REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE
MINISTÈR DE L'ENSEIGNEMENT SUPÉRIEUR ET DE LA RECHERCHE SCIENTIFIQUE
UNIVERSITÉ CONSTANTINE 1
Faculté des sciences exactes
Département des Mathématiques

N° d'ordre:.....78/DS/2013......................

Série:.......12/Math/2013.........................

THÈSE

Présentée pour l'obtention du diplôme de Docteur en sciences en Mathématiques

Option: Equations différentielles et applications

Par

Mohammed-Salah Abdelouahab

Intitulée

Les systèmes chaotiques à dérivées fractionnaires

Soutenue le:..*28/11/2013*............................

Devant le jury

Président:	M. Denche	*Professeur, Université Constantine 1*
Rapporteur:	N. Hamri	*Professeur, Centre Universitaire de Mila*

Examinateurs:

- E. Zeraoulia — *Professeur, Université de Tebessa*
- M. Deghdak — *Professeur, Université Constantine 1*
- K. Haouam — *Maître de conférences, Université de Tebessa*
- A. Boukabou — *Maître de conférences, Université de Jijel*

Année Universitaire 2012-2013

Acknowledgements

There are many people to whom I am very obliged for their support.

Firstly I would like to express my great thanks and gratitude to my supervisor Professor Nasr-Eddine Hamri for his guidance and his steady faith in me, throughout the realization of this work.
I am thankful to Professor Mohammed Denche for the honor he made me, of having kindly agreed to chair my thesis commitee.
I wish also to express all my gratitude to Professor Elhadj Zeraoulia, to Professor Messoud Deghdak, to Doctor Kamel Haouam and to Doctor Abdelkrim Boukabou for their willingness to read and examine my thesis, and for their help during the realization of this work.
I wish to acknowledge appreciated support from Professor René Lozi of the Laboratory J.A. Dieudonné-UMR du CNRS n^o6621 University Nice France, for all the facilities he afforded to me in this Laboratory.
I am thankful to Professor Abdelfetah Charef for his help and appreciated support.

During the realization of this thesis there, many colleagues encouraged and helped me, but I particularly thank R. Bouden, L. Mermoul, T. Mermoul, M. Bouchoucha, H. Djemmada, H. Khiat, H. Dambri and M. Kedjbour.

I am also thankful to my father, to my wife and our sons for their everyday patience, understanding, love and support.

Dedicated to the memory of my mother.

Contents

List of Figures

General introduction

The subject of fractional calculus is, in a certain sense, as old as classical calculus [64] as we know it today: Its origins date back to the end of the seventeenth century, also the time when Newton and Leibniz developed the theory of differential and integral calculus. Namely, Leibniz introduced the symbol $\frac{d^n f}{dt^n}$ to denote the nth derivative of a function f in a letter to de l'Hospital (with the assumption that $n \in \mathbf{N}$), de l'Hospital replied: "What does $\frac{d^n f}{dt^n}$ mean if $n = \frac{1}{2}$?"
Leibniz wrote prophetically, "Thus it follows that $d^{\frac{1}{2}}x$ will be equal to $x\sqrt[2]{dx : x}$, an apparent paradox, from which one day useful consequences will be drawn."
The fact that de l'Hospital specifically asked for $n = \frac{1}{2}$ (i.e. a fraction or rational number), gave rise to the name of this part of mathematics. But it was not until the first half of the 19th century that the theory of generalized operators achieved a level in its development suitable as a point of departure for the modern mathematician. By then the theory had been extended to include operators D^α, where α could be rational or irrational, positive or negative, real or complex. Thus the name fractional calculus became somewhat of a misnomer. A better description might be differentiation and integration to an arbitrary order. However, we shall adhere to tradition and refer to this theory as the fractional calculus. It was Liouville [47] who expanded functions in series of exponentials and defined the αth derivative of such a series by operating term-by-term as though α were a positive integer. Riemann [63] proposed a different definition that involved a definite integral and was applicable to power series with non-integer exponents. Evidently it was Grünwald and Krug who first unified the results of Liouville and Riemann. Grünwald [33], disturbed by the restrictions of Liouville's approach, adopted as his starting point the definition of a derivative as the limit of a difference quotient and arrived at definite-integral formulas for the αth derivative. Krug [40], working through Cauchy's integral formula for ordinary derivatives, showed that Riemann's definite integral had to be interpreted as having a finite lower limit

while Liouville's definition, in which no distinguishable lower limit appeared, corresponded to a lower limit $-\infty$. These theoretical beginnings was accomplished by a parallel development of the applications of the fractional calculus to various problems. The first of these was the discovery by Abel [5] in 1823 that the solution of the integral equation for the tautochrone could be accomplished via an integral transform, which, benefits from being written as a semi-derivative. An important next step in the application of fractional derivatives was the operational calculus of Heaviside, developed to solve certain problems of electromagnetic theory. Namely Heaviside [37] introduced fractional differentiation in his investigation of transmission line theory; and this concept has been extended by Gemant [32] for use in problems of elasticity.

Although the birthday of fractional calculus date back to the end of the seventeenth century and the first steps of the theory itself and some applications traced back to the first half of the nineteenth century, the subject only really came to life over the last few decades. A particular feature is that fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional models in comparison with classical integer-order models, in which such effects are in fact neglected, another feature is that engineers and scientists have developed new models that involve fractional differential equations in mechanics (theory of viscoelasticity and viscoplasticity), bio-chemistry (modelling of polymers and proteins), electrical engineering (transmission of ultrasound waves), medicine (modelling of human tissue under mechanical loads), etc. The first book which was entirely devoted to a systematic presentation of the ideas, methods and applications of the fractional calculus is the book written by K. U. Oldharn and J. Spanier [60].

The first conference devoted to the topic of fractional calculus took place in 1974, in New Haven, USA. Circumstances have changed considerably since then. in the last decades the general interest in such a tool has experienced a continuing growth and at present many conferences, symposia, workshops, or special sessions are found, as well as papers and special issues in recognized journals, devoted to the theoretical and application aspects of fractional calculus.

On the other hand, in recent years, many scientists have become aware of the potential use of chaotic dynamics in engineering applications, such as electrical engineering, information processing, secure communications, etc...

With the development of the fractional-order algorithm, the dynamics of fractional-order systems have received much attention. Studying chaos in fractional-order dynamical systems is an interesting topic as well. It is well known that chaos cannot occur in continuous integer order systems of total order less than three due to the Poincaré-Bendixon theorem. It has been shown that many fractional-order dynamical systems behave chaotically with total order less then three. The thesis consists of two parts.

The first part is devoted to the fractional calculus and it contains two chapters (chapter 1 and chapter 2).
In chapter 1 some preliminary concepts are introduced, including the Laplace transform and their basic properties, special functions (the gamma and the beta function, the Mitag-Leffler function) which play the most important role in the theory of fractional derivatives and fractional differential equations.

In chapter 2 three approaches (Riemann-Liouville, Grünwald-Litnicov and Caputo approaches) to the generalization of the notions of derivation and integration are considered. In the end of this chapter some methods of treatment of the fractional differential equations are introduced including numerical algorithm. The second part is devoted to the concept of fractional-order dynamical systems and applications, it is divided into two chapters (chapter 3 and chapter 4).
In chapter 3 a generalization of notion of dynamical systems (Fractional-order dynamical systems) is considered including stability theory, periodic solutions, Bifurcations and chaos, it is shown that all most classical criterion and tools for the study of dynamical systems have been reformulated in a general sitting and used for the study of fractional-order dynamical systems.
In chapter 4 we present our three papers [1–3] given as applications of the mathematical tools introduced in the previous chapters. The thesis is ended by a general conclusion and perspectives.
More than 80 references are listed and cited in this thesis, even if it cannot be a complete bibliography for this area of interest. We can find many other references related to this topic.

Part I

Fractional calculus

Chapter 1

Preliminaries

In this chapter we, briefly, introduce some necessary but relatively simple mathematical tools that will arise in the study of the concepts of fractional calculus. These are the Laplace transform, the Gamma function, the Beta function and the Mittag-Leffler Function.

1.1 The Laplace transform

"What we know is not much. What we do not know is immense."

Pierre-Simon Laplace

The Laplace transform is a powerful tool that we shall exploit in our investigation of fractional differential equations. Our purpose in this section is to present the definition and some basic properties of the Laplace transform then we derive some transforms and inverse transforms of functions that arise frequently in this study. We denote the Laplace transform of a function $f(t)$ by the symbol $\mathcal{L}\{f(t)\}$, or when convenient, by $F(s)$. More detailed information may be found in [29, 30].
The Laplace transform of a function $f(t)$ of a real variable $t \in \mathbb{R}^+$ is formally defined by

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st} f(t)dt, \quad (s \in \mathbb{C}). \tag{1.1}$$

If the integral in (1.1) is convergent at $s_0 \in \mathbb{C}$, then it converges absolutely for $s \in \mathbb{C}$ such that $Re(s) > Re(s_0)$.

The inverse Laplace transform is given for $t \in \mathbb{R}^+$ by the formula

$$\mathcal{L}^{-1}\{g(s)\}(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} g(s) ds, \quad (\gamma = Re(s_0)). \tag{1.2}$$

Obviously, $\mathcal{L}$ and $\mathcal{L}^{-1}$ are linear integral operators. The direct and the inverse Laplace transforms are inverse to each other for "sufficiently good" functions f and g

$$\mathcal{L}^{-1}\mathcal{L}\{f\} = f \text{ and } \mathcal{L}\mathcal{L}^{-1}\{g\} = g.$$

1.1.1 Existence conditions for the Laplace transform

Theorem 1.1.
Let f be a continuous or piecewise continuous function in every finite interval $(0, T)$. If $f(t)$ is of exponential order e^{at}, then the Laplace transform of $f(t)$ exists for all s such that $Re(s) > a$.

Proof.
Suppose that f, is of exponential order e^{at}, then there exists a positive constant K such that for all $t > T$

$$|f(t)| \leqslant Ke^{at}.$$

We have

$$\begin{aligned} \left| \int_0^\infty e^{-st} f(t)dt \right| &\leq \textstyle\int_0^\infty e^{-st}|f(t)|dt, \\ &\leq K \textstyle\int_0^\infty e^{-t(s-a)}dt = \frac{K}{s-a}, \end{aligned}$$

for $Re(s) > a$.

This complete the proof. □

1.1.2 Basic properties of the Laplace transform

a) Heavisides first shifting property

Theorem 1.2.
For a real constant a *we have*

$$\mathcal{L}\{e^{-at}f(t)\}(s) = F(s+a)$$

where $F(s) = \mathcal{L}\{f(t)\}(s)$.

