expression (7.2.2) follows immediately from the SDE (7.1.1). To derive the equation (7.2.3) we apply Itô formula to SDE (7.1.1), with $f(x) = x^2$, so

$$dX^2(t) = \left(a_2(t)X^2(t) + b_2(t)X(t) + c_2(t)\right)dt + 2\left(\gamma(t)X^2(t) + \beta(t)X(t)\right)dW(t), \tag{7.2.5}$$

where

$$a_2(t) = 2\alpha(t) + \gamma^2(t), b_2(t) = 2\left(\mu(t) + \gamma(t)\beta(t)\right), c_2(t) = \beta^2(t). \tag{7.2.6}$$

and hence, Equation (7.2.5) can be written as

$$X^2(t) = X_0^2 + \int_0^t \left(a_2(s)X^2(s) + b_2(s)X(s) + c_2(s)\right)ds + 2\int_0^t \left(\gamma(s)X^2(s) + \beta(s)X(s)\right)dW(s). \tag{7.2.7}$$

Let $m_2(t) = E\left\{X^2(t)\right\}$ and taking the expectation each side of (7.2.7), then

$$m_2(t) = m_2(0) + \int_0^t (a_2(s)m_2(s) + b_2(s)m_1(s) + c_2(s))\, ds. \tag{7.2.8}$$

Differentiating with respect to t we obtain

$$\frac{dm_2(t)}{dt} = a_2(t)m_2(t) + b_2(t)m_1(t) + c_2(t),\ t \geq 0, \tag{7.2.9}$$

since $R_1(t) = m_2(t) - m_1^2(t)$, then we $\frac{dR_1(t)}{dt} = \frac{dm_2(t)}{dt} - 2m_1(t)\frac{dm_1(t)}{dt}$. Using (7.2.2), (7.2.9) and the fact that $m_2(t) = R_1(t) + m_1^2(t)$ we obtain the following differential equation

$$\frac{dR_1(t)}{dt} = a_2(t)R_1(t) + (a_2(t) - 2\alpha(t))\, m_1^2(t) + (b_2(t) - 2\mu(t))\, m_1(t) + c_2(t).$$

By general results of linear ordinary differential equations (see, e.g., [40], chap. 1) we obtain

$$R_1(t) = \phi_1(t)\left(R_1(0) + \int_0^t \phi_1^{-1}(s)\left[\gamma(s)m_1(s) + \beta(s)\right]^2 ds\right)$$

and the expression (7.2.3) follows. To prove (7.2.4), we have by SDE (7.1.1), $dE\left\{X(t)X(s)\right\} = \alpha(t)E\left\{X(t)X(s)\right\}dt + \mu(t)E\left\{X(s)\right\}dt$, so $E\left\{X(t)X(s)\right\} = R_1(t,s) + m_1(t)m_1(s)$ and hence

$$dR_1(t,s) + m_s(t)dm_1(t) = \alpha(t)R_1(t,s)dt + \left[\alpha(t)m_1(t) + \mu(t)\right]dtm_1(s).$$

Using the expression (7.2.2) we get $dR_1(t,s) + m_1(s)dm_1(t) = \alpha(t)R_1(t,s)dt + dm_1(t)m_1(s)$ which implies $dR_1(t,s) = \alpha(t)R_1(t,s)dt$, and by general results on linear ordinary differential equations (see, e.g., [40], chap. 1) the expression (7.2.4) holds true. □ □

Corollary 7.2.1. *In time-invariant case and under the condition (7.2.1), we have the following results*

$$m_1(0) = -\frac{\mu}{\alpha},\ R_1(0) = \frac{(\alpha\beta - \mu\gamma)^2}{\alpha^2\left|2\alpha + \gamma^2\right|},\ R_1(h) = R_1(0)e^{\alpha|h|},\ h \in \mathbb{R}. \tag{7.2.10}$$

from which the parameters of certain specifications can be deduced as functions of the empirical moments.

7.2.2 Moments properties of $\left(X^k(t)\right)_{t\geq 0}$

Now, we examine the second order properties of the process $\left(X^k(t)\right)_{t\geq 0}$ for any integer $k \geq 2$ in which the condition

$$2\alpha(t) + (2k-1)\gamma^2(t) < 0,\ \text{for all } t \geq 0, \tag{7.2.11}$$

must be imposed.

Theorem 7.2.2. *Consider the process $\left(X^k(t)\right)_{t\geq 0}, k \geq 2$, then under the condition (7.2.11), the mean, covariance and variance functions are given respectively by*

$$m_k(t) = \varphi_k(t)\varphi_k^{-1}(s) \tag{7.2.12}$$
$$\times\left\{m_k(s) + k\int_s^t \varphi_k(s)\varphi_k^{-1}(u)\left((\mu(u) + (k-1)\gamma(u)\beta(u))\, m_{k-1}(u) + \frac{1}{2}\beta^2(u)(k-1)m_{k-2}(u)\right)du\right\}$$

$$R_k(t,s) \tag{7.2.13}$$
$$= \varphi_k(t)\varphi_k^{-1}(s)\{R_k(s) + k\int_s^t \varphi_k(s)\varphi_k^{-1}(u)\left[(\mu(u)+\gamma(u)\beta(u)(k-1))\, Cov(X^{k-1}(u),X^k(s))\right]$$
$$+\frac{1}{2}\beta^2(u)(k-1)Cov\left(X^{k-2}(u),X^k(s)\right)du\}, t\geq s$$

$$R_k(t) = \phi_k(t)\phi_k^{-1}(s)R_k(s)+ \tag{7.2.14}$$
$$\int_s^t \phi_k(t)\phi_k^{-1}(u)[\gamma^2(u)m_k^2(u) + 2k\left(\mu(u)+(2k-1)\gamma(u)\beta(u)\right)m_{2k-1}(u)$$
$$-2k\left(\mu(u)+(k-1)\gamma(u)\beta(u)\right)m_k(u)m_{k-1}(u) + k(2k-1)\beta^2(u)m_{2k-2}(u) - k(k-1)\beta^2(u)m_k(u)m_{k-2}(u)]du,$$

Proof. Set $m_k(t) = E\left\{X^k(t)\right\}, k\geq 2$ and by applying the Itô formula to SDE (7.1.1) for $f(x)=x^k$, we get

$$dX^k(t) = \left(a_k(t)X^k(t)+b_k(t)X^{k-1}(t)+c_k(t)X^{k-2}(t)\right)dt+\left(\gamma(t)kX^k(t)+\beta(t)kX^{k-1}(t)\right)dw(t), \tag{7.2.15}$$

with

$$a_k(t) = \alpha(t)k+\frac{1}{2}\gamma^2(t)k(k-1), b_k(t) = \mu(t)k+\gamma(t)\beta(t)k(k-1), c_k(t) = \frac{1}{2}\beta^2(t)k(k-1). \tag{7.2.16}$$

We can write (7.2.15) as

$$X^k(t) = X^k(0) + \int_0^t \left(a_k(t)X^k(s)+b_k(t)X^{k-1}(s)+c_k(t)X^{k-2}(s)\right)ds$$
$$+\int_0^t \left(\gamma(t)kX^k(s)+\beta(t)kX^{k-1}(s)\right)dw(s). \tag{7.2.17}$$

Therefore, taking the expected value of each side of (7.2.17), if we put $m_k(t) = E\left\{X^k(t)\right\}$ we find

$$m_k(t) = m_k(0) + \int_0^t \left(a_k(s)m_k(s)+b_k(s)m_{k-1}(s)+c_k(s)m_{k-2}(s)\right)ds. \tag{7.2.18}$$

Differentiating with respect to t we obtain

$$\frac{dm_k(t)}{dt} = a_k(t)m_k(t)+b_k(t)m_{k-1}(t)+c_k(t)m_{k-2}(t), k\geq 2, t>0 \tag{7.2.19}$$

$$m_k(0) = E\left\{X^k(0)\right\}, k\geq 2, m_0(t)=1, \forall t\geq 0. \tag{7.2.20}$$

By solving the above of differential equations for any $k\geq 2$ the formula (7.2.12) follows. Since $R_k(t) = m_{2k}(t) - (m_k(t))^2$, then by differentiating $R_k(t)$ with respect to t we find $\frac{dR_k(t)}{dt} = \frac{dm_{2k}(t)}{dt} - 2m_k(t)\frac{dm_k(t)}{dt}$ and use (7.2.19) for $k\geq 2$ and the fact that $m_{2k}(t) = R_k(t)+m_k^2(t)$ we obtain the following differential equation

$$\frac{dR_k(t)}{dt}$$
$$= a_{2k}(t)R_k(t) + \left(a_{2k}(t)-2a_k(t)\right)m_k^2(t) + b_{2k}(t)m_{2k-1}(t)$$
$$-2b_k(t)m_k(t)m_{k-1}(t) + c_{2k}(t)m_{2k-2}(t) - 2c_k(t)m_k(t)m_{k-2}(t),$$

where from (7.2.16) the coefficients of the above equation can be given as

$$\begin{aligned}
a_{2k}(t) &= 2\alpha(t)k + \gamma^2(t)k(2k-1), \\
a_k(t) &= \alpha(t)k + \frac{1}{2}\gamma^2(t)k(k-1), \\
b_{2k}(t) &= 2\left(\mu(t)k + \gamma(t)\beta(t)k(2k-1)\right), \\
b_k(t) &= \left(\mu(t)k + \gamma(t)\beta(t)k(k-1)\right), \\
c_{2k}(t) &= \beta^2(t)k(2k-1), \\
2c_k(t) &= \beta^2(t)k(k-1).
\end{aligned}$$

By general results on linear ordinary differential equations the expression (7.2.14) follows. To prove the formula (7.2.13), we observe that for any $t \geq s$

$$dE\left\{X^k(t)X^k(s)\right\} = a_k(t)E\left\{X^k(t)X^k(s)\right\}dt + b_k(t)E\left\{X^{k-1}(t)X^k(s)\right\}dt + c_k(t)E\left\{X^{k-2}(t)X^k(s)\right\}dt$$

By general results on linear ordinary differential equations we obtain

$$E\left\{X^k(t)X^k(s)\right\} = \varphi_k(t)\varphi_k^{-1}(s)$$
$$\times\left\{E\left\{X^{2k}(s)\right\} + \int_s^t \varphi_k(s)\varphi_k^{-1}(u)\left[b_k(u)E\left\{X^{k-1}(u)X^k(s)\right\} + c_k(u)E\left\{X^{k-2}(u)X^k(s)\right\}\right]du\right\}.$$