Proof.
By definition we have

$$\mathcal{L}\{e^{-at}f(t)\}(s) = \int_0^\infty e^{-(s+a)t}f(t)dt = F(s+a).$$

□

b) Scaling property

For a constant $a \neq 0$, we have

$$\mathcal{L}\{f(at)\}(s) = \frac{1}{|a|}F\left(\frac{s}{a}\right).$$

c) The Laplace transform of derivatives

To find the Laplace transform of a derivative, we integrate the expression (1.1) by parts. Then, we obtain

$$\int_0^\infty e^{-st}f(t)dt = \left[\frac{-f(t)e^{-st}}{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}f'(t)dt.$$

Evaluating the limits and multiplying by s gives the following

$$s\mathcal{L}\{f(t)\}(s) = f(0) + \mathcal{L}\{f'(t)\}(s).$$

This gives the Laplace transform of $f'(t)$ as follows

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0).$$

This can be continued for higher order derivatives (replacing $f(t)$ by $f'(t)$ in the above equation) and gives the following expression for the Laplace transform of the nth derivative of $f(t)$.

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n\mathcal{L}\{f(t)\}(s) - \sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0). \tag{1.3}$$

d) Convolution Property

If $\mathcal{L}\{f(t)\}(s) = F(s)$ and $\mathcal{L}\{g(t)\}(s) = G(s)$, then

$$\mathcal{L}\{f(t) * g(t)\}(s) = F(s)G(s). \tag{1.4}$$

Or, equivalently,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t),$$

where $f(t) * g(t)$ is the convolution of $f(t)$ and $g(t)$ defined by the integral

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

1.1.3 The Laplace transform of some usual functions

1. $\mathcal{L}\{1\}(s) = \int_0^\infty e^{-st}dt = \dfrac{1}{s}$,
2. $\mathcal{L}\{e^{at}\}(s) = \int_0^\infty e^{-(s-a)t}dt = \dfrac{1}{s-a}$, for $s > a$
3. $\mathcal{L}\{\sin(at)\}(s) = \int_0^\infty e^{-st}\sin(at)dt = \dfrac{a}{s^2+a^2}$,
4. $\mathcal{L}\{\cos(at)\}(s) = \int_0^\infty e^{-st}\cos(at)dt = \dfrac{s}{s^2+a^2}$,
5. $\mathcal{L}\{t^n\}(s) = \int_0^\infty t^n e^{-st}dt = \dfrac{n!}{s^{n+1}}$.

1.2 Special functions

In this section, we deal with definitions and some basic properties of the special functions (Gamma, Beta and Mittag-Leffler) these later are essential elements in our coming chapters.

1.2.1 Gamma function

One of the important basic functions of the fractional calculus is the Euler's Gamma function, which generalizes the factorial $n!$, and allows n to take also non-integer and even complex values.

This function plays an important role in the theory of differ-integration. A comprehensive definition of $\Gamma(x)$ is that provided by the Euler limit [60]

$$\Gamma(x) = Lim_{N\to\infty}[\frac{N!N^x}{x[x+1][x+2]...[x+N]}],$$

But the so-called Euler integral definition:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad (x>0), \tag{1.5}$$

is often more useful, although it is restricted to positive values of x. An integration by parts applied to the definition (1.5) leads to the recurrence relationship

$$\Gamma(x+1) = x\Gamma(x).$$

Since $\Gamma(1) = 1$, this recurrence shows that for a positive integer n, we have

$$\Gamma(n+1) = n\Gamma(n) = n[n-1]\Gamma(n-1) = ... = n[n-1]...2.1.\Gamma(1) = n!.$$

Rewritten as

$$\Gamma(x-1) = \frac{\Gamma(x)}{x-1}. \quad x-1>0$$

Using this relation, the Euler Gamma function is extended to negative arguments for which definition (1.5) is inapplicable. The graph of the gamma function is shown in Figure(1.1).

1.2.2 Beta function

The function that is closely related to the gamma function is the complete beta function $B(x,y)$. For positive values of the two parameters, x and y, this function is defined by the Beta integral:

$$B(x,y) = \int_0^1 t^{x-1}[1-t]^{y-1}dt. \quad (x>0,\ y>0) \tag{1.6}$$

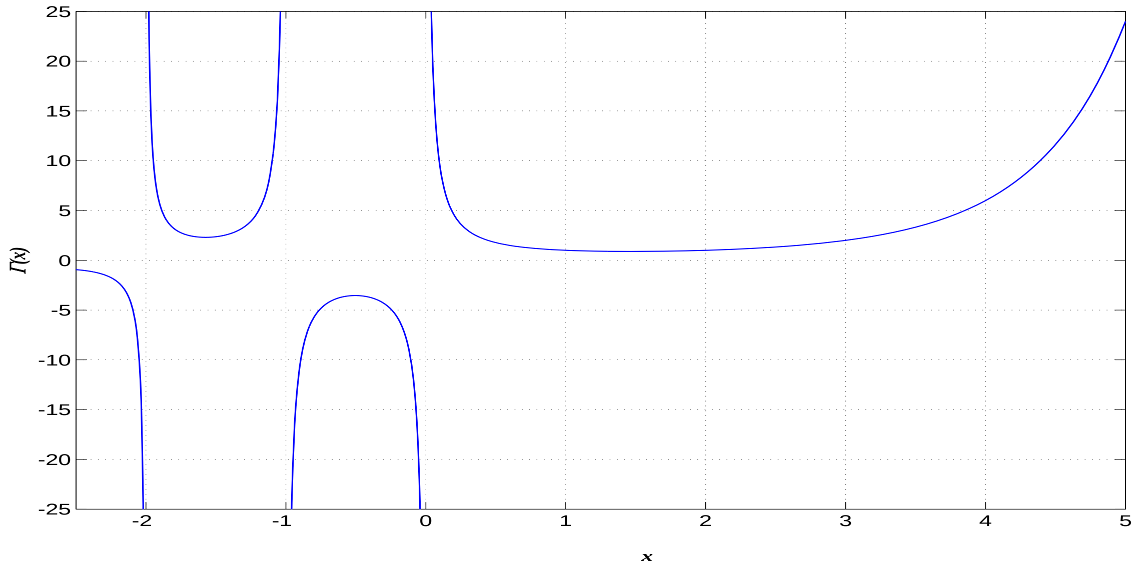

FIGURE 1.1: Graphical representation of Euler Gamma function

The beta function $B(x, y)$ is symmetric with respect to its arguments x and y, that is, $B(x, y) = B(y, x)$.
This follows from (1.6) by the change of variables $1 - t = u$, that is

$$B(x, y) = \int_0^1 u^{y-1}[1-u]^{x-1} du = B(y, x).$$

Using the Laplace transform, we can prove that this function is connected with the Gamma function by the relation

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \quad (x, y \notin \mathbb{Z}_0^-)$$

Clearly this relationship extended the beta function to negative non-integer arguments for which the definition (1.6) is inapplicable.
With help of the Beta function we can establish the following two important relationships for the Gamma function. The first one is

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad (0 < x < 1),$$

for example $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
The second one is the Legendre formula

$$\Gamma(x)\Gamma(x + \frac{1}{2}) = \sqrt{\pi} 2^{2x-1}\Gamma(2x), \quad (2x \notin \mathbb{Z}_0^-).$$

1.2.3 Mittag-Leffler function

"The mathematicians best work is art, a high perfect art, as daring as the most secret dreams of imagination, clear and limpid. Mathematical genius and artistic genius touch one another."

Gosta Mittag-Leffler

The exponential function e^z, plays a very important role in the theory of integer-order differential equations. Its one-parameter generalization, is the Mittag-Leffler function defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (z \in \mathbb{C}; \quad \alpha > 0). \tag{1.7}$$

This function was introduced by Mittag-Leffler [53, 54], and studied also by Wiman [80].

$E_\alpha(z)$ is an entire function of z. In particular we have

$$E_1(z) = e^z \quad and \quad E_2(z) = cosh(\sqrt{z}).$$

Graphical representations of this function for some values of α are shown in figure(1.2)

Now we shall give some informations about the asymptotic behaviour of this function [26].

Theorem 1.3.
Let $\alpha > 0$, $r > 0$, $\varphi \in [-\pi, \pi]$, then the following statements hold
a) $\lim_{r\to\infty} E_\alpha(re^{i\varphi}) = 0$ *if* $|\varphi| > \alpha\pi/2$.
b) $\lim_{r\to\infty} |E_\alpha(re^{i\varphi})| = \infty$ $|\varphi| < \alpha\pi/2$.
c) $E_\alpha(re^{i\varphi})$ *remains bounded for* $r \to \infty$ *if* $|\varphi| = \alpha\pi/2$.

The following theorem describe the interconnection between the one-parameter Mittag-Leffler function and the Laplace transform operation.

Theorem 1.4.
Let $\alpha > 0$, $\lambda \in \boldsymbol{C}$ and define $x(t) = E_{-\alpha}(\lambda t^\alpha)$, then the Laplace transform of x is given by

$$\mathcal{L}x(s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda} \quad (\lambda \in \boldsymbol{C}; \; Re(s) > 0; \; |\lambda s^{-\alpha}| < 1). \tag{1.8}$$

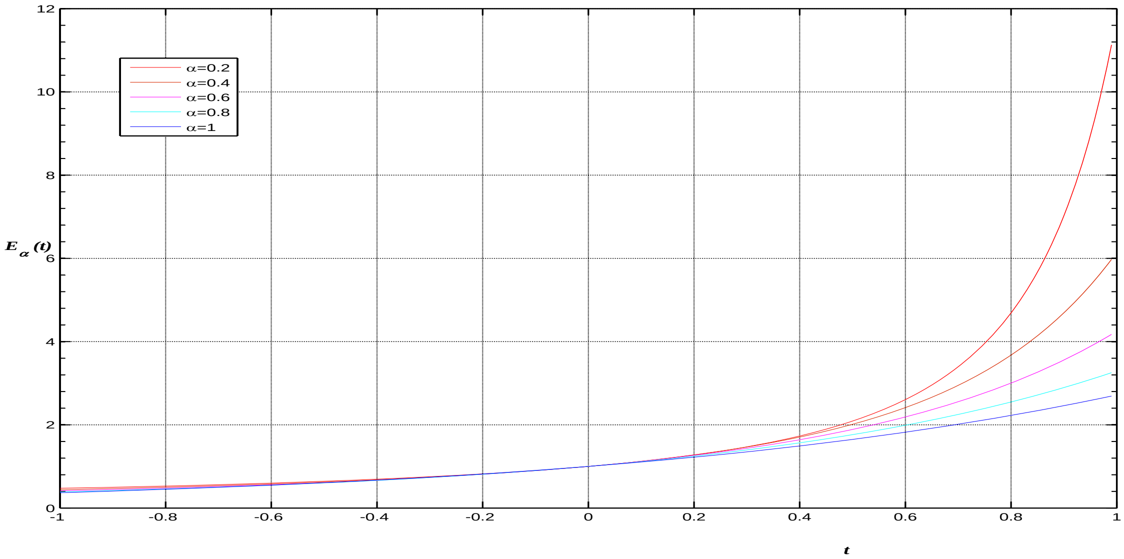

FIGURE 1.2: Graphical representation of the one parameter Mittag-Lefller function for some value of α.

Proof.
Writing down the series expansion of $x(t)$ in powers of t^α gives

$$x(t) = E_\alpha(-\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(-\lambda t^\alpha)^k}{\Gamma(\alpha k + 1)}.$$

Applying the Laplace transform in a term-wise manner yields

$$\begin{aligned}\mathcal{L}x(s) &= \sum_{k=0}^{\infty} \frac{\mathcal{L}\left(-\lambda t^\alpha\right)^k}{\Gamma\left(\alpha k + 1\right)}, \\ &= \frac{1}{s}\sum_{k=0}^{\infty} \left(-\lambda s^{-\alpha}\right)^k, \\ &= \frac{s^{\alpha-1}}{s^\alpha + \lambda}.\end{aligned}$$

□

Differentiating (1.8) n times with respect to λ leads to the following relation

$$\mathcal{L}[t^{\alpha n} E^{(n)}(-\lambda t^\alpha)](s) = \frac{n! s^{\alpha-1}}{(s^\alpha + \lambda)^{n+1}}, \quad (\lambda \in \mathbf{C};\ Re(s) > 0;\ |\lambda s^{-\alpha}| < 1). \quad (1.9)$$

Now, let us introduce an important theorem called "Final value theorem" which gives information about the asymptotic behaviour of the function $f(t)$ directly from his Laplace's transform $F(s)$.

Theorem 1.5. *(Final value theorem)*
Let $F(s)$ be the Laplace transform of the function $f(t)$. If all poles of $sF(s)$ are in the open left-half plane, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

Using theorems (1.4 and 1.5) we obtain a statement on the asymptotic behaviour of the function $x(t) = E_\alpha(-\lambda t^\alpha)$ as its argument tends to infinity:

Theorem 1.6.
Let $\alpha > 0$, $r > 0$, $\varphi \in [-\pi, \pi]$ and $\lambda = r\exp(i\varphi)$. Define $x(t) = E_\alpha(-\lambda t^\alpha)$. Then, the following two statements hold
a) $\lim_{t\to\infty} x(t) = 0$ *if* $|\varphi| < \alpha\pi/2$,
b) $x(t)$ *is unbounded as* $t \to \infty$ *if* $|\varphi| > \alpha\pi/2$.

The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$, generalizing the one in (1.7), is defined by the series expansion [62]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (z \in \mathbb{C};\ \alpha > 0,\ \beta > 0). \tag{1.10}$$

When $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with $E_\alpha(z)$

$$E_{\alpha,1}(z) = E_\alpha(z).$$

From the definition (1.10) we can get

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z,$$

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{e^z - 1}{z}.$$

The hyperbolic sine and cosine are also particular cases of the two-parameter Mittag-Leffler function:

$$E_{2,1}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh(z),$$

$$E_{2,2}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+2)} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \frac{\sinh(z)}{z}.$$

The Mittag-Leffler function satisfies the following differentiation formulas

$$\left(\frac{d}{dz}\right)^n [z^{\beta-1} E_{n,\beta}(\lambda z^n)] = z^{\beta-n-1} E_{n,\beta-n}(\lambda z^n) \quad (n \in \mathbb{N};\ \lambda \in \mathbb{C}).$$

Chapter 2

Fractional integrals and fractional derivatives

In his discovery of calculus, Leibniz first introduced the idea of a symbolic method and used the symbol $\frac{d^n y}{dx^n} = D^n y$ for the nth derivative, where n is a non-negative integer. L'Hospital asked Leibniz about the possibility that n be a fraction. 'What if $n = \frac{1}{2}$?' Leibniz replied: 'It will lead to a paradox.' But he added prophetically, 'From this apparent paradox, one day useful consequences will be drawn '.

From this brief historical introduction we can say that fractional calculus grows out of the classical definitions of the integral and derivative operators, in much the same way fractional exponents is an outgrowth of exponents with integer value.

The meaning of integer exponents is a repeated multiplication of a numerical value, this concept can clearly become confused when considering exponents of non integer value, it is the notation that makes the jump seem obvious. While one can not imagine the multiplication of a quantity a fractional number of times, there seems no practical restriction to placing a non-integer into the exponential position. Similarly, the common formulation for the fractional integral (derivative) can be derived directly from a traditional expression of the repeated integration (differentiation) of a function, and provides an interpolation between integer-order integrals (derivatives).

There are several types of fractional integrals and fractional derivatives. In this chapter we give definitions and some basic properties of three different types (the choice has been reduced to those definitions which are related to applications).

2.1 Riemann-Liouville fractional integral

2.1.1 Definition

According to Riemann-Liouville approach the notion of fractional integral of order $\alpha(\alpha > 0)$ is a natural consequence of the well known formula (usually attributed to Cauchy) that reduces the calculation of the n-fold integral of a function $f(t)$ to a single integral of convolution type.

We begin by a review of the n-fold integral of a function f assumed to be continuous on the interval $[a, b]$, where $b > x$.

First recalling that if $G(x, t)$ is jointly continuous on $[a, b] \times [a, b]$, then we have

$$\int_a^x dx_1 \int_a^{x_1} G(x_1, t)dt = \int_a^x dt \int_t^x G(x_1, t)dx_1. \tag{2.1}$$

In particular when $G(x, t)$ is a function of a variable t only, that is if $G(x, t) \equiv f(t)$ then (2.1) can be written as

$$\begin{aligned} \int_a^x dx_1 \int_a^{x_1} f(t)dt &= \int_a^x f(t)dt \int_t^x dx_1, \\ &= \int_a^x (x-t)f(t)dt, \end{aligned}$$

this is the formula of two-fold integral reduced to a single integral. Similar computation gives the following formula of 3-fold integral reduced to a single integral

$$\int_a^x dx_1 \int_a^{x_1} dx_2 \int_a^{x_2} f(t)dt = \int_a^x \frac{(x-t)^2}{2} f(t)dt.$$

By induction we deduce the Cauchy formula of n-fold integral

$$\begin{aligned} J_a^n f(x) = {}_aD_x^{-n} f(x) &= \int_a^x dx_1 \int_a^{x_1} dx_2 ... \int_a^{x_{n-1}} f(t)dt, \\ &= \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt. \quad x > a, \ n \in \mathbb{N}^* \end{aligned}$$

Using the Gamma function this formula can be rewritten as

$$J_a^n f(x) = {}_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt. \quad x > a, \ n \in \mathbb{N}^* \quad (2.2)$$

Since the gamma function is an analytic expansion of the factorial for all positive real values (section 1.2.1), one can replace n by a real positive number α, in (2.1.1), then one defines the Riemann-Liouville fractional integral of order $\alpha > 0$ as follows

$$J_a^\alpha f(x) = {}_a^{RL}D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a, \ \alpha > 0. \quad (2.3)$$

Example 2.1.

Let $f(x) = (x-a)^\beta$ for a fixed $\beta > -1$ and $\alpha > 0$, we have

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-a)^\beta (x-t)^{\alpha-1} dt.$$

Using the substitution $t = a + s(x-a)$ and the Beta function we get

$$\begin{aligned} J_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} (x-a)^{\alpha+\beta} \int_0^1 s^\beta (1-s)^{\alpha-1} ds \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\alpha+\beta}. \end{aligned}$$

Theorem 2.1.
Let $f \in L_1[a,b]$ and $\alpha > 0$, then the integral $J_a^\alpha f(x)$ exists for almost every $x \in [a,b]$. Moreover, the function $J_a^\alpha f$ itself is also an element of $L_1[a,b]$.

Proof. We have

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt = \int_{-\infty}^{+\infty} \Phi_1(x-t)\Phi_2(t)dt.$$

where

$$\Phi_1(u) = \begin{cases} u^{\alpha-1} & \text{for } \ 0 < u \leq b-a, \\ 0 & \text{else,} \end{cases}$$

and

$$\Phi_2(u) = \begin{cases} f(u) & \text{for } a \le u \le b, \\ 0 & \text{else.} \end{cases}$$

Clearly $\Phi_1,\ \Phi_2 \in L_1(\mathbb{R})$, and thus the desired result follows. □

2.1.2 Some basic properties

- If f is a continuous function for $x \geq a$ then, we have [62]

$$\lim_{\alpha \to 0} J_a^\alpha f(x) = f(x), \tag{2.4}$$

so we can put

$$J_a^0 f(x) = f(x).$$

- Let f be a continuous function for $x \geq a$, we have

$$J_a^\alpha \left(J_a^\beta f(x)\right) = J_a^{\alpha+\beta} f(x). \tag{2.5}$$

In fact we have

$$\begin{aligned} J_a^\alpha \left(J_a^\beta f(x)\right) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\ J_a^\beta f(t)dt, \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha-1}dt \int_a^t (t-s)^{\beta-1} f(s)ds, \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(s)ds \int_s^x (x-t)^{\alpha-1}(t-s)^{\beta-1}dt, \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-s)^{\alpha+\beta-1} f(s)ds, \\ &= J_a^{\alpha+\beta}\left(f(x)\right). \end{aligned}$$

where the integral

$$\begin{aligned} \int_s^x (x-t)^{\alpha-1}(t-s)^{\beta-1}dt &= (x-s)^{\alpha+\beta-1} \int_0^1 (1-y)^{\alpha-1} y^{\beta-1} dy, \\ &= B(\alpha,\beta)((x-s)^{\alpha+\beta-1}, \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-s)^{\alpha+\beta-1}, \end{aligned}$$

is evaluated using the substitution

$$t = s + y(x-s),$$

and the definition of the beta function.