Since $E\left\{X^{2k}(s)\right\} = R_k(s) + m_k^2(s)$ and

$$\begin{aligned}
E\left\{X^k(t)X^k(s)\right\} &= Cov(X^k(t), X^k(s)) + E\left\{X^k(t)\right\}E\left\{X^k(s)\right\} = R_k(t,s) + m_k(t)m_k(s) \\
E\left\{X^{k-1}(u)X^k(s)\right\} &= Cov\left(X^{k-1}(u), X^k(s)\right) + m_{k-1}(s)m_k(s), E\left\{X^{k-2}(u)X^k(s)\right\} \\
&= Cov\left(X^{k-2}(u), X^k(s)\right) + m_{k-2}(s)m_k(s)
\end{aligned}$$

then, using the formula (7.2.12) we obtain

$$\begin{aligned}
&R_k(t,s) + m_k(t)m_k(s) \\
&= \varphi_k(t)\varphi_k^{-1}(s)\left\{R_k(s) + \int_s^t \varphi_k(s)\varphi_k^{-1}(u)\left(b_k(u)Cov\left(X^{k-1}(u), X^k(s)\right) + c_k(u)Cov\left(X^{k-2}(u), X^k(s)\right)\right)du\right\} \\
&+ \varphi_k(t)\varphi_k^{-1}(s)\left\{m_k(s) + \int_s^t \phi(s)\phi^{-1}(u)\left(b_k(u)m_{k-1}(u) + c_k(u)m_{k-2}(u)\right)du\right\}m_2(s) \\
&= \varphi_k(t)\varphi_k^{-1}(s)\left\{R_k(s) + \int_s^t \varphi_k(s)\varphi_k^{-1}(u)\left(b_k(u)Cov\left(X^{k-1}(u), X^k(s)\right) + c_k(u)Cov\left(X^{k-2}(u), X^k(s)\right)\right)du\right\} \\
&+ m_k(t)m_k(s),
\end{aligned}$$

which implies

$$\begin{aligned}
&R_k(t,s) \\
&= \varphi_k(t)\varphi_k^{-1}(s)\{R_k(s) + k\int_s^t \varphi_k(s)\varphi_k^{-1}(u)((\mu(u) + \gamma(u)\beta(u)(k-1))\,Cov\left(X^{k-1}(u), X^k(s)\right) \\
&+ \frac{1}{2}\beta^2(u)(k-1)Cov\left(X^{k-2}(u), X^k(s)\right))du\},
\end{aligned}$$

and the expression (7.2.13) holds. □ □

In time-invariant case and under the condition (7.2.11), we can computing the moments of order k of the *SDE* (7.1.1). Indeed,

Theorem 7.2.3. *Assume that the parameters of the SDE (7.1.1) are constant and satisfying the condition (7.2.11) , then the moments of order k are given by*

1. *If* $\beta \neq 0$, *then* $m_1(0) = -\frac{b_1}{a_1}, m_2(0) = -\frac{b_2}{a_2}m_1(0) - \frac{c_2}{a_2}, m_3 = -\frac{b_3}{a_3}m_2 - \frac{c_3}{a_3}m_1$ *and* $\forall k \geq 2$

$$m_k(0) = (-1)^{k+1}\left(\frac{a_3m_3(0)+b_3m_2(0)}{a_3}\right)\left\{\prod_{j=2}^{k-2}\frac{b_{j+2}}{a_{j+2}}\right\}\sum_{(k-4,2)} Z_i + (-1)^k m_2(0)\left\{\prod_{j=1}^{k-2}\frac{b_{j+2}}{a_{j+2}}\right\}\sum_{(k-3,1)} Z_i$$

where the parameters a_k, b_k, c_k *are given by the equations (7.2.16),* $Z_k = -\frac{c_{k+2}a_{k+1}}{b_{k+2}b_{k+1}}$, $k \geq 2$ *and*

$$\sum_{(k-4,2)} Z_i = \sum_{\substack{d_1,\dots,d_{k-4}=0\\ d_{i+1}d_i=0\\ i=1,\dots,k-5}}^{1}\prod_{i=1}^{k-4} Z_{i+2}^{d_i} \sum_{(k-3,1)} Z_i = \sum_{\substack{d_1,\dots,d_{k-3}=0\\ d_{i+1}d_i=0\\ i=1,\dots,k-4}}^{1}\prod_{i=1}^{k-3} Z_{i+1}^{d_i}, k \geq 2$$

with the convenient $\sum\limits_{(0,k)} Z_i = \sum\limits_{(-1,k)} Z_i = 1$, $\sum\limits_{(-2,k)} Z_i = 0, \forall k \in \mathbb{N}, \sum_{j=m}^{m-n} x_j = 0$ *and* $\prod_{j=m}^{m-n} x_j = 1$, $\forall m, n \in \mathbb{N}$..

2. *If* $\beta = 0$, *we obtain* $m_k(0) = (-1)^k \prod_{j=1}^{k}\left(\frac{b_j}{a_j}\right)$, $\forall k \geq 1$.

Proof. To prove Theorem 7.2.3, we use the same approach as Popenda [57]. Indeed

1. If $\beta \neq 0$, and under the condition (7.2.11), the process $X^k(t)$ is second order stationary, then the moments are independent of t, it means $m_k(t) = m_k(0)$ for all $k \geq 1$ which implies from the ordinary equation (7.2.19)

$$a_k m_k(0) + b_k m_{k-1}(0) + c_k m_{k-2}(0) = 0, \tag{7.2.21}$$

where the parameters a_k, b_k and c_k are given by the equations (7.2.16), in particular for $k = 1$, $m_1(0) = -\frac{b_1}{a_1}$ and for $k = 2$ we have $m_2(0) = -\frac{b_2}{a_2}m_1(0) - \frac{c_2}{a_2}$. Now, since the equation (7.2.21) is a linear homogeneous second order equation its general solution is thus given by

$$\begin{aligned} m_k(0) &= (-1)^k M_1 \frac{c_3}{a_3}\left\{\prod_{j=2}^{k-2}\frac{b_{j+2}}{a_{j+2}}\right\}\sum_{(k-4,2)} Z_i + (-1)^k M_2\left\{\prod_{j=1}^{k-2}\frac{b_{j+2}}{a_{j+2}}\right\}\sum_{(k-3,1)} Z_i \\ &= (-1)^k M_1 \frac{c_3}{a_3}\left\{\prod_{j=2}^{k-2}\frac{b_{j+2}}{a_{j+2}}\right\}\sum_{(k-4,2)} Z_i + (-1)^k M_2\left\{\prod_{j=1}^{k-2}\frac{b_{j+2}}{a_{j+2}}\right\}\sum_{(k-3,1)} Z_i, k \geq 2, \end{aligned}$$

where $Z_n = -\frac{c_{n+2}a_{n+1}}{b_{n+2}b_{n+1}}, n \geq 2$, and the constants M_1 and M_2 can be given as $M_2 = m_2(0)$ and $M_1 = -\frac{a_3m_3(0)+b_3m_2(0)}{c_3}$.

2. If $\beta = 0$, the parameter $c_k = 0$ for all $k \geq 1$, then in time-invariant case we obtain the difference equation $a_k m_k(0) + b_k m_{k-1}(0) = 0$ which implies $m_k(0) = -\frac{b_k}{a_k} m_{k-1}(0), \forall k \geq 1$ and the result holds. □

□

Example 7.2.4. *The following table illustrate some finite-order moments for the $COBL(1,1)$ process*

β	m_1	m_2	m_3	m_4
$\beta \neq 0$	$-\frac{\mu}{\alpha}$	$-\frac{2(\gamma\beta+\mu)}{(2\alpha+\gamma^2)} m_1 - \frac{\beta^2}{(2\alpha+\gamma^2)}$	$-\frac{\gamma\beta+\mu}{(\alpha+\gamma^2)} m_2 - \frac{\beta^2}{(\alpha+\gamma^2)} m_1$	$-\frac{2(3\gamma\beta+\mu)}{(2\alpha+3\gamma^2)} m_3 - \frac{3\beta^2}{(2\alpha+3\gamma^2)} m_2$
$\beta = 0$	$-\frac{\mu}{\alpha}$	$\frac{2\mu^2}{\alpha(2\alpha+\gamma^2)}$	$-\frac{2\mu^3}{\alpha(2\alpha+\gamma^2)(\alpha+\gamma^2)}$	$\frac{4\mu^4}{\alpha(2\alpha+\gamma^2)(\alpha+\gamma^2)(2\alpha+3\gamma^2)}$

Table(1): The first forth-order moments of $COBL(1,1)$

Example 7.2.5. *The following table illustrated some finite-order moments for the GOU process defined by $dX(t) = (\mu - \alpha X(t))dt + \beta dW(t)$ with $\alpha > 0$ and $\beta \neq 0$,*

m_1	m_2	m_3	m_4	*Kurtosis*	*Skewness*
$\frac{\mu}{\alpha}$	$\frac{2\mu^2+\alpha\beta^2}{2\alpha^2}$	$\frac{\mu\left(2\mu^2+3\alpha\beta^2\right)}{2\alpha^3}$	$\frac{4\mu^4+12\alpha\beta^2\mu^2+3\alpha^2\beta^4}{4\alpha^4}$	$-\frac{12\mu^2}{\alpha\beta^2}$	$-\left(\frac{2}{\alpha}\right)^{\frac{3}{2}}\left(\frac{\mu}{\beta}\right)^3$

Table(2): First forth order moment of GOU process

This moments maybe used for identifying the process GOU.

7.3 Yule-Walker estimates

7.3.1 An overview

Let $(X(t))_{t\in\mathbb{R}}$ be a time invariant $CAR(p)$, i.e.,

$$X^{(p)}(t) + \alpha_{p-1}X^{(p-1)}(t) + ... + \alpha_0 X(t) = \sigma w(t), \tag{7.3.1}$$

where $(w(t))_{t\in\mathbb{R}}$ denotes a standard Bm. It is well known (see for instence Hyndman [34]) that if $(X(t))_{t\in\mathbb{R}}$ is second-order stationary, then the Yule-Walker estimates of $CAR(p)$ is carried out on the derivative autocovariance function $(DACF)$ $D_{j,k}(h)$ defined by

$$D_{j,k}(h) = Cov(X^{(j)}(t+h), X^{(k)}(t)), 0 \leq j,k \leq p-1. \tag{7.3.2}$$

which are closely related to the autocovariance function (ACF), i.e., $R_1(h) = Cov(X(t+h), X(t))$ according to the following equations

$$D_{j,k}(h) = (-1)^k R_1^{(j+k)}(h), h \geq 0 \text{ and } 0 \leq j,k \leq p-1, \tag{7.3.3}$$

and $D_{j,k}(-h) = (-1)^j R_1^{(j+k)}(h)$, where $R_1^{(j+k)}(h)$ denotes the $(j+k)-$th derivative of the ACF. This result can be extended to $D_{j,p}(h)$ which is defined by the Itô integral

$$D_{j,p}(h) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X^{(j)}(t+h) dX^{(p-1)}(t), \tag{7.3.4}$$

where the limit exists in the sense of mean-square convergence. Hence, $D_{j,p}(h) = (-1)^p R_1^{(j+p)}(h), 0 \leq j \leq p-2, h \neq 0$, $D_{p-1,p}(h) = (-1)^p R_1^{(2p-1)}(h)$ and $D_{p-1,p}(0) = -\frac{\sigma^2}{2}$. The so-called Yule-Walker equations for $CAR(p)$ processes is then obtained by multiplying both sides of Equation (7.3.1) by $X^{(j)}(t+h)$ and taking expectations to get

$$\alpha_0 D_{j,0}(h) + \alpha_1 D_{j,1}(h) + ... + \alpha_{p-1} D_{j,p-1}(h) + D_{j,p}(h) = 0, j = 0, 1, ..., p-1. \tag{7.3.5}$$

So, by replacing $X^{(j)}(t)$ by $X(t-j)$ and letting $h = 0$, we obtain the traditional discrete-time Yule- Walker equations. The Yule- Walker equations (7.3.5) can be written in the matrix form

$$\Gamma_p(h)\underline{\alpha} + \underline{\mathbf{D}}_p(h) = 0, \tag{7.3.6}$$

where $\Gamma_p(h)$ is matrix of variance-covariance of the vector $\left(X(t), X^{(1)}(t), ..., X^{(p-1)}(t)\right)'$, i.e., $\Gamma_p(h) = [D_{i,j}(h)]_{0 \leq i,j \leq p-1}$, $\underline{\mathbf{D}}_p(h) = (D_{0,p}(h), D_{1,p}(h), ..., D_{p-1,p}(h))'$ $h \geq 0$ and $\underline{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{p-1})'$. As already pointed by Hyndman [34] that $\Gamma_p(0)$ non-singular, and thus we have

$$\underline{\alpha} = -\Gamma_p^{-1}(0)\underline{\mathbf{D}}_p(0). \tag{7.3.7}$$

which is analogue to the Yule-Walker method for estimating a discrete-time autoregressive models. In end, the Yule-Walker estimator of $\underline{\alpha}$ is obtained by replacing the covariances by their sample estimates and hence

$$\widehat{\underline{\alpha}} = -\widehat{\Gamma}_p^{-1}(0)\widehat{\mathbf{D}}_p(0), \tag{7.3.8}$$

in which the elements of $\widehat{\Gamma}_p(0)$ are the sample estimates of $\widehat{D}_{j,k}(0)$ that are given by

$$\widetilde{D}_{j,k}(0) = \frac{1}{T}\int_0^T X^{(j)}(t)X^{(k)}(t)dt,\ 0 \leq j,k \leq p-1, \tag{7.3.9}$$

and the elements of the vector $\widehat{\mathbf{D}}_p(0)$ are given by the sample estimates

$$\widetilde{D}_{j,p}(0) = \frac{1}{T}\int_0^T X^{(j)}(t)dX^{(p-1)}(t), 0 \leq j \leq p-1, \tag{7.3.10}$$

which converges in probability to $D_{j,k}(0)$ as $T \to \infty$ (see Yaglom [70], pp. 231-33). Finally, the estimate of σ^2 is thus

$$\widehat{\sigma}^2 = -2\widehat{D}_{p-1,p}(0). \tag{7.3.11}$$

Remark 7.3.1. *Hyndman [34] has showed that the Yule-Walker estimates satisfy the least squares criteria for all order p. Moreover he has showed that the asymptotic distribution of the Yule-Walker estimators for $\widehat{\underline{\alpha}}$ coincide with that of Maximum Likelihood estimator.*

Remark 7.3.2. *In discrete-time Yule-Walker estimators for $CAR(p)$, Souchet and Guyon [61] have proved the weak consistency of $\widehat{\underline{\alpha}}$ and of $\widehat{\sigma}^2$, their asymptotic normality and their efficiency.*

7.3.2 Extension to $COBL(1,1)$ processes

In time−invariant version of the SDE (7.1.1) and under the condition (7.2.11), Lebreton and Musiela in [45] have showed that there exists a wide-sense Bm process $(w^*(t))_{t\geq 0}$ uncorrelated with $X(0)$ such that $(X(t))_{t\geq 0}$ admits the following $CAR(1)$ representation

$$dX(t) = (\alpha X(t) + \mu)dt + \sigma^* dw^*(t), \tag{7.3.12}$$

where $\sigma^{*2} = \gamma^2 R_1(0) + (\gamma m_1(0) + \beta)^2$. In the sequel, we shall assume $\beta = 0$ (this assumption can be fulfilled by the transformation $Y(t) = \dfrac{\mu}{(\gamma\mu - \alpha\beta)}(\beta + \gamma X(t)))$ and $\mu \neq 0$, then by (7.2.10) and the results in Table (1) of example 7.2.4, we have

$$m_1(0) = -\frac{\mu}{\alpha}, m_2(0) = \frac{2\mu^2}{\alpha\left(2\alpha + \gamma^2\right)},\ R_1(0) = \frac{\mu^2\gamma^2}{\alpha^2|2\alpha + \gamma^2|}, R_1(h) = R_1(0)e^{\alpha|h|}.,\ \sigma^{*2} = \gamma^2 m_2(0), \tag{7.3.13}$$

In order to apply the Yule-Walker method for estimating the vector $\underline{\alpha} = \left(\alpha, \mu, \gamma^2\right)'$ of parameters of this model, we define

$$D_{0,0}(h) = R_1(h) = Cov\left(X(t+h), X(t)\right), D_{0,1}(h) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \left(X(t+h) - m_1(0)\right)dX(t), \tag{7.3.14}$$

where the limit exists in $\mathbb{L}_2$−sense. Then it follows that for $h \neq 0, D_{0,1}(h) = -R_1^{(1)}(h)$, and since $R_1(h)$ is not differentiable at $h = 0$, $D_{0,1}(0)$ is computed using Itô formula so we obtain $D_{0,1}(0) = -\dfrac{\sigma^{*2}}{2}$ which is the right derivative of $-D_{0,0}(h)$ at $h = 0$ (see Doob [24], 1953, p. 544). The Yule-Walker equations for the above $CAR(1)$ process (7.3.12) is

$$D_{0,1}(h) = \alpha D_{0,0}(h), \forall h \geq 0, \tag{7.3.15}$$

so for $h = 0$ we have $\alpha = \dfrac{D_{0,1}(0)}{D_{0,0}(0)}$, Now, we assume that we are able to observe between the times 0 and T a time-invariant sample function $(X(t))_{t\geq 0}$ of the solution process SDE (7.1.1), then $D_{0,0}(0)$ and $D_{0,1}(0)$ may be estimated by the sample covariances

$$\widehat{D}_{0,0}(0) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \left(X(t) - \widehat{m}_1(0)\right)^2 dt \text{ and } \widehat{D}_{0,1}(0) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \left(X(t) - \widehat{m}_1(0)\right)dX(t), \tag{7.3.16}$$

where $\widehat{m}_1(0) = \dfrac{1}{T}\int_0^T X(t)dt$ is the estimate of the mean $m_1(0)$ which is strongly consistent and asymptotically normal, and that $\widehat{D}_{0,0}(0) \to D_{0,0}(0)$ and $\widehat{D}_{0,1}(0) \to D_{0,1}(0)$ in probability. (see Yaglom [70], pp. 231-33). Moreover, since the asymptotic properties of Yule-Walker estimators coincide with those obtained by of maximum likelihood estimators, then by the close analogy between the continuous-time Yule-Walker estimators and the discrete-time one, the Yule-Walker estimator of the parameters α, μ, σ^* and γ^2 are now gathered in the following proposition

Proposition 7.3.3. *Consider the time−invariant version of the SDE (7.1.1) with $CAR(1)$ representation (7.3.12). Then, the estimators of α,μ, σ^*, and $|\gamma|$ are given respectively by*

$$\widehat{\alpha} = \frac{\widehat{D}_{0,1}(0)}{\widehat{D}_{0,0}(0)}, \widehat{\mu} = -\widehat{m}_1(0)\widehat{\alpha},\ \widehat{\sigma^*} = \sqrt{-2\widehat{\alpha}\widehat{D}_{0,0}(0)} \textit{ and } |\widehat{\gamma}| = \sqrt{\frac{-2\widehat{\alpha}\widehat{D}_{0,0}(0)}{\widehat{D}_{0,0}(0) + \widehat{m}_1^2(0)}}. \tag{7.3.17}$$

Proof. The estimator of μ follows immediately from the mean of the process $X(t)$ and the Yule-Walker estimator of the parameter α maybe deducted from (7.3.15). The estimator of σ^2 is then given by $\widehat{\sigma^*}^2 = -2\widehat{\alpha}\widehat{D}_{0,0}(0)$. Since $\widehat{R}_1(0) = \widehat{D}_{0,0}(0)$ we can obtain an estimate $|\widehat{\gamma}|$ for $|\gamma|$ using the relationships in (7.3.13) and the last expression of $\widehat{\sigma^*}^2$. □ □