- If $F(s)$ is the Laplace transform of the function $f(x)$ then the Laplace transform of the Riemann-Liouville fractional integral $J_a^\alpha f(x)$ is given by

$$\mathcal{L}(J_a^\alpha f(x))(s) = \frac{F(s)}{s^\alpha}. \tag{2.6}$$

For the proof of (2.6) we introduce the following causal function

$$\Phi_\alpha(t) = \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha > 0,$$

where the suffix + is just denoting that the function is vanishing for $t < 0$. Clearly this function is locally absolutely integrable in $\mathbb{R}^+$, and the Laplace transform of $\Phi_\alpha(t)$ is given by

$$\mathcal{L}(\Phi_\alpha(t)) = \frac{1}{s^\alpha}.$$

Notice that the Riemann-Liouville fractional integral of $f(t)$ could be expressed as the convolution of the two functions $\Phi_\alpha(t)$ and $f(t)$ namely

$$\begin{aligned} J_a^\alpha f(x) &= \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)dt, \\ &= \int_a^x \Phi_\alpha(x-t) f(t)dt, \\ &= \Phi_\alpha(t) * f(t). \end{aligned}$$

Based on the convolution property of the Laplace transform (1.4), one deduce that

$$\begin{aligned} \mathcal{L}(J_a^\alpha f(x)) =& \mathcal{L}(\Phi_\alpha(t))\mathcal{L}(f(t)), \\ =& \frac{F(s)}{s^\alpha}. \end{aligned} \tag{2.7}$$

2.2 Riemann-Liouville fractional derivatives

2.2.1 Definition

After shedding light on some basic properties of the Riemann-Liouville integral operators. Now we come to the corresponding differential operators. First, we recall the following identity which holds for a function f having a continuous nth derivative on the interval $[a, b]$

$$D^n f = D^m J^{m-n} f, \tag{2.8}$$

where $n,\ m \in \mathbb{N}$, such that $m > n$.

Now suppose that n is not an integer. In view of previous sections the right-hand side of (2.8) is meaningful. Hence, we come to the following definition of the Riemann-Liouville fractional differential operator.

Definition 2.2.
Let $\alpha \in \mathbb{R}_+$ and $m = \lceil \alpha \rceil$ (the smallest integer that exceeds α).
The operator ${}_a^{RL}D_t^{\alpha}$ defined by

$${}_a^{RL}D_t^{\alpha} f = D^m J_a^{m-\alpha} f, \tag{2.9}$$

is called the Riemann-Liouville fractional differential operator of order α.

Equivalently, we have

$${}_a^{RL}D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-\alpha-1} f(s) ds, & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t). & \alpha = m \end{cases}$$

For $\alpha = 0$ we set ${}_a^{RL}D_t^{0} = I$, the identity operator, and whenever $\alpha \in \mathbb{N}$ the new operator ${}_a^{RL}D_t^{\alpha}$ coincides with the classical differential operator D^{α}.

Remark 2.3.
The Riemann-Liouville fractional operator, is not local.

Example 2.2.
Let $f(t) = c$, *then*

$$\begin{aligned} {}^{RL}_{a}D^{\alpha}_{t} f(t) &= D^m (J^{m-\alpha}_a f(t)), \\ &= \frac{c}{\Gamma(m-\alpha)} D^m (\int_a^t (t-s)^{m-\alpha-1} ds), \\ &= \frac{c(t-a)^{-\alpha}}{\Gamma(1-\alpha)}. \end{aligned}$$

Example 2.3.
Let $f(t) = (t-a)^{\beta}$ *for a fixed* $\beta > -1$ *and* $\alpha > 0$. *Then, in view of example (2.1), we have*

$$\begin{aligned} {}^{RL}_{a}D^{\alpha}_{t} f(t) &= D^m (J^{m-\alpha}_a f(t)), \quad m = \lceil \alpha \rceil \\ &= \frac{\Gamma(\beta+1)}{\Gamma(m-\alpha+\beta+1)} D^m (x-t)^{m-\alpha+\beta}. \end{aligned}$$

One can distinguish two cases. Namely: ***If*** $(\alpha - \beta) \in \mathbb{N}$, *then the right-hand side is the m-th derivative of a polynomial of degree* $m - (\alpha - \beta)$, *this implies that*

$${}^{RL}_{a}D^{\alpha}_{t} f(t) = 0.$$

But if $(\alpha - \beta) \notin \mathbb{N}$, *then*

$${}^{RL}_{a}D^{\alpha}_{t} f(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-s)^{\beta-\alpha}.$$

2.2.2 Some basic properties

The law of exponents

In section (2.1.2), we have proved the rule of composition for Riemann-Liouville fractional integrals. That is if f is a continuous function for $x \geq a$ and $\alpha > 0$, $\beta > 0$, then

$$J^{\alpha}_a \left(J^{\beta}_a f(x) \right) = J^{\alpha+\beta}_a f(x). \tag{2.10}$$

However, this rule may not be generalized to the case of fractional derivatives without imposing some additional restrictions on f. To show that (2.10) does not necessarily hold for all α and β when replacing J^{α}_a and J^{β}_a by ${}^{RL}_{a}D^{\alpha}_{t}$ and ${}^{RL}_{a}D^{\beta}_{t}$, let introduce the following example

Let $f(t) = t^{\frac{1}{2}},\ \alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$ then

$$
\begin{aligned}
{}^{RL}_{\ 0}D_t^{\alpha}(f(t)) &= \frac{1}{2}\sqrt{\pi},\\
{}^{RL}_{\ 0}D_t^{\beta}(f(t)) &= 0,\\
{}^{RL}_{\ 0}D_t^{\alpha}(\,{}^{RL}_{\ 0}D_t^{\beta}(f(t))) &= 0,\\
{}^{RL}_{\ 0}D_t^{\beta}(\,{}^{RL}_{\ 0}D_t^{\alpha}(f(t))) &= -\frac{1}{4}t^{-\frac{3}{2}},\\
{}^{RL}_{\ 0}D_t^{\alpha+\beta}(f(t)) &= -\frac{1}{4}t^{-\frac{3}{2}}.
\end{aligned}
$$

Obviously in this example we have

$${}^{RL}_{\ 0}D_t^{\alpha}(\,{}^{RL}_{\ 0}D_t^{\beta}(f(t))) \neq {}^{RL}_{\ 0}D_t^{\alpha+\beta}(f(t)).$$

In the following we shall state, precisely, some conditions under which the law of exponents holds.

- **Composition with integer-order derivatives.**
  The composition of Riemann-Liouville fractional derivatives with integer order derivatives appears in many applied problems, so it is convenient to introduce it here. Let us consider the n-th derivative of the Riemann Liouville fractional derivative of real order α, we have

$$
\begin{aligned}
D^n(\,{}^{RL}_{\ a}D_t^{\alpha}f(t)) &= \frac{1}{\Gamma(m-\alpha)}D^{n+m}\left(\int_a^t (t-s)^{m-\alpha-1}f(s)ds\right),\\
&= \frac{1}{\Gamma(n+m-(n+\alpha))}D^{n+m}\left(\int_a^t (t-s)^{n+m-(n+\alpha)-1}f(s)ds\right),\\
&= {}^{RL}_{\ a}D_t^{n+\alpha}f(t).
\end{aligned}
$$

To consider the fractional derivatives of the n-th integer derivative we recall the following relationships

$$
\begin{aligned}
{}_aD_t^{-n}f^{(n)}(t) &= \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}f^{(n)}(s)ds,\\
&= f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)(x-a)^j}{\Gamma(j+1)},
\end{aligned}
$$

and

$$ {}_a^{RL}D_t^{\alpha} g(t) = {}_a^{RL}D_t^{\alpha+n} \left({}_aD_t^{-n} g(t) \right). $$

Using the above relations we obtain

$$ \begin{aligned} {}_a^{RL}D_t^{\alpha}(f^{(n)}(t)) &= {}_a^{RL}D_t^{\alpha+n} \left({}_aD_t^{-n} f^{(n)}(t) \right), \\ &= {}_a^{RL}D_t^{\alpha+n} \left(f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)} \right), \\ &= {}_a^{RL}D_t^{\alpha+n} f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(t-a)^{j-n-\alpha}}{\Gamma(j-n-\alpha+1)}. \end{aligned} $$

From this results we see that the Riemann-Liouville fractional operator ${}_a^{RL}D_t^{\alpha}$ commutes with the integer operator D^n only if f^j vanishes in the lower terminal a for all $j = 0, 1, 2, ..., n-1$.

- **Composition with fractional-order Derivatives.**

Now, let us consider the composition of two fractional Riemann-Liouville operators ${}_a^{RL}D_t^{\alpha}$ and ${}_a^{RL}D_t^{\beta}$, we put $m = \lceil \alpha \rceil$ and $n = \lceil \beta \rceil$ then

$$ \begin{aligned} {}_a^{RL}D_t^{\alpha}({}_a^{RL}D_t^{\beta} f(t)) &= D^m \left({}_a^{RL}D_t^{-(m-\alpha)} \left({}_a^{RL}D_t^{\beta} f(t) \right) \right), \\ &= D^m \left({}_a^{RL}D_t^{\alpha+\beta-m} f(t) \right. \\ &\left. - \sum_{j=1}^{n} \left[{}_a^{RL}D_t^{\beta-j} f(t) \right]_{t=a} \frac{(t-a)^{m-\alpha-j}}{\Gamma(m-\alpha-j+1)} \right), \\ &= {}_a^{RL}D_t^{\alpha+\beta} f(t) - \sum_{j=1}^{n} \left[{}_a^{RL}D_t^{\beta-j} f(t) \right]_{t=a} \frac{(t-a)^{-\alpha-j}}{\Gamma(-\alpha-j+1)}, \end{aligned} \tag{2.11} $$

and

$$ {}_a^{RL}D_t^{\beta}({}_a^{RL}D_t^{\alpha} f(t)) = {}_a^{RL}D_t^{\beta+\alpha} f(t) - \sum_{j=1}^{m} \left[{}_a^{RL}D_t^{\alpha-j} f(t) \right]_{t=a} \frac{(t-a)^{-\beta-j}}{\Gamma(-\beta-j+1)}. \tag{2.12} $$

From this relationships we deduce that in general case the Riemann-Liouville fractional operators ${}_a^{RL}D_t^{\alpha}$ and ${}_a^{RL}D_t^{\beta}$ do not commute, except for the case $\alpha = \beta$. When $\alpha \neq \beta$ it commutes only if both sums in the right-hand sides of (2.11) and (2.12) vanish, that is if

$$f^{j}(a)=0, \quad for \;\; all \;\; j=0,1,2,...,\max(n-1,m-1).$$

The Laplace transform In order to evaluate the Laplace transform of the Riemann-Liouville fractional derivative ${}_{0}^{RL}D_{t}^{\alpha}f(t)$ we write it in the form:

$${}_{0}^{RL}D_{t}^{\alpha}f(t)=g^{(m)}(t),$$

where

$$g(t)={}_{0}^{RL}D_{t}^{-(m-\alpha)}f(x)=\frac{1}{\Gamma(m-\alpha)}\int_{0}^{t}(t-s)^{m-\alpha-1}f(s)ds. \;\; m=\lceil\alpha\rceil$$