Note

In practice, the observations are collected at discrete times, so, to use the above estimators it is necessary to derive a discretized version of SDE (7.1.1). For this purpose, suppose that the data are observed at times $0 = t_1 < t_2 < ... < t_n = T$ and let $\Delta_i = t_{i+1} - t_i$ and $\Delta = \sup_i \Delta_i$. Then, for Δ small enough, it seems reasonable to estimate $D_{0,0}(0)$ and $D_{0,1}(0)$ by the numerical integrals

$$\widetilde{D}_{0,0}(0) = \frac{1}{T}\sum_{i=0}^{n-1}\left(X(t_i) - \widetilde{m}_1(0)\right)^2 \Delta_i, \text{ and } \widetilde{D}_{0,1}(0) = \frac{1}{T}\sum_{i=0}^{n-1}\left(X(t_i) - \widetilde{m}_1(0)\right)\left(X(t_{i+1}) - X(t_i)\right).$$

where $\widetilde{m}_1(0) = \frac{1}{T}\sum_{i=0}^{n-1} X(t_i)\Delta_i$. Now, define a discrete form of the Yule-Walker estimators by replacing $\widehat{D}_{0,0}(0)$, $\widehat{D}_{0,1}(0)$ by $\widetilde{D}_{0,0}(0)$ and $\widetilde{D}_{0,1}(0)$ respectively in (7.3.17), it follows that the estimators of the parameters α,μ,σ^* and $|\gamma|$ in discrete form are given respectively by

$$\widetilde{\alpha} = \frac{\widetilde{D}_{0,1}(0)}{\widetilde{D}_{0,0}(0)},\ \widetilde{\mu} = -\widetilde{m}_1(0)\widetilde{\alpha},\ \widetilde{\sigma^*} = \sqrt{-2\widetilde{\alpha}\widetilde{D}_{0,0}(0)} \text{ and } |\widetilde{\gamma}| = \sqrt{\frac{-2\widetilde{\alpha}\widetilde{D}_{0,0}(0)}{\widetilde{D}_{0,0}(0) + \widetilde{m}_1^2(0)}}.$$

and their asymptotic properties can be easily deduced.

7.4 Some Monte Carlo results

We provide in this section some simulations results for the Yule-Walker estimator and their asymptotic behavior already discussed in previous section for estimating the unknown vector $\underline{\theta} = (\alpha, \mu, \beta)$ involved in the model. The true values of $\underline{\theta}$ is denoted with $\underline{\theta}_0$ is chosen to satisfy the condition (7.2.1). For this purpose, we simulated 500 independent trajectories from a second-order stationary series according to the SDE (7.1.1) of length $n \in \{1000, 2000, 3000\}$ with standard Bm. The results of simulation experiments for estimating the vector $\underline{\theta}$ are reported in tables below in which the line "Mean of" correspond to the average of the parameters estimates over the 500 repetitions. In order to show the performance of the method compared with the MLE method, we have reported (results between bracket) the root-mean square errors ($RMSE$) of each estimates. The study of changes in parameter values with sampling interval Δ is also fruitfully used for the robustness of estimates an for the optimal choice of Δ. So, we have reported the variation of Yule-Walker estimates with sampling interval of each experiment.

7.4.1 GOU

The first design of our experiment consists to estimate the parameter of the Gaussian Ornstein-Uhlenbeck (GOU) process, i.e.,

$$dX(t) = (\mu - \alpha X(t))\,dt + \beta dw(t), \tag{7.4.1}$$

in which $\alpha > 0$ and $\beta \neq 0$. Its exact discretization based on Itô solution is given by

$$X((t+1)\Delta) - \frac{\mu}{\alpha} = e^{-\alpha\Delta}\left(X(t\Delta) - \frac{\mu}{\alpha}\right) + \beta\sqrt{\frac{1-e^{-2\alpha\Delta}}{2\alpha}}e((t+1)\Delta)$$

where $(e(t))_t$ is a Gaussian white noise independent of $(X(t))_t$, and the vector $\underline{\theta}$ of interest is however $\underline{\theta} = (\alpha, \beta, \mu)'$. So, $E\{X\} = \frac{\mu}{\alpha}$, $Var\{X\} = \frac{\beta^2}{2\alpha}$, $Cov(X((t+1)\Delta), X(t\Delta)) = e^{-\alpha\Delta}Var\{X\}$. The results of simulation of such model are reported in Table(3)

Length	1000		2000		3000	
Method	YW	ML	YW	ML	YW	ML
Mean($\tilde{\alpha}$)	2.1820	2.3882	2.0931	2.1953	2.0541	2.1196
($RMSE$)	(0.7044)	(0.7393)	(0.4448)	(0.4493)	(0.3499)	(0.3628)
Mean($\tilde{\beta}$)	1.4958	1.4988	1.5005	1.5019	1.5044	1.5049
($RMSE$)	(0.0431)	(0.0420)	(0.0287)	(0.0284)	(0.0214)	(0.0214)
Mean($\tilde{\mu}$)	1.0625	1.2228	1.0314	1.1072	1.0191	1.0699
($RMSE$)	(0.5332)	(0.6587)	(0.3828)	(0.4296)	(0.3115)	(0.3393)

design(1): $\alpha = 2.0$, $\beta = 1.5$ and $\mu = 1.0$

Mean($\tilde{\alpha}$)	1.6133	1.7403	1.5725	1.6479	1.5481	1.6000
($RMSE$)	(0.4548)	(0.4943)	(0.3369)	(0.3440)	(0.2808)	(0.2910)
Mean($\tilde{\beta}$)	0.6460	0.6332	0.5781	0.5758	0.5544	0.5538
($RMSE$)	(0.0171)	(0.0245)	(0.0107)	(0.0127)	(0.0078)	(0.0084)
Mean($\tilde{\mu}$)	2.0240	2.3109	2.0323	2.1938	2.0226	2.1342
($RMSE$)	(0.6088)	(0.6342)	(0.4551)	(0.4537)	(0.3826)	(0.3912)

design(2): $\alpha = 1.5$, $\beta = 0.5$ and $\mu = 2.0$

Mean($\tilde{\alpha}$)	1.1838	1.4081	1.0885	1.1935	1.0537	1.1221
($RMSE$)	(0.5440)	(0.6127)	(0.3417)	(0.3530)	(0.2591)	(0.2784)
Mean($\tilde{\beta}$)	0.9873	0.9910	0.9942	0.9958	0.9987	0.9991
($RMSE$)	(0.0456)	(0.0447)	(0.0250)	(0.0257)	(0.0173)	(0.0176)
Mean($\tilde{\mu}$)	0.5508	0.7124	0.5256	0.6002	0.5154	0.5667
($RMSE$)	(0.3690)	(0.5271)	(0.2602)	(0.3161)	(0.2107)	(0.2483)

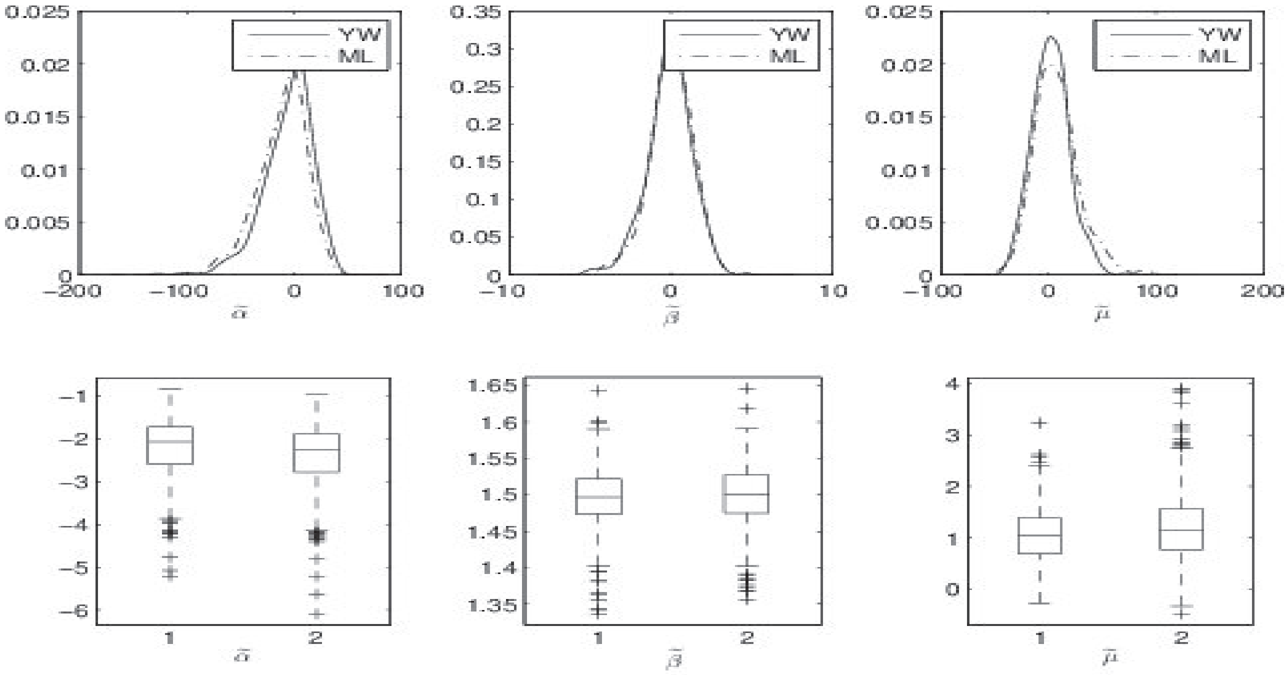
design(3): $\alpha = 1.0$, $\beta = 1.0$ and $\mu = 0.5$

Mean($\tilde{\alpha}$)	0.9016	1.1261	0.8290	0.9362	0.8000	0.8684
($RMSE$)	(0.4549)	(0.5388)	(0.2974)	(0.3288)	(0.2293)	(0.2513)
Mean($\tilde{\beta}$)	1.0503	1.0363	1.0260	1.0235	1.0200	1.0187
($RMSE$)	(0.0558)	(0.0706)	(0.0290)	(0.0316)	(0.0202)	(0.0216)
Mean($\tilde{\mu}$)	-1.0589	-1.3955	-1.0305	-1.2182	-1.0180	-1.1427
($RMSE$)	(0.6416)	(0.7296)	(0.4322)	(0.4785)	(0.3504)	(0.3793)

design(4): $\alpha = 0.75$, $\beta = 1.0$ and $\mu = -1.0$

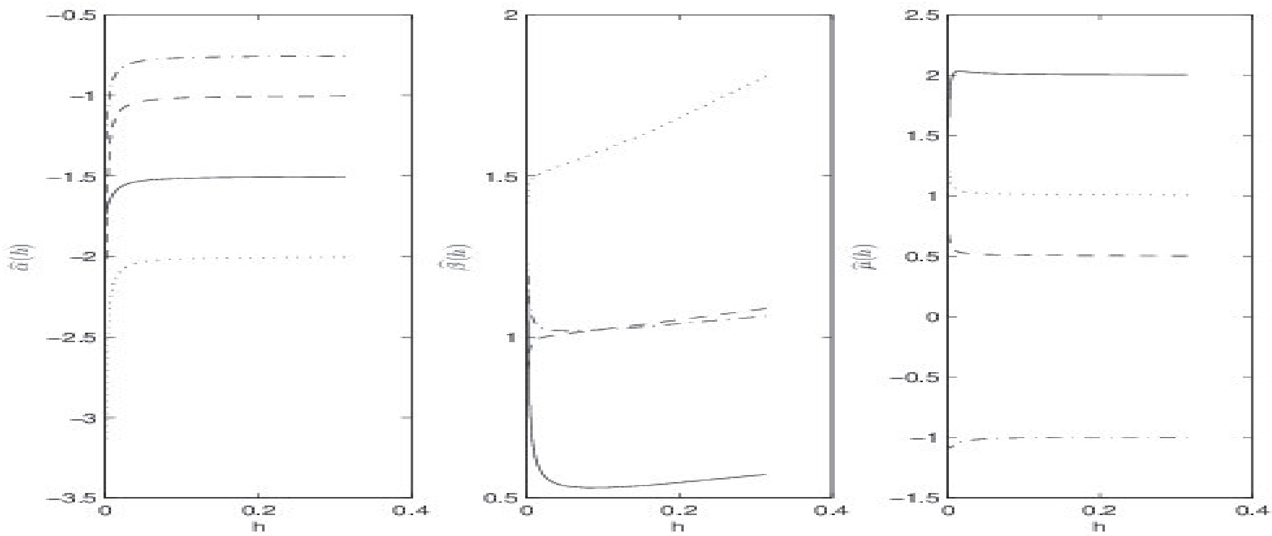
Table(3): The results of simulation by the Yule-Walker estimator and MLE for GOU.

The empirical densities of each parameters according to two methods for *GOU* model followed by their box plot summary of the statistical properties of each estimates are shown in Figure(1)



Fig(1). The plots of empirical densities of each parameters according to two methods for *GOU* model

The variation of the Yule-Walker estimator for *GOU* model with the sampling interval are shown in figure (2)



Fig(2). Variation of $\underline{\theta}$ with sampling interval according to the four designs in Table(3)

7.4.2 COBL(1,1)

In the second design, we consider the $COBL(1,1)$ generated by the following SDE

$$dX(t) = (\alpha X(t) + \mu)\, dt + \gamma X(t) dw(t), t \geq 0, X(0) = X_0 \tag{7.4.2}$$

The vector $\underline{\theta}$ of interest is thus $\underline{\theta} = (\alpha,\ \gamma, \mu)'$. The $CARMA$ representation (7.3.12) becomes $dX(t) = (\alpha X(t) + \mu)\, dt + \xi dw^*(t)$ where $\xi^2 = \gamma^2 \left(R_1(0) + m_1^2\right)$. The Euler discretization yields

$$X(t+\Delta) - X(t) = (\alpha X(t) + \mu)\, \Delta + \xi(w^*(t+\Delta) - w^*(t))$$

while the exact discretization is given by

$$X(t+\Delta) - X(t) = -\frac{\mu}{\alpha}\left(1 - e^{\alpha\Delta}\right) - \left(1 - e^{\alpha\Delta}\right) X(t) + \zeta e^{\alpha t} \int_t^{t+\Delta} e^{-\alpha s} dw^*(s)$$

so we obtain

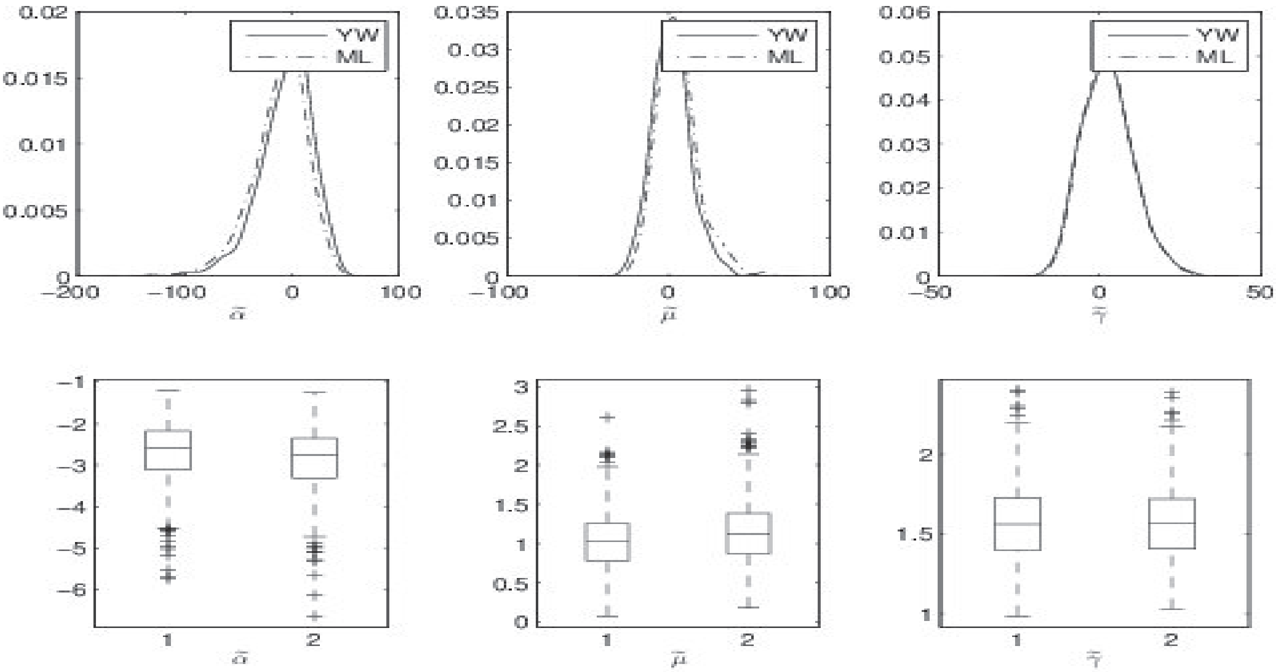
$$X((t+1)\Delta) + \frac{\mu}{\alpha} = e^{\alpha\Delta}\left(X\,(t\Delta) + \frac{\mu}{\alpha}\right) + \zeta\sqrt{\frac{1 - e^{2\alpha\Delta}}{-2\alpha}} e((t+1)\Delta)\,, \tag{7.4.3}$$

$(e\,(t))_t$ is a Gaussian white noise independent of $(X\,(t))_t$. Equation (7.4.3) means that the exact discretization of $COBL(1,1)$ is an $AR(1)$ model with coefficient $e^{\alpha\Delta} > 0$, so we have $E\{X\} = -\frac{\mu}{\alpha}$, $Var\{X\} = \frac{\zeta^2}{-2\alpha}$, $Cov(X((t+1)\Delta),\ X(t\Delta)) = e^{\alpha\Delta} Var\{X\}$. The results of simulation of $COBL(1,1)$ are reported in the following table

Length	1000		2000		3000	
Method	YW	ML	YW	ML	YW	ML
Mean($\tilde{\alpha}$)	-2.6791	-2.8777	-2.5958	-2.6963	-2.5549	-2.6192
($RMSE$)	(0.7614)	(0.7858)	(0.4836)	(0.4859)	(0.3855)	(0.3957)
Mean($\tilde{\mu}$)	1.0444	1.1651	1.0235	1.0832	1.0134	1.0529
($RMSE$)	(0.3697)	(0.4153)	(0.2562)	(0.2716)	(0.2075)	(0.2184)
Mean($\tilde{\gamma}$)	1.5767	1.5777	1.5447	1.5456	1.5292	1.5296
($RMSE$)	(0.2477)	(0.2386)	(0.1706)	(0.1672)	(0.1348)	(0.1330)
design(1): $\alpha = -2.5$ $\mu = 1.0$ and $\gamma = 1.5$						
Mean($\tilde{\alpha}$)	-1.6404	-1.7991	-1.5807	-1.6672	-1.5511	-1.6092
($RMSE$)	(0.5245)	(0.5661)	(0.3631)	(0.3695)	(0.2935)	(0.3057)
Mean($\tilde{\mu}$)	0.5155	0.5973	0.5111	0.5551	0.5069	0.5369
($RMSE$)	(0.1769)	(0.1863)	(0.1246)	(0.1249)	(0.1019)	(0.1053)
Mean($\tilde{\gamma}$)	0.6118	0.6029	0.5566	0.5551	0.5383	0.5378
($RMSE$)	(0.0686)	(0.0499)	(0.0427)	(0.0356)	(0.0326)	(0.0288)
design(2): $\alpha = -1.5$, $\mu = 0.5$ and $\gamma = 0.5$						
Mean($\tilde{\alpha}$)	-1.1803	-1.3998	-1.0879	-1.1915	-1.0536	-1.1213
($RMSE$)	(0.5353)	(0.6021)	(0.3387)	(0.3499)	(0.2580)	(0.2770)
Mean($\tilde{\mu}$)	1.0925	1.3991	1.0466	1.1930	1.0278	1.1281
($RMSE$)	(0.6346)	(0.8310)	(0.4354)	(0.5002)	(0.3482)	(0.3985)
Mean($\tilde{\gamma}$)	1.1314	1.1276	1.0664	1.0661	1.0427	1.0424
($RMSE$)	(0.2910)	(0.2696)	(0.1908)	(0.1833)	(0.1518)	(0.1479)
design(3): $\alpha = -1.0$, $\mu = 1.0$ and $\gamma = 1.0$						
Mean($\tilde{\alpha}$)	-1.4279	-1.6370	-1.3382	-1.4394	-1.3034	-1.3695
($RMSE$)	(0.5751)	(0.6307)	(0.3663)	(0.3746)	(0.2829)	(0.2998)
Mean($\tilde{\mu}$)	0.8048	0.9843	0.7776	0.8651	0.7663	0.8257
($RMSE$)	(0.3924)	(0.4733)	(0.2673)	(0.2921)	(0.2138)	(0.2354)
Mean($\tilde{\gamma}$)	1.1138	1.1100	1.0582	1.0578	1.0373	1.0371
($RMSE$)	(0.2487)	(0.2292)	(0.1645)	(0.1575)	(0.1281)	(0.1245)
design(4):$\alpha = -1.25$, $\mu = 0.75$ and $\gamma = 1.0$						