Using the notation $\mathcal{L}(f(x))(s)=F(s), \;\; \mathcal{L}(g(x))(s)=G(s)$ and the formula for the Laplace transform of integer-order derivative (1.3) we get

$$\mathcal{L}\left\{{}_{0}^{RL}D_{t}^{\alpha}f(x)\right\}(s)=s^{m}G(s)-\sum_{k=0}^{m-1}s^{k}g^{(m-k-1)}(0). \tag{2.13}$$

$G(s)$ can be evaluated by the formula of the Laplace transform of the Riemann-Liouville fractional integral (2.7) namely

$$G(s)=s^{-(m-\alpha)}F(s). \tag{2.14}$$

On the other hand, we have

$$g^{(m-k-1)}(t)=\frac{d^{m-k-1}}{dt^{m-k-1}}\,{}_{0}^{RL}D_{t}^{-(m-\alpha)}f(t)={}_{0}^{RL}D_{t}^{\alpha-k-1}f(t). \tag{2.15}$$

Substituting (2.14) and (2.15) into (2.13), then we obtain the formula for the Laplace transform of the Riemann-Liouville fractional derivative

$$\mathcal{L}\left\{{}_{0}^{RL}D_{t}^{\alpha}f(t)\right\}(s)=s^{\alpha}F(s)-\sum_{k=0}^{m-1}s^{k}\left[{}_{0}^{RL}D_{t}^{\alpha-k-1}f(t)\right]_{t=0}. \tag{2.16}$$

The practical application of this Laplace transform is limited by the absence of physical interpretation of the limit values of fractional derivative at the lower terminal $t=0$, [62].

2.3 Grünwald-Letnikov fractional derivative

2.3.1 Definition

As presented above, the Riemann-Liouville formulation approaches the problem of fractional calculus from the repeated integral, but the Grünwald-Letnikov formulation approaches the problem from the derivative side by observing that the derivative of integer order m and the m-fold integral are two notions closer to each other than one usually assumes, namely they are particular cases of the following general expression

$$ {}_aD_t^p f(t) = \lim_{\substack{h \to 0 \\ nh = t-a}} h^{-p} \sum_{k=0}^{n} (-1)^k \begin{pmatrix} p \\ k \end{pmatrix} f(t-kh), \tag{2.17}$$

where

$$\begin{pmatrix} p \\ k \end{pmatrix} = \frac{p(p-1)(p-2)...(p-k+1)}{k!},$$

are the binomial coefficients.

The expression (2.17) represents the derivative of order m if $p = m$ and the m-fold integral if $p = -m$, this expression is used to define a fractional derivative and fractional integral by directly replacing $p \in \mathbb{N}$ in (2.17), by an arbitrary real α, provided that the binomial coefficient can be understood as using the Gamma function in place of the standard factorial. Also, the upper limit of the summation goes to infinity as $\dfrac{t-a}{h}$.
We end up with the generalized form of the Grünwald-Letnikov fractional derivative

$$ {}_a^{GL}D_t^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{k=0}^{\frac{t-a}{h}} (-1)^k \left(\frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} \right) f(t-kh). \tag{2.18}$$

It is conceivable, that like the definition of Riemann-Liouville for the fractional integral may be used to define the fractional derivative, the above form of the G-L derivative could be altered for use in an alternate definition of the fractional integral. The most natural alteration of this form is to consider the G-L derivative for negative α. But in (2.17), there is a problem that $\begin{pmatrix} -p \\ k \end{pmatrix}$ is not defined using factorials. We have

$$\begin{pmatrix} -p \\ k \end{pmatrix} = \frac{-p(-p-1)(-p-2)...(-p-k+1)}{k!},$$
$$= (-1)^k \frac{(p+k-1)!}{(p-1)!k!}. \quad (2.19)$$

The factorial in (2.19) may be generalized for negative real, using the Gamma function, thus

$$\begin{pmatrix} -p \\ k \end{pmatrix} = (-1)^k \frac{\Gamma(p+k)}{\Gamma(p)k!}. \quad (2.20)$$

Now we can rewrite (2.18) for $-\alpha$, and this leads to the G-L fractional integral

$${}^{GL}_{a}D_t^{-\alpha} f(t) = \lim_{h\to 0} h^{\alpha} \sum_{k=0}^{\frac{t-a}{h}} \left(\frac{\Gamma(\alpha+k)}{k!\Gamma(\alpha)} \right) f(t-kh). \quad (2.21)$$

2.3.2 Link to the Riemann-Liouville approach

If we assume that the derivatives $f^{(k)}(t)$, $(k = 1, 2, ..., m)$ are continuous in the interval $[a, T]$ and m is an integer such that $m > \alpha$, we can rewrite (2.18) as follows

$${}^{GL}_{a}D_t^{\alpha} f(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds. \quad (2.22)$$

Also, the right hand side of (2.22) can be written as

$$\frac{d^m}{dt^m} \left\{ \sum_{k=0}^{m-1} \frac{f^{(k)}(a)(t-a)^{m-\alpha+k}}{\Gamma(m-\alpha+k+1)} + \frac{1}{\Gamma(2m-\alpha)} \int_a^t (t-s)^{2m-\alpha-1} f^{(m)}(s) ds \right\}, \quad (2.23)$$

and after m integration by part, we obtain the expression of the Riemann-Liouville derivative

$$\frac{d^m}{dt^m}\left\{\frac{1}{\Gamma(m-\alpha)}\int_a^t (t-s)^{m-\alpha-1}f(t)dt\right\} = \frac{d^m}{dt^m}\left\{ {}_a^{RL}D_t^{-(m-\alpha)}f(t)\right\},$$
$$= {}_a^{RL}D_t^{\alpha}f(t). \tag{2.24}$$

So, under the above assumptions we have

$${}_a^{RL}D_t^{\alpha}f(t) = {}_a^{GL}D_t^{\alpha}f(t).$$

Therefore, the properties that we have seen in the Riemann-Liouville definition for the fractional derivative remain valid for the Grünwald-Letnikov definition, under a suitable assumptions.

Remark 2.4.
The Riemann-Liouville definition of the fractional integral and derivative is suitable to find the analytic solution for relatively simple functions. Conversely, the Grünwald-Letnikov definition is adopted for numerical computations.

2.4 Caputo fractional derivative

2.4.1 Definition

As it is mentioned in section (2.2), the Laplace transform of the Riemann-Liouville fractional derivative include the limit values of fractional derivative at the lower terminal $t = 0$, so the initial conditions required for the solution of fractional order differential equations are themselves of a non-integer order. Also, the fractional derivative of a constant is not a 0.
In the mathematical sense, when solving non-integer order differential equations, it is possible to use this definition given the proper initial conditions as it happens. However in the physical world, these properties of the RL definition presents, a serious problem. Today, we are well versed with the interpretation of the physical world in the equations of integer order, and we do not have a practical knowledge of the world in a fractional order. Our mathematical tools go in excess of practical limitations of our comprehension.

The Italian mathematician Caputo proposed a solution to this conflict in 1967, [17]. By introducing a new definition, in which he attempts to find a link between what is possible and what is practical. The aim of the slight modification of the concept of fractional derivative is to allow the use of integer order initial conditions in the solution of fractional differential equations. In addition, the Caputo derivative of a constant is 0, as we will see below. In order to achieve this goal, Caputo proposes the same operations as in Riemann-Liouville definition but in the reverse order, namely to get the Riemann-Liouville derivative of order $\alpha > 0$, of a function f, first one must integrate f by the fractional order $m - \alpha$, after that, differentiate the resulting function by the integer order m. While in the Caputo approach, first one must differentiate f by the integer order m, then integrate $f^{(m)}$ by the fractional order $m - \alpha$.

Definition 2.5.
Let $\alpha \geq 0$, and $m = \lceil \alpha \rceil$. Then, we define the Caputo's fractional operator ${}_a^C D_t^\alpha$ by

$$\begin{aligned} {}_a^C D_t^\alpha f(t) &= J_a^{m-\alpha} \frac{d^m}{dt^m} f(t), \\ &= \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \end{aligned}$$

whenever $\frac{d^m}{ds^m} f \in L_1[a, b]$.

Remark 2.6.
As in the case of the Riemann-Liouville operators, we see that the Caputo derivatives are not local either.

2.4.2 Some basic properties

1. **Linearity**

   Let $\lambda, \gamma \in \mathbb{R}$. From the definition of ${}_a^C D_t^\alpha$ it follows directly that

$$ {}_a^C D_t^\alpha (\lambda f(t) + \gamma g(t)) = \lambda \, {}_a^C D_t^\alpha (f(t)) + \gamma \, {}_a^C D_t^\alpha (g(t)). $$

2. **Interpolation**

   When $\alpha \in \mathbb{N}$, we have $m = \alpha$, then the definition (2.5) implies that

$$ {}_a^C D_t^\alpha f = J_a^0 \frac{d^m}{dt^m} f = \frac{d^m}{dt^m} f. $$

This means that: similarly to the Riemann-Liouville and Grünwald-Litnikove approaches, the Caputo approach provides also an interpolation between integer-order derivatives.

3. **Composition**

   Let $n \in \mathbb{N}$ and $m = \lceil \alpha \rceil$, we have

$$ {}_a^cD_t^{\alpha}({}_a^cD_t^{m} f(t)) = {}_a^cD_t^{\alpha+n} f(t). $$

   Namely

$$ \begin{aligned} {}_a^cD_t^{\alpha}({}_aD_t^{n} f(t)) &= {}_aD_t^{-(m-\alpha)}\, {}_aD_t^{m}({}_aD_t^{n} f(t)), \\ &= {}_aD_t^{-(m-\alpha)}\, {}_aD_t^{m+n} f(t), \\ &= {}_aD_t^{-(m+n-(\alpha+n))}\, {}_aD_t^{m+n} f(t), \\ &= {}_a^cD_t^{\alpha+n} f(t). \end{aligned} $$

4. **Laplace transform**

   We begin by writing the derivative in the form:

$$ {}_0^CD_t^{\alpha} f(t) = J^{m-\alpha} g(t), $$

   where

$$ g(x) = f^{(m)}(x), \quad m = \lceil \alpha \rceil. $$

   Using the formula for the Laplace transform of Riemann-Liouville fractional integral (2.7), and the formula for the Laplace transform of integer-order derivative (1.3) we get

$$ \mathcal{L}\left\{ {}_0^CD_t^{\alpha} f(t) \right\}(s) = S^{-(m-\alpha)} G(s) = s^{\alpha} F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0). \qquad (2.25) $$

   Clearly, the Laplace transform of the Caputo fractional derivative involves the values of $f(x)$ and its derivatives at the lower terminal $x = 0$, for which a certain physical interpretation exists, so we expect that the fractional Caputo derivative can be useful for solving applied problems.