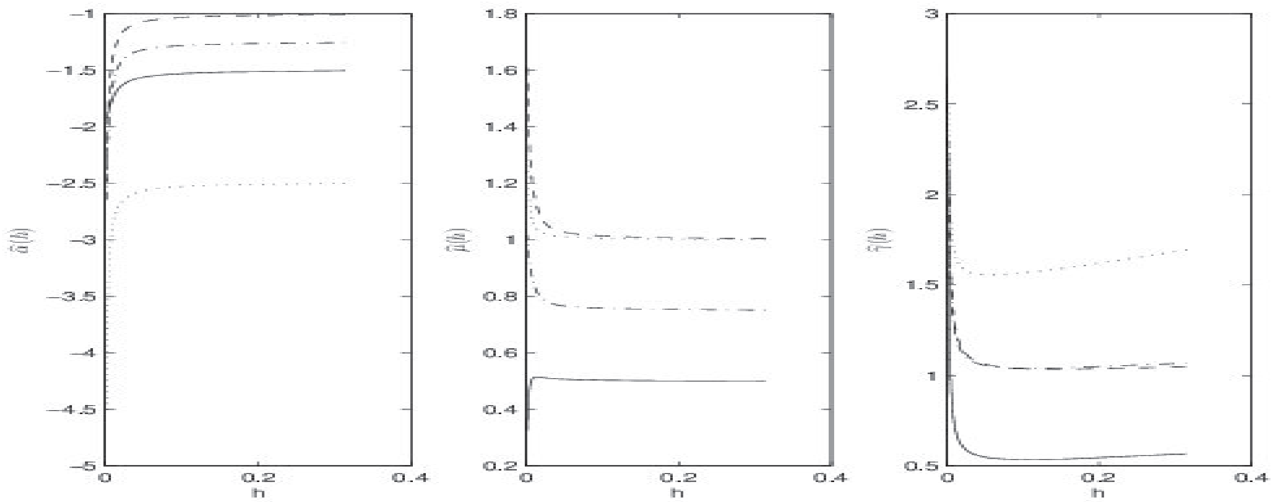
Table(4): The results of simulation by the Yule-Walker estimator and MLE of COBL(1,1).

The plots of asymptotic density, box plots of each parameters in $\underline{\theta}$ are summarized in the following figure



Fig(3). The plots of empirical densities of each parameters according to two methods for $COBL(1,1)$

The variation of the Yule-Walker estimator for $COBL(1,1)$ model with the sampling interval are shown in figure (4)



Fig(4). Variation of $\underline{\theta}$ with sampling interval according to the four designs in Table(4)

7.4.3 Discussion

Now, a few comments are in order. Inspection of Table(3) reveals that the results of Yule-Walker and of MLE methods are reasonably close on each other and also for their $RMSE$ with sone non significant deviation. It is also observed that the $RMSE$ of the estimates $\widehat{\beta}$ in four designs is more important than the others parameters. These observations maybe seen regarding the plots of empirical densities of their estimates and their elementary statistics summarized in box plots of the two methods which represents a strong similarities. Regarding now the variation of $\underline{\widehat{\theta}}$ with sampling interval for GOU model, it is worth noting that with exception of some variations observed at the neighborhood of origin, the estimates remains unchanged during their trajectories and confused with its true values. This finding maybe interpreted by the robustness of the Yule-Walker method. For the experiment of the $COBL$ model, it is observed that generally the results reported in Table (4) are in accordance with the asymptotic theory. Moreover, the values of the estimates and their $RMSE$ are very close one from the other except that values of $RMSE$ of $\widehat{\gamma}$ in design(2) are more important compared with others parameters. The plot of empirical densities of the estimated parameters and their box plots shows a strong similarity between the Yule-Walker estimates and MLE one. In end the analyze of the variation of $\underline{\widehat{\theta}}$ with the sampling interval is the same as for GOU model.

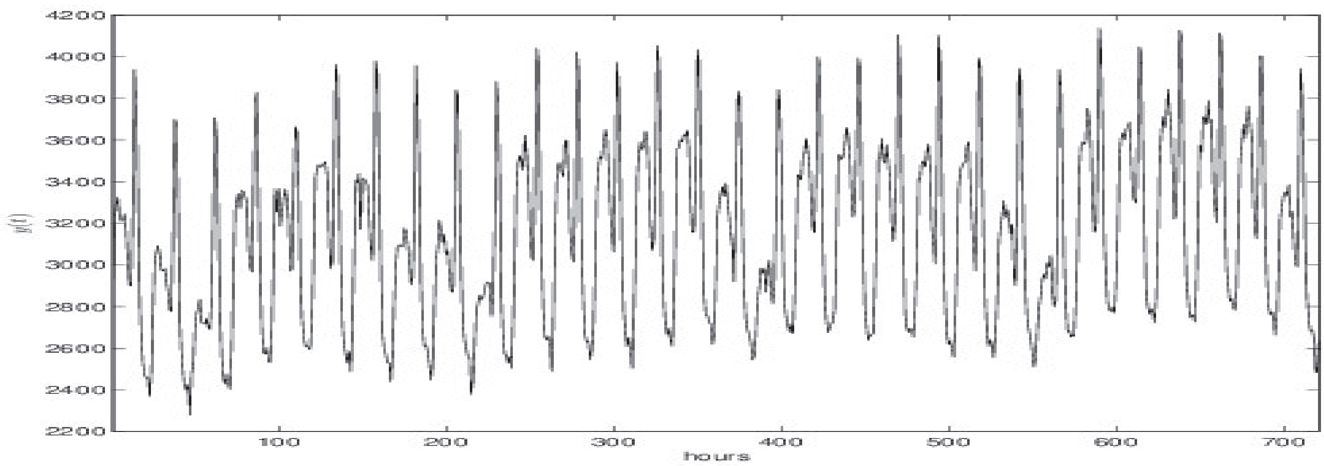
7.5 Application to real data

The proposed method is now applied for real data which consist the evolution of average Algerian electricity consumption each 15mn noted $(y(t))$ throughout the month of September 2001, with length $n = 2880$ observations, Some descriptive statistics of such series are given in table (5)

	mean	Std.Dev	Median	Max	Min	Skewness	Kurtosis	J. Bera
$10^{-3} * y(t)$	3.1791	0.4801	3.1770	4.4200	2.3460	0.4714	2.5413	0.00010

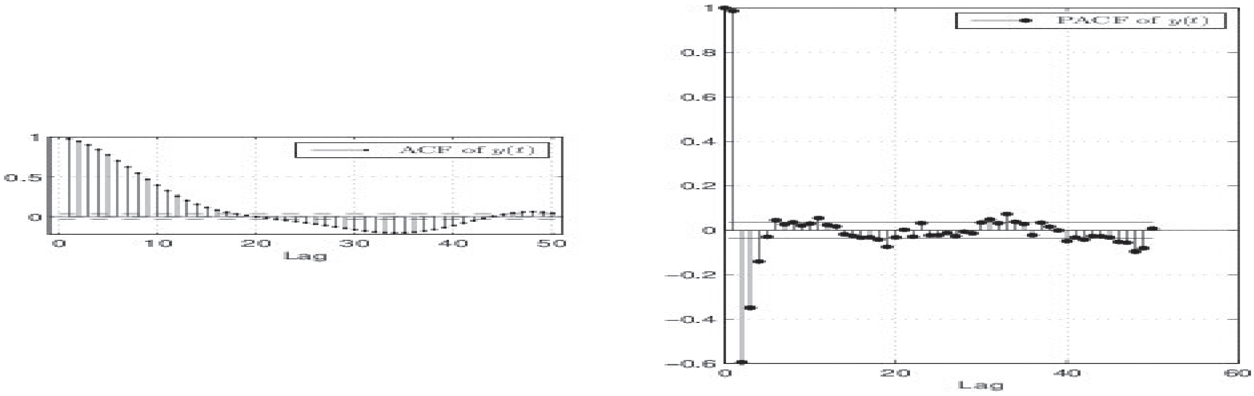
Table (5): Descriptive statistics of the series $(y(t))_{t\geq 1}$

Its graphic is reported in figure Fig(5)



Fig(5) The graphic of original series.

A first preliminary examination based on the structure of (partial) autocorrelation functions figures displayed in Fig (6) and the J. Bera test reported in Table (5) shows that a discrete nonlinear model with appropriate parameters would well describe the series $(y(t))$.



Fig(6). The ACF and $PACF$ function of the series $y(t)$

For this purpose, we propose a $COBL(1,1)$ model for medelling this series. The parameters corresponding to the fitted according to (7.3.12) model are gathered in following table

$\underline{\theta}$	α	μ	γ
$10^{-3} * YW$ method	-0.0015	4.6387	0.0003
$10^{-3} * ML$ method	-0.0013	4.0356	0.0002

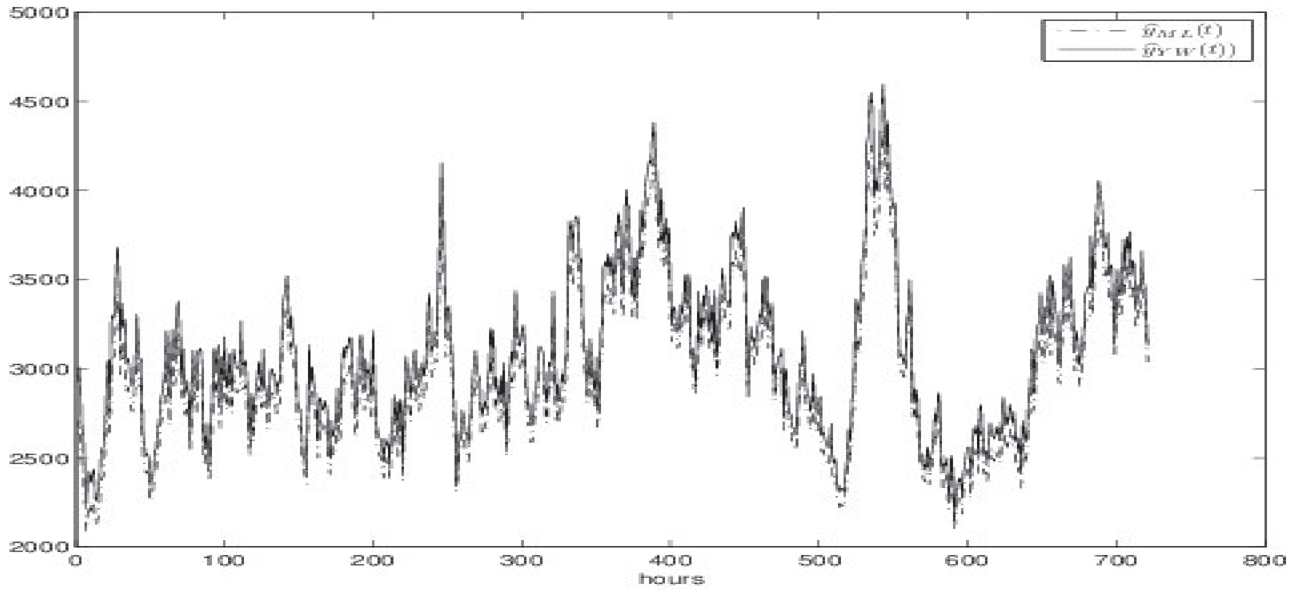
Table (6): Parameters of adjusted series according to $COBL\,(1,1)$

The descriptive statistics of fitted series according to the Table (6) are summarized in the following table

The series	mean	Std.Dev	Median	Max	Min	Skewness	Kurtosis
$10^{-3} * \widehat{y}_{YW}(t)$	3.0908	0.4452	3.0404	4.6524	2.1627	0.6297	3.1363
$10^{-3} * \widehat{y}_{ML}(t)$	0.9500	0.4244	2.8996	4.3863	2.0871	0.5681	3.0014

Table (7) : Descriptive statistics of the series $(\widehat{y}_{YW}(t))_{t\geq 1}$ and $(\widehat{y}_{ML}(t))_{t\geq 1}$

The results in Table (7) of fitted series according to (7.3.12) model, reveal a noticeable resemblance with the results of the original series displayed in Table (5). Moreover, the graphics stacking of the series $(\widehat{y}_{YW}(t))_{t\geq 1}$ and $(\widehat{y}_{ML}(t))_{t\geq 1}$ are shown the following figures below



Fig(7). Plot of fitted series.

7.6 Conclusion

In first part of this chapter, we have studied the higher-order moments of a diffusion process with time-varying coefficients via Itô formula. In particular, in time-invariant case, an explicit expression of the moments for any order are given. In the second part, we have proposed the Yulk-Walter type estimator for estimating a such processes. The method proposed is based on the $CARMA$ representation. Finally, we investigated the empirical study of our estimators via monte Carlo simulation in order to highlight the theoretical results. The method is also applied to modelling the electricity consumption sampled each 15mn in Algeria.

Bibliography

[1] Ait-Sahalia, Y. (2002). Maximum likelihood estimation of discretely sampled diffusion: A closed-form approximation approach. Econometrica, 70(1), pp. 223-262.

[2] Arató, M . (1982) Linear Stochastic systems with Constant coefficients: a Statistical Approach, Lecture notes in control and information sciences 45. Berlin: Springer.

[3] Arnold, L. (1974). Stochastic differential equations, theory and applications, *J. Wiley, New york.*

[4] Applebaum, D. (2004). Lévy processes and stochastic calculus, *Cambridge University Press.*

[5] Beran, J. (1994). Statistics for Long-Memory Processes. *Champan and Hall, London.*

[6] Bergstrom, A.R. (1990). Continuous Time Econometrics Modelling. Oxford, U.K.

[7] Bernet, ∅. (2000). Stochastic differential equations: An introduction with applications. Springer-Verlag.

[8] Bendr , C., P. Parczewski, (2010). Approximating a geometric fractional Brownian motion and related processes via discrete Wick calculus Bernoulli 16(2), 389-417.

[9] Bibi, A.(2006). Evolutionary transfer functions of bilinear processes with time varying coefficients. *Comp. and Maths. with App.* 52, 331-338.

[10] Bibi, A. and F. Merahi (2017). Moment method estimation of first-order continuous-time bilinear processes. To appear in Com. Stat. Comp. Simul.

[11] Bibi, A. and F. Merahi (2015). A note on $\mathbb{L}_2$-structure of continuous-time bilinear processes with time-varying coefficients. Inter. J. of Statist. Prob. Vol. 4, No. 3, pp. 150-160.

[12] Brillinger, D. R. (1975). Time series: Data analysis and theory. New York, Hold, Reinhart.

[13] Billingsley, P. (1995) *Probability and measure.* (3rd Edition) Wiley-Interscience.

[14] Brockwell, P., E. Chadraa and A. Lindner (2006). Continuous-time GARCH processes. *Annals. Prob.* 16(2), 790-826

[15] Brockwell, P. J. (2001). Continuous-time ARMA processes. *Handbook of statistics.* 19, 249-276. North holland, Amsterdam.

[16] Brockwell, P. J. and Davis, R. A. (1987). Time series : Theory and methods. *Springer*, New York.

[17] Broze,L. O. Scaillet and J-M. Zakoïan (1988). Quasi-indirect inference for diffusion processes. Econometrics Theory 4(2), Issue 2, pp. 161-186.

[18] Chan, K. C., G. Karolyi, F.A., Longstaff, and A.B., Sanders (1992). An Empirical Comparison of Alternative Models of the Short-Term Interest Rate. Journal of finance, 47(3), pp. 1209-1227.

[19] Dacuna-Castte, D. and Florens-Zmirou, D. (1 986). Estimation of the coefficient of a diffusion from discrete observations. Stochastics 19, 263-284.

[20] Dai, W and Heyde, C. C (1996). Itô's formula with respect to fractional Brownian motion and its application. *Journal of Applied Mathematics and Stochastic Analysis,* 9 Number 4, 439-448.

[21] Duncan, T.E. (2006) Some bilinear stochastic equations with a fractional Brownian motion. In J. Menaldi and E. Rofman and A Sulem, editors: Stochastic Processes, Optimization, and Control Theory Applications in Financial Engineering, Queueing Networks, and Manufacturing Systems, pp. 97-108. Springer.

[22] Duncun, T.E., B. Pasik-Duncun (2002). Parameter identification for some linear systems with fractional Brownian motion. Proc. IFAC, 15th Triennial World Congress, Barcelona, 21–26 July.

[23] Dobrushin, R.L., (1979). Gaussian and their subordinated self-similar random generalized fields. *Ann. of Prob.* 7, pp. 1-28.

[24] Doob, J. L. (1953). Stochastic Process. Wiley, New York.

[25] Dunford, N., and J.T. Schwartz (1963). Linear operators, Part II. J. Wiley & Sons.

[26] Dzhaparidze, K.O., and A.M. Yaglon (1983) Spectrum parameter estimation in time series analysis. Developments in Statistics, Vol. 4, Ch. 1, 1-96, New York: Academic Press

[27] Florens-Zmirou, D. (1989). Approximate discrete time schemes for statistics of diffusion processes. Statistics 20, 547-557.

[28] Giraitis, L., and P.M., Robinson (2001) Whittle estimation of ARCH models Econometrics theory, 17, 608-631.

[29] Goncalves, E. and Martins, C.M. and N. Mendes-Lopes,N. (2014). The Taylor property in non-negative bilinear models.

[30] Granger, C.WJ., and Andersen, A.P. (1978). An introduction to bilinear time series models. Vandenhoeck and Rpurecht, Gottingen.

[31] Guyon, X., Souchet, S. (2002), Estimation de Yule-Walker d'un CAR(p) observé à temps discret, Ann. I. H. Poincaré - PR 38, 6 (2002) pp. 1093-1100.

[32] Haug, S., C. Klüppelberg, A. Lindner and M. Zapp (2007). Method of moment estimation in the COGARCH(1,1) model. Econometrics Journal, 10, pp. 320-341.

[33] Hansen, L. P. (1982), Large Sample Properties of Generalized Method of Moments Estimators, Econometrica,5 0, 1029-1054.

[34] Hyndman, R.J. (1993) Yule–Walker estimates for continuous-time autoregressive models, JTSA 14 (3), pp. 281–296.

[35] Hannan, E. (1973) The asymptotic theory of linear time series models. J. App. Prob. 10, 130-145.

[36] Ibragimov, I.A., and Y.A. Rozanov (1978). Gaussian random processes. Springer-Verlag.

[37] Ibragimov, I.A., and Y.V. Linnik (1971). Independent and stationary sequences of random variables. Wolters- Noordho Publishing, Groningen.

[38] Igloti, E. and G. Terdik (1999). Bilinear stochastic systems with fractional Brownian motion input, *The Annals of Applied Probability,*Vol.9, No1, 46-77.

[39] Kallsen, J. and J. Muhle-Karbe (2011). Method of moment estimation in time-changed Levy models. Statistics&Decisions, 28, pp. 169-194.

[40] Kelley W.G., and A. Peterson (2010). The theory of differential equations. Springer Verlag.

[41] Kessler, M. (1997). Estimation of an ergodic diffusion from discrete observations. Scand. J. Statistics. 24, pp. 211-229.

[42] Kluppelberg, C., A.Lindner and R. Maller (2004). A continuous time GARCH process driven by a Lévy process: Stationarity and second order behaviour. *J. Appl. Probab.* 41 pp. 601-622.

[43] Kutoyants, A. Yury (2004). Statistical inference for ergodic diffusion processes. Springer-Verlag London.

[44] Le breton, A. and M. Musiela (1980). A study of one-dimensional bilinear differential model for stochastic processes, R.R. no. 221 Lab. I.M.A.G,*Univ. Grenoble 1, France.*

[45] Le breton, A. and M. Musiela (1984). A study of one-dimensional bilinear differential model for stochastic processes, *probaability and mathemtical statistics,* Vol.4, Fase.1, 91-107.

[46] Le breton, A. and M. Musiela (1983). A look at a bilinear model for multidimensional stochastic systems in continuous time. Statistics & Decisions 1, pp. 285-303.