2.4.3 Link to the Riemann-Liouville approach

Let $\alpha > 0$ and f a function having a continuous derivatives $f^{(k)}(t)$, $(k = 1, 2, ..., m)$ in the interval $[a, T]$, where $m = \lceil \alpha \rceil$, then from (2.24) we have

$$\begin{aligned} {}^{RL}_{a}D^{\alpha}_{t} f(t) &= \sum_{k=0}^{m-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha)} \int_{a}^{t} (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \\ &= \sum_{k=0}^{m-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} + {}^{C}_{a}D^{\alpha}_{t} f(t). \end{aligned} \tag{2.26}$$

Clearly if $f^{(k)}(a) = 0$, $(k = 0, 1, 2, ..., m-1)$ then

$${}^{RL}_{a}D^{\alpha}_{t} f(t) = {}^{C}_{a}D^{\alpha}_{t} f(t).$$

2.5 Fractional differential equations

This section will be devoted to the study of Caputo's fractional differential equations. In the first subsection we aboard the existence and uniqueness questions for the initial value problems with a most general class of fractional equations, then in the second subsection we move to the analytical resolution of linear equations. Whereas in the third subsection we shall deal with numerical resolution.

2.5.1 Initial value problems

We begin with initial value problem of the form

$$\begin{cases} {}^{c}_{0}D^{\alpha}_{t} x(t) = f(t, x), \\ x^{k}(0) = x_{0}^{(k)}, \quad k = 0, 1, 2, ..., m-1 \end{cases} \tag{2.27}$$

where as usual we have set $m = \lceil \alpha \rceil$.
The existence and uniqueness theory for such equations have been presented in [26].

Theorem 2.7.
Let $\alpha > 0$ *and* $m = \lceil \alpha \rceil$.
Moreover, let $x_0^0, x_0^1, ..., x_0^{m-1} \in \mathbb{R}, K > 0$ *and* $T^* > 0$.

Define

$$G = \left\{ (t,x) : t \in [0,T^*], |x - \sum_{k=0}^{m-1} t^k x_0^{(k)}/k! |\leq K \right\},$$

and let the function $f : G \to \mathbb{R}$ *be continuous.*
Furthermore, define $M = sup_{(t,z)\in G}|f(t,z)|$ *and*

$$T = \begin{cases} T^* & \text{if } M = 0, \\ \min\left\{T^*, (K\Gamma(\alpha+1)/M)^{1/\alpha}\right\} & \text{else.} \end{cases} \tag{2.28}$$

Then, there exists a function $x \in C[0,T]$ *solving the initial value problem (2.27).*

Theorem 2.8.
Let $\alpha > 0$ *and* $m = \lceil \alpha \rceil$*.*
Moreover, let $x_0^0, x_0^1, ..., x_0^{m-1} \in \mathbb{R}, K > 0$ *and* $T^* > 0$*.*
Define G *as in theorem (2.7) , and let the function* $f : G \to \mathbb{R}$ *be continuous and satisfying a Lipschitz condition with respect to the second variable, i.e;*

$$|f(t,x_1) - f(t,x_2)| \leq L|x_1 - x_2|, \tag{2.29}$$

with some constant $L > 0$*. Then define* T *as in theorem (2.7), there exists a uniquely defined function* $x \in C[0,T]$ *solving the initial value problem (2.27).*

Corollary 2.9.
Assumes the hypotheses of the theorem (2.7 and 2.8), except that the set G*, i.e; the domain of definition of the function* f *is now taken to be* $G = \mathbb{R}^2$*.*
Moreover, we assume that f *is continuous and that there exist constants* $c_1 \geq 0$*,* $c_2 \geq 0$ *and* $0 \leq \mu < 1$ *such that*

$$|f(t,x)| \leq c_1 + c_2|x|^\mu \quad \text{for all} \quad (t,x) \in G.$$

Then, there exists a uniquely function $x \in C[0,\infty)$*, solving the initial value problem (2.27).*

For the proof of theorem (2.7 and 2.8) one can refer to [26](chapter 6).

Remark 2.10.
- In real applications, we have usually $0 < \alpha \leq 1$. In this case, the set G defined in the theorem (2.7) is just the simple rectangle

$$G = [0,T] \times \left[x_0^{(0)} - K, x_0^{(0)} + K\right].$$

-For simplicity of the presentation we only treat the scalar case here. However, all the results in this section can be extended to vector-valued functions x (i.e. systems of equations) without any problems.

It is well know that if

$$f : [0, a] \times [b, c] \to \mathbb{R},$$

is a continuous function and satisfy a Lipschitz condition with respect to the second variable and y, z are two solutions of the differential equation of order 1

$$\frac{dx(t)}{dt} = f(t, x),$$

subject to the initial conditions $y(0) = y_0$, $z(0) = z_0$ where $y_0 \neq z_0$. Then, for all t where both $y(t)$ and $z(t)$ exist, we have $y(t) \neq z(t)$. But a similar statement does not hold for equations of higher order, for example the equation

$$\frac{d^2x}{dt^2} = -x(t),$$

has solutions $x_1(t) = 0$, $x_2(t) = \cos t$ and $x_3(t) = \sin t$ clearly the graphs of these solutions cross each other. Similar effects arise for fractional equations and we have the following result

Theorem 2.11.
Let $0 < \alpha < 1$ and assume that $f : [0, a] \times [b, c] \to \mathbb{R}$, is a continuous function and satisfy the Lipschitz condition (2.29) with respect to the second variable and y, z are two solutions of the fractional differential equation of order α

$${}_0^C D_t^\alpha x(t) = f(t, x),$$

subject to the initial conditions $y(0) = y_0$, $z(0) = z_0$ where $y_0 \neq z_0$. Then, for all t where both $y(t)$ and $z(t)$ exist, we have $y(t) \neq z(t)$.

2.5.2 Initial value problems for linear equations

It is a common observation in many areas of mathematics that the linearity assumption allows to derive more precise statements. So, in this section we restrict our attention to linear fractional differential equations which are very important in many applications. Explicit expressions for solutions of such equations can be obtained and used for the study of stability property.

2.5.2.1 One dimensional case

For simplicity, we begin by the scalar (one dimensional) case

Theorem 2.12.
Let $\alpha > 0$ and $m = \lceil \alpha \rceil$, $\lambda \in \mathbb{R}$ and $q \in C[0,T]$. The solution of the initial value problem

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) = \lambda x(t) + q(t), \\ x^{(k)}(0) = x_0^{(k)}, \quad k = 0, 1, ...m-1, \end{cases} \tag{2.30}$$

is given by

$$x(t) = \sum_{k=0}^{m-1} x_0^{(k)} u_k(t) + \tilde{x}(t), \tag{2.31}$$

with

$$\tilde{x}(t) = \begin{cases} J_0^{\alpha} q(t), & if \lambda = 0, \\ \frac{1}{\lambda} \int\limits_0^t q(t-\tau) u_0'(\tau) d\tau & if \lambda \neq 0, \end{cases} \tag{2.32}$$

where $u_k(t) = J_0^k e_\alpha(t)$, $k = 0, 1, ...m-1$ and $e_\alpha(t) = E_\alpha(\lambda t^\alpha)$.

Remark 2.13.
In the special case $0 < \alpha < 1$, the solution is given by

$$\begin{aligned} x(t) &= x_0^{(0)} E_\alpha(\lambda t^\alpha) + \alpha \int\limits_0^t q(t-\tau) \tau^{\alpha-1} E_\alpha'(\lambda \tau^\alpha) d\tau, \\ &= x_0^{(0)} E_\alpha(\lambda t^\alpha) + \alpha \int\limits_0^t (t-\tau)^{\alpha-1} E_\alpha'(\lambda (t-\tau)^\alpha) q(\tau) d\tau. \end{aligned}$$

In the limit case $\alpha \to 1^-$ we obtain the classical formula

$$x(t) = x_0^{(0)} e^{\lambda t} + \int_0^t e^{\lambda(t-\tau)} q(\tau) d\tau.$$

Proof. (Theorem (2.12)).
In the case $\lambda = 0$, we have $e_\alpha(t) = E_\alpha(0) = 1$. Then, $u_k(t) = t^k/k!$, for every k. Thus, the direct differentiation of a given $x(t)$ affirms the claim.
In the case $\lambda \neq 0$, the proof will be divided into two facts:
The first is that the functions u_k satisfy the homogeneous differential equation

$${}_0^C D_t^{\alpha} u_k = \lambda u_k \quad (k = 0, 1, ..., m-1),$$

with initial conditions $u_k^{(j)}(0) = \delta_{kj}$ (Kronecker's delta) for $j, k = 0, 1, ..., m-1$.
The second fact is that the function $\tilde{x}$ is a solution of (2.30).
Then the proof will be achieved by the superposition principal.
a) we have

$$e_\alpha(t) = \sum_{j=0}^{\infty} \frac{\lambda^j t^{\alpha j}}{\Gamma(1 + j\alpha)}.$$

Then

$$u_k(t) = J_0^k e_\alpha(t) = \sum_{j=0}^{\infty} \frac{\lambda^j t^{\alpha j + k}}{\Gamma(1 + j\alpha + k)}, \tag{2.33}$$

applying the operator ${}_0^C D_t^\alpha$ to both sides of (2.33) yields

$$\begin{aligned} {}_0^C D_t^\alpha u_k(t) &= \sum_{j=1}^{\infty} \frac{\lambda^j t^{\alpha(j-1)+k}}{\Gamma(1 + (j-1)\alpha + k)}, \\ &= \sum_{j=0}^{\infty} \frac{\lambda^{j+1} t^{\alpha j + k}}{\Gamma(1 + j\alpha + k)}, \\ &= \lambda \sum_{j=0}^{\infty} \frac{\lambda^j t^{\alpha j + k}}{\Gamma(1 + j\alpha + k)}, \\ &= \lambda u_k(t). \end{aligned}$$

Moreover, for $j = k$, we have

$$u_k^{(k)}(0) = D^k J_0^k e_\alpha(0) = 1.$$

For $j < k$, we have

$$u_k^{(j)}(0) = D^j J_0^k e_\alpha(0) = J_0^{k-j} e_\alpha(0) = \frac{1}{\Gamma(k-j)} \int_0^0 (0 - \tau)^{k-j-1} e_\alpha(\tau) d\tau = 0.$$

And for $j > k$, we have

$$u_k^{(j)}(0) = D^j J_0^k e_\alpha(0) = D_0^{j-k} e_\alpha(0) = 0,$$

since

$$\begin{aligned} D_0^{j-k} e_\alpha(t) &= D_0^{j-k} \sum_{l=0}^{\infty} \frac{\lambda^l t^{l\alpha}}{\Gamma(1 + l\alpha)}, \\ &= \sum_{l=1}^{\infty} \frac{\lambda^l t^{l\alpha + k - j}}{\Gamma(1 + l\alpha + k - j)}. \end{aligned}$$

b)We have

$$\tilde{x}(t) = \frac{1}{\lambda}\int_0^t q(t-\tau)u_0'(\tau)d\tau = \frac{1}{\lambda}\int_0^t q(t-\tau)e_\alpha'(\tau)d\tau = \frac{1}{\lambda}\int_0^t q(\tau)e_\alpha'(t-\tau)d\tau.$$

Since q is continuous and e'_α is at least improperly integrable, then the integral exists and it is a continuous function of t, thus $\tilde{x}(0) = 0$. Using the well know rules for differentiation of parameter integrals we obtain

$$\begin{aligned} D\tilde{x}(t) &= \frac{1}{\lambda}\int_0^t q(\tau)e''_\alpha(t-\tau)d\tau + \frac{1}{\lambda}q(t)e'_\alpha(0) \\ &= \frac{1}{\lambda}\int_0^t q(\tau)e''_\alpha(t-\tau)d\tau \end{aligned}$$

because $e'_\alpha(0) = 0$, this formula can be generalized for $(k = 0, 1, ..., m-1)$ as follows

$$D^k\tilde{x}(t) = \frac{1}{\lambda}\int_0^t q(\tau)e_\alpha^{(k+1)}(t-\tau)d\tau,$$

then $D^k\tilde{x}(0) = 0$ for $(k = 0, 1, ..., m-1)$. Thus $\tilde{x}$ Fulfills the required homogeneous initial conditions. Now, it remains to show that $\tilde{x}$ solves the non-homogeneous differential equation. We have

$$\begin{aligned} e'_\alpha(t) &= \lambda\alpha t^{\alpha-1}E'_\alpha(\lambda t^\alpha) = \lambda\alpha t^{\alpha-1}\sum_{j=1}^{\infty}\frac{j(\lambda t^\alpha)^{j-1}}{\Gamma(1+j\alpha)}, \\ &= \lambda t^{\alpha-1}\sum_{j=1}^{\infty}\frac{(\lambda t^\alpha)^{j-1}}{\Gamma(j\alpha)} = \sum_{j=1}^{\infty}\frac{\lambda^j t^{j\alpha-1}}{\Gamma(j\alpha)}, \end{aligned}$$

then

$$\begin{aligned} \tilde{x}(t) &= \frac{1}{\lambda}\int_0^t q(\tau)e'_\alpha(t-\tau)d\tau = \frac{1}{\lambda}\int_0^t q(\tau)\sum_{j=1}^{\infty}\frac{\lambda^j(t-\tau)^{j\alpha-1}}{\Gamma(j\alpha)}d\tau, \\ &= \sum_{j=1}^{\infty}\frac{\lambda^{j-1}}{\Gamma(j\alpha)}\int_0^t q(\tau)(t-\tau)^{j\alpha-1}d\tau = \sum_{j=1}^{\infty}\lambda^{j-1}J_0^{j\alpha}q(t) \quad . \end{aligned}$$

Thus

$$\begin{aligned} {}_0^C D_t^\alpha \tilde{x}(t) &= \sum_{j=1}^{\infty}\lambda^{j-1}\,{}_0^C D_t^\alpha J_0^{j\alpha}q(t) = \sum_{j=1}^{\infty}\lambda^{j-1}J_0^{(j-1)\alpha}q(t), \\ &= \sum_{j=0}^{\infty}\lambda^j J_0^{(j)\alpha}q(t) = q(t) + \sum_{j=1}^{\infty}\lambda^j J_0^{j\alpha}q(t) = q(t) + \lambda\tilde{x}(t). \end{aligned}$$

Notice here that in view of the convergence property of the series expansion for e'_α and the continuity of q, the interchange between summation and integration is possible. □

2.5.2.2 Multidimensional case

First, let us give the general solution for the commensurate fractional order linear homogeneous system

$$ {}_0^C D_t^\alpha X(t) = AX(t), \quad 0 < t \leq a, \tag{2.34}$$

where $X \in \mathbb{R}^n$, $a > 0$, and $A \in \mathbb{R}^n \times \mathbb{R}^n$. To derive this general solution the author of [57], proceeds by analogy with treatment of homogeneous integer order linear systems with constant coefficients where the exponential function $Exp(t)$ is replaced by the Mittag-Leffler function $E_\alpha(t^\alpha)$. Hence, we seek solutions of the form

$$X(t) = uE_\alpha(\lambda t^\alpha), \tag{2.35}$$

the constant λ and the vector u are to be determined. Substituting (2.35) in (2.34) gives

$$u\lambda E_\alpha(\lambda t^\alpha) = AuE_\alpha(\lambda t^\alpha). \tag{2.36}$$

Thus

$$(A - \lambda I)u = 0, \tag{2.37}$$

because $E_\alpha(\lambda t^\alpha) \neq 0$. Therefore, the vector X in (2.35) is a solution of the system (2.34) on condition that λ is an eigenvalue and u an associated eigenvector of the matrix A. Now, if all k-fold eigenvalues of A have k eigenvectors, then we know that the set of all these eigenvectors is linearly independent and thus it forms a basis of $\mathbf{C}^n$. Hence, the following result holds.

Theorem 2.14.
Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of the matrix A and $u^{(1)}, ..., u^{(n)}$ be the corresponding eigenvectors. Then, the general solution of the fractional differential equation (2.34)is given by

$$X(t) = \sum_{k=1}^{n} C_k u^{(k)} E_\alpha(\lambda_k t^\alpha), \tag{2.38}$$

with certain constants $C_k \in \boldsymbol{C}$.

Example 2.4.
Let us consider the system

$$ {}_{0}^{C}D_t^{\alpha} X(t) = AX(t), $$

where $0 < \alpha < 1$ *and* $A = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$.
The eigenvalues of the matrix A *are* $\lambda_1 = 1$ *and* $\lambda_2 = -2$ *and their corresponding eigenvectors are* $u^{(1)} = [1, 1]^T$ *and* $u^{(2)} = [1, 4]^T$ *respectively. Thus, the general solution of the given system is*

$$ X(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} E_\alpha(t^\alpha) + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} E_\alpha(-2t^\alpha), $$

where c_1 *and* c_2 *are arbitrary constants.*

Remark 2.15.
If the matrix A has a repeated eigenvalue λ, of algebraic multiplicity k and geometric multiplicity m (i.e: with m linearly independent eigenvectors $u^{(1)}, ..., u^{(m)}$), then we envisage two cases.
a) If $m = k$, then

$$ X^{(1)} = u^{(1)} E_\alpha(\lambda t^\alpha), ..., X^{(k)} = u^{(k)} E_\alpha(\lambda t^\alpha), $$

are k linearly independent solutions of the homogeneous system (2.34).
b) If $m < k$, then, the theorem (2.14) is not applicable and we must resort to a different representation of the general solution.

Definition 2.16.
Let λ be an eigenvalue of multiplicity k, of the $n \times n$ matrix A. Then for $i = 1, ..., k$, any nonzero solution v of

$$ (A - \lambda I)^i v = 0 \quad with \quad (A - \lambda I)^{i-1} v \neq 0, $$

is called a generalized eigenvector of order i, of the matrix A. The set of generalized eigenvectors $v^{(1)}, ..., v^{(k)}$ is linearly independent and is called a Jordan chain.

Notice that an ordinary eigenvector u can be considered as a generalised eigenvector of order 1. The generalized eigenvectors $v^{(1)}, ..., v^{(k)}$ can be determined by solving the following successive sequence of linear equations, in which $v^{(r)}$ is known and $v^{(r+1)}$ is unknown:

$$\begin{aligned}
(A-\lambda I)v^{(1)} &= 0,\\
(A-\lambda I)v^{(2)} &= v^{(1)},\\
(A-\lambda I)v^{(3)} &= v^{(2)},\\
&\quad . \quad . \quad .\\
(A-\lambda I)v^{(k)} &= v^{(k-1)}.
\end{aligned}$$

In the case (b) of remark (2.15) the generalized eigenvalues will be useful for creating the fundamental set of solutions of the homogeneous system (2.34) as shown in the following theorem.

Theorem 2.17.
For each k-fold eigenvalue λ, of the matrix A we have k linearly independent solutions $X^{(1)}, ..., X^{(k)}$ of the homogeneous linear system (2.34) which can be represented in the form

$$X^{(l)}(t) = \sum_{s=0}^{l-1} v^{(s+1)} t^{(l-1-s)\alpha} E_{\alpha}^{(l-1-s)}(\lambda t^{\alpha}), \quad l = 1, ..., k. \tag{2.39}$$

The combination of these solutions for all eigenvalues leads to n linearly independent solutions of the system (2.34).

Remark 2.18.
Let $X(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$ be the solution of the initial value problem, consisting of the homogeneous system (2.34) and the initial condition $X(0) = X_0$. Then, the initial value problem for the non-homogeneous fractional order system

$$\begin{cases} {}_0^C D_t^{\alpha} X(t) = AX(t) + B(t), \quad 0 < t \leq a, \\ X(0) = X_0, \end{cases}$$

where

$$B(t) = [b_1(t), b_2(t), ..., b_n(t)]^T,$$

has the solution

$$Y(t) = [y_1(t), y_2(t), ..., y_n(t)]^T,$$

such that

$$y_i(t) = x_i(t) + \int_0^t x_i(\tau - t) b_i(\tau) d\tau.$$

2.5.3 Numerical algorithms for fractional equations

Two sets of the numerical methods have been mainly used in the literature, to solve fractional-order differential equations, namely, the frequency-domain methods [67] and the time-domain methods [24, 25, 27]

The frequency-domain methods have been primarily most frequently used to investigate chaos in fractional order systems [35, 42]. Unfortunately, it has been shown that these approaches are not always reliable for detecting chaos in such systems [70, 72]. Therefore, a great deal of effort has been recently expended over the last years in attempting to find robust and stable numerical as well as analytical time-domain methods for solving fractional differential equations of physical interest. The Adomian decomposition method [4], homotopy perturbation method [59], homotopy analysis method [15], differential transform method [55] and variational iteration method [58] are relatively new approaches to provide an analytical approximate solution to linear and nonlinear fractional differential equations.