[47] Leon, J.A., and V. Perez-Abreu (1993). Strong solutions of stochastic bilinear equations with anticipating drift in the first Wiener chaos. In Cambanis, S., Ghosh, J.K., Karandikar, R. and Sen, P.K. (eds.). Stochastic processes: A Festschrift in Honor of Gopinath Kallianpur. Springer Verlag, pp. 235-243.

[48] Lin, S.J (1995). Stochastic analysis of fractional Brownian motions, *An International Journal of Probability and Stochastic,* Vol.55, 121-140.

[49] Lipcer, R. S and Shiryayev, A.N (1978). Statistics of random processes. I, II, *Springer-Verlag, Berlin.*

[50] Major, P. (1981). Multiple Wiener-Itô integrals. *Lecture Notes in Mathematics* 849, Springer-Verlag, New York.

[51] Mishura,Y. (2008). Stochastic calculus for fractional Brownian motion and related processes. Lecture Notes in Mathematics No. 1929. Springer-Verlag

[52] Mohler. R. R. (1988). Nonlinear time series and signal processing. Lecture notes in control and information sciences N0.106. Springer Verlag.

[53] Mohler, R.R (1973). Bilinear control processes. *Academic Press.*

[54] Oesook, L. (2012). Exponential Ergodicity and $\beta-$Mixing Property for Generalized Ornstein-Uhlenbeck Processes. Theoretical Economics Letters, Vol. 2, pp.21-25.

[55] Peccati, G., and M.S. Taqqu (2011). Wiener Chaos: Moments, Cumulants and Diagrams: A survey with computer implementation. Springer.

[56] Pham, D.T and A. Le Breton (1991) Levinson-Durbin-type algorithms for continuous-time autoregressive models and applications. Math. of Cont., Signals and Systems, Vol. 4(1), pp 69–79

[57] Popenda, J. (1987). One expression for the solutions of second order difference equations, Proc. Amer. Math. Soc., 100, 87-93.

[58] Prakasa Rao, B. L. S. (2010). Statistical Inference for Fractional Diffusion Processes. Wiley.

[59] Rémillar, B. (2013). Statistical methods for financial engineering. CRC Press. Taylor & Francis Group

[60] Shen, G., X. Yin and L.Yan (2016). Least squares estimation for Ornstein-Uhlenbeck processes driven by the weighted fractional Brownian motion. Acta Mathematica Scientia, Ser. B Engl. Ed., 36(2):394-408.

[61] Souchet, S and Guyon, X (2002). Estimation de Yule-Walker d'un CAR(p) observé à temps discret, *Ann. I. H. Poincaré-* PR 38, 6 (2002) 1093-1100

[62] Subba Rao, T., and G. Terdik (2003). On the theory of discrete and continuous bilinear time series models, *Handbook of Statistics* 21, 827-870.

[63] Swishchuk, A. (2013). Modeling and pricing swaps for financial and energy markets with stochastic volatilities. World Scientific, Publishing Co. Pte. Ltd. Singapore.

[64] Terdik, G. (1990). Stationary solutions for bilinear systems with constant coefficients. Progress in Probability, Vol. 18, pp. 197-206.

[65] Terdik, G. and Subba, R (1989). On Wiener-Itô representation and the best linear predictions for bilinear time series. J. Appl. Prob. 26, pp. 274-286.

[66] Terdik, G. (1985). Transfer functions and conditions for stationarity of bilinear models with Gaussian residuals. Proc. Roy. Soc. London A 400, 315-330.

[67] Tsai, H., and K.S. Chan (2005). Quasi-maximum likelihood estimation for a class of continuous-time long memory processes. J. Time Ser.Analys. Vol. 26, pp. 691-713.

[68] van der Vaart, A. W. (1998). Asymptotic statistics. Combridge University Press.

[69] Wegman, E.J., Schwartz, S.C. and Tomas, J.B. (eds). (1989). Topics in non-Gaussian gignal processing. Springer-Verlag, New York.

[70] Yagloma, . M. (1987). Correlation Theory of Stationary and Related Random Functions, Vol. 1, Basic Results. Springer, New York.

ABSTRACT

In this thesis, we are studying a class of continuous-time bilinear processes ($COBL(1,1)$) generated by some stochastic differential equations where we have investigate some probabilistic properties and statistical inference. We use Itô approach for studying the $\mathbb{L}_2$ structure of the $COBL(1,1)$ process and its powers for any order with time varying coefficients. Furthermore we prove that these results can be obtained by using the transfer functions approach, moreover, by the spectral representation of the process, we give also conditions for the stability of moments, in particular the moments of the quadratic process provide us to checking the presence of the so called Taylor property for $COBL(1,1)$ process. In a second part of this thesis, we use the results of the first part and we propose some methods of estimation for involving unknown parameters, so, we starting by the moments method (MM) to estimate the parameters by two methods, taking into consideration the relation that exists between the moments of the process and its quadratic version and those associate with the incremented processes where we have showed that the resulting estimators are strongly consistent and asymptotically normal under certain conditions. Using the linear representation of $COBL(1,1)$ process, we are able to propose three other methods, one is in frequency domain and the rest are in time domain and we prove the asymptotic properties of the proposed estimators. Simulation studies are presented in order to illustrate the performances of the different estimators, furthermore, this methods are used to model some real data such as the exchanges rate of the Algerian Dinar against the US-dollar and against the single European currency and the electricity consumption sampled each 15mn in Algeria.

Keywords: Continuous-time bilinear processes, Spectral representation, Itô's solution, Stationarity, long memory property, Taylor effect, Quadratic processes, $(G)\,MM$ estimation, Yule-Walker estimates, Maximum Likelihood estimates, Strong consistency, Asymptotic Normality.

RÉSUMÉ

Dans cette thèse, nous étudions une classe d'équations différentielles stochastiques non linéaires (COBL (1,1)) où nous étudions ses propriétés probabilistes et leur inférence statistique. Ainsi, nous utilisons l'approche d'Itô pour étudier la structure $\mathbb{L}_2$ du processus $COBL(1,1)$ à coefficients dépendent du temps et ses puissances pour tout ordre. Plus précisément, en utilisant la représentation spectrale du processus, nous donnons des conditions de stabilité des moments, en particulier les moments du processus quadratique. Nous étudions également la présence de la propriété de Taylor pour cette classe de processus. Dans la deuxième partie de la thèse, nous utilisons les résultats de la première partie pour proposer des méthodes d'estimation des paramètres des paramètres inconnus impliqués dans le modèle $COBL(1,1)$. A cet effet, nous commençons par la méthode des moments (MM) pour estimer les paramètres par deux méthodes en considérant d'une part la relation qui existe entre les moments du processus et sa version quadratique et d'autre part avec les moment des processus des incréments associés, les estimateurs proposés sont fortement consistants et asymptotiquement normaux sous certaines conditions imposé es. En utilisant la représentation linéaire du processus COBL (1,1), nous proposons cependant trois autres méthodes d'estimations, l'un dans le domaine fréquentiel et les autres sont dans le domaine temporel et nous prouvons la consistance forte et la normalité asymptotique des estimateurs que nous introduisons. Des études de simulation sont présentées afin d'illustrer les performances des différents estimateurs étudiés. De plus, ces méthodes sont utilisées pour modéliser des données réelles telles que le taux de changes du Dinar algérien par rapport au dollar US et par rapport à la monnaie unique européenne et à la consommation algérienne de l'électricité échantillonnée chaque 15mn.

Mots clés: Processus bilinéaire à temps continu, Représentation spectrale, Solution de Itô, Stationnarité, Propriété de longue mémoire, Propriété de Taylor, Processus quadratique, Estimation GMM, Estimations de maximum vraisemblance, Consistance forte, Normalité Asymptotique.

ملخّص

في هذه الأطروحة، قمنا بدراسة فئة من المعادلات التفاضلية العشوائية و التي هي المعادلة التفاضلية العشوائية ثنائية الخطِّية حيث قمنا بدراسة خصائصها الإحتمالية والإستدلال الإحصائي. قمنا باستعمال مقاربة إيتو لدراسة بنية الدرجة الثانية لهذا النمط العشوائي مع معاملات متغيّرة مع الزّمن ، وتبين لنا أن هذه النتائج يمكن الحصول عليها عن طريق منهجية الدّوال التّردّديّة، وبعبارة أخرى باستخدام التمثيل الطيفي لهذا للنّمط العشوائي، نعطي أيضا شروط استقرار العزوم ، ولا سيما عزوم النّمط العشوائي التّربيعي و التي توفر لنا التحقق من وجود خاصّية تايلور بالنّسبة لنمط عشوائي ثنائي الخطّيّة. في الجزء الثاني من الرسالة استخدمنا نتائج الجزء الأول لاقتراح بعض طرق التقدير لهذا النّمط الذي ينطوي على معاملات غير معلومة ، لذلك بدأنا بطريقة العزوم لتقدير المعاملات بطريقتين إذا أخذنا بعين الاعتبار العلاقة القائمة بين عزوم النّمط ثنائي الخطّيَّة ونمطه التربيعي مع عزوم نمط الزيادات المرفق، المقدرين الناتجين تتميّز بالكفاءة القويّة و التّقارب الطّبيعي في ظل بعض الشروط المفروضة. نتيجة مهمّة تتمثّل في التّمثيل الخطّي للنّمط العشوائي ثنائية الخطِّية أستخدمت لاقتراح ثلاث مقدّرات أخرى ، مقدّر واحد في المجال التّردّدي ومقدّرين آخرين في المجال الزمني، وقمنا بإثبات التّقارب الطّبيعي لهذه المقدِّرات التّي قمنا بتقديمها. وقدّمنا دراسة محاكاة لتوضيح أداء مختلف المقدرين المدروسين، وعلاوة على ذلك، تستخدم هذه الأساليب لنمذجة بعض البيانات الحقيقية مثل سعر التبادل للدينار الجزائري مقابل الدولار الأمريكي و مقابل العملة الأوروبية الموحدة وعيّنات استهلاك الكهرباء كل 15 دقيقة في الجزائر.

الكلمات المفتاحية: النمط العشوائي ثنائي الخطِّية بأزمنة مستمرّة ، تمثيل طيفي، حل إيتو، الإستقراريّة ، خاصية ذاكرة طويلة، خاصية تايلور، نمط عشوائي تربيعي، مقدّر العزوم المعمّم، تقدير الإحتمال الأقصى، الكفاءة القويّة ، التّقارب الطّبيعي