An efficient method for solving fractional differential equations in term of Caputo type fractional derivative, is the predictor-corrector scheme or more precisely, PECE (Predict, Evaluate, Correct, Evaluate) [27, 28], which represents a generalization of Adams-Bashforth-Moulton algorithm. This method is described as follows. Let consider the following fractional order initial value problem

$$\begin{cases} {}_0^C D_t^\alpha x &= f(t, x), \quad 0 \leq t \leq T, \\ x^{(k)}(0) &= x_0^k, \quad k = 0, 1, 2, ..., \lceil \alpha \rceil - 1 \end{cases} \tag{2.40}$$

which is equivalent to the Volterra integral equation

$$x(t) = \sum_{k=0}^{n-1} x_0^k \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{(\alpha-1)} f(\tau, x(\tau)) d\tau. \tag{2.41}$$

Set $h = T/N$ and $t_j = jh$, $(j = 0, 1, 2, ..., N)$ with T being the upper bound of the interval on which we are looking for the solution. Then, the corrector formula for equation (2.41) is given by

$$x_h(t_{n+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} x_0^k \frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, x_h^p(t_{n+1})) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, x_h(t_j)), \tag{2.42}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n. \end{cases} \tag{2.43}$$

By using a one-step Adams-Bashforth rule instead of a one-step Adams-Moulton rule, the predictor $x_h^p(t_{n+1})$ is given by

$$x_h^p(t_{n+1}) = \sum_{k=0}^{n-1} x_0^k \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, x_h(t_j)), \tag{2.44}$$

where

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}((n-j+1)^\alpha - (n-j)^\alpha), \quad 0 \le j \le n. \tag{2.45}$$

The error estimate of this method is

$$\varepsilon = \max_{0 \le j \le n} |x(t_j) - x_h(t_j)| = O(h^p), \tag{2.46}$$

where $p = \min(2, 1+\alpha)$.

Now, the basic algorithm for the fractional Adams-Bashforth-Moulton method is completely described.

For numerical resolution of fractional differential equations in term of Riemann-Liouville derivative we adopt the algorithm derived from the GrünwaldLetnikov definition (2.18). This approach is based on the fact that for a wide class of functions, two definitions GL (2.18) and RL (2.9) are equivalent. The relation for the explicit numerical approximation of the αth derivative at the points kh $(k = 1, 2, ...)$ has the following form

$$^{RL}_{\ 0}D_t^\alpha x(kh) \approx \frac{1}{h^\alpha} \sum_{j=0}^{k} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} x((k-j)h), \tag{2.47}$$

where h is the time step of the calculation and $(-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} = C_j^\alpha, \quad (j = 0, 1, ...)$ are binomial coefficients. For their calculation we can use the following expression

$$C_0^\alpha = 1, \quad C_j^\alpha = (\frac{j - \alpha - 1}{j})C_{j-1}^\alpha. \tag{2.48}$$

The described numerical method is a so-called Power Series Expansion (PSE) of a generating function.

For $t \gg a$ the number of addends in the fractional-derivative approximation (2.42) (2.47) becomes enormously large. However, it follows from the expression for the coefficients in the Grünwald-Letnikov definition (2.17) that for large t the role of the history of behaviour of the function $f(t)$ near the lower terminal $t = a$ can be neglected under certain assumption. Those observations lead Podlubny [62], to the formulation of the short memory principle which mean taking into account the behaviour of $f(t)$ only in the short interval $[t - L, t]$, where L is the memory length

$$_aD_t^\alpha f(t) \approx \ _{t-L}D_t^\alpha f(t), \quad (t > a + L). \tag{2.49}$$

Clearly, the fractional derivative with lower limit a is approximated by the fractional derivative with moving lower limit $t - L$, therefore the number of addends in (2.49) is always less than L/h.
If $f(t) \leq M$ for all $t \in [a, b]$ then the error of approximation is given by [62]

$$\Delta(t) = | \ _aD_t^\alpha f(t) - \ _{t-L}D_t^\alpha f(t)| \leq \frac{M}{L^\alpha |\Gamma(1 - \alpha)|}, \quad (a + L \leq t). \tag{2.50}$$

Thus, in order to obtain a good approximation (i.e; $\Delta(t) \leq \epsilon$) we must choose the memory length L which satisfies

$$L \geq \left(\frac{M}{\epsilon |\Gamma(1 - \alpha)|}\right)^{1/\alpha}. \tag{2.51}$$

Part II

Fractional systems and applications

Chapter 3

Fractional-order dynamical systems

Fractional systems, can be considered as a generalization of integer order systems [51, 60]. In this chapter we will focus our attention on the qualitative study (stability theory, periodic behavior, bifurcation and chaos) of a fractional-order dynamical system given in the following form

$$ {}_0^C D_t^{\alpha} x = f(t, x), \tag{3.1}$$

where $x \in \mathbb{R}^n$, f is defined on a suitable subset $U \subset \mathbb{R}^{n+1}$ and $\alpha = [\alpha_1, \alpha_2, ..., \alpha_n]^T$ are the fractional orders, $0 \leq \alpha_i \leq 1$, $(i = 1, 2, ..., n)$ (we adopt this restriction of fractional order α because fractional equations in this range require only one initial condition to guarantee the uniqueness of the solution). When $\alpha_1 = \alpha_2 = ... = \alpha_n$, the system (3.1) is called a commensurate order system, otherwise it is an incommensurate order system. If f depends explicitly on t then (3.1) is called non-autonomous system otherwise it is called autonomous system. The constant a is an equilibrium point of the Caputo fractional dynamical system (3.1), if and only if $f(t, a) = 0$, for all t.

3.1 Stability theory of fractional systems

A well known and important area of research in theory of dynamical system is the stability theory, the stability of fractional system is different from that in the integer one. When talking about stability, one is interested in the behaviour of solutions of (3.1) for $t \to \infty$. Therefore we will only consider problems whose solutions x exist on $[0, \infty)$. Moreover, some additional assumptions are required in this section. The first assumption is that f is defined on a set $G = [0, \infty) \times \{w \in \mathbb{R}^n : \|w\| < W\}$ with some $0 < W \leq \infty$. The norm of G may be an arbitrary norm on $\mathbb{R}^n$. The second assumption is that f is continuous on its domain of definition and that it satisfies a Lipschitz condition there. This asserts that the initial value problem consisting of (3.1) and the initial condition $x(0) = x_0$ has a unique solution on the interval $[0, b)$ with some $b \leq \infty$ if $\|x_0\| \leq W$. And finally we assume that the function $x(t) = 0$ is a solution of (3.1) for $t \geq 0$. Under these assumptions we may formulate the followings main concepts.

Definition 3.1.
Under the hypothesis mentioned above, The solution $x(t) = 0$ of the system (3.1) is said to be

- Stable if: for any $\epsilon > 0$ there exists some $\delta > 0$ such that the solution of the initial value problem consisting of (3.1) and the initial condition $x(0) = x_0$ satisfies $\|x(t)\| < \epsilon$ for all $t \geq 0$ whenever $\|x_0\| < \delta$.

- Asymptotically stable if: it is stable and there exists some $\gamma > 0$ such that $\lim_{t\to\infty} x(t) = 0$ whenever $\|x_0\| < \gamma$.

Remark 3.2.
A solution y of the differential equation ${}_0^C D_t^\alpha x = g(t, x)$ is said to be (asymptotically) stable if and only if the zero solution of ${}_0^C D_t^\alpha z = f(t, z)$ with $f(t, z) = g(t, z + y(t)) - g(t, y(t))$ is (asymptotically) stable.

Definition 3.3. (Exponential stability).
The solution $x(t) = 0$ of the system (3.1) is said to be (locally) exponentially stable if there exist two real constants $\alpha, \lambda > 0$ such that

$$\|x(t)\| \leq \alpha \|x(t_0)\| e^{-\lambda t} \quad \text{for all } t > t_0, \tag{3.2}$$

whenever $\|x(t_0)\| < \delta$. It is said to be globally exponentially stable if (3.2) holds for any $x(t_0) \in \mathbb{R}^n$.

A generalization of exponential stability is the Mittag-Leffler stability which is more useful for fractional system.

Definition 3.4. (Mittag-Leffler stability).
The solution $x(t) = 0$ of (3.1) is said to be Mittag-Leffler stable if

$$\|x(t)\| \leq \{m[x(t_0)] E_\alpha(-\lambda(t - t_0)^\alpha)\}^b,$$

where t_0 is the initial time, $\alpha \in (0,1)$ the fractional order, $\lambda \geqslant 0$, $b > 0$, $m(0) = 0$, $m(x) \geqslant 0$ and $m(x)$ is locally Lipschitz on $x \in \mathbf{B} \subset \mathbb{R}^n$ with constant Lipschitz m_0.

Definition 3.5. (Generalized Mittag-Leffler stability).
The solution $x(t) = 0$ of (3.1) is said to be generalized Mittag-Leffler stable if

$$\|x(t)\| \leq \{m[x(t_0)](t-t_0)^{-\gamma} E_{\alpha,1-\gamma}(-\lambda(t-t_0)^{\alpha})\}^b,$$

where t_0 is the initial time, $\alpha \in (0,1)$ the fractional order, $-\alpha < \gamma \leqslant 1-\alpha$, $\lambda \geqslant 0$, $b > 0$, $m(0) = 0$, $m(x) \geqslant 0$ and $m(x)$ is locally Lipschitz on $x \in \mathbf{B} \subset \mathbb{R}^n$ with constant Lipschitz m_0.

Notice here that the Mittag-Leffler stability and Generalized Mittag-Leffler stability imply asymptotic stability.

As mentioned in [48], the stabilities of fractional-order systems are not of exponential type. Thus, a new definition called power law stability $t^{-\beta}$ was introduced in [61], which is a special case of the Mittag-Leffler stability [46] and it is defined as follows.

Definition 3.6. (Power law stability $t^{-\beta}$).
The trajectory $x(t) = 0$ of the system (3.1) is $t^{-\beta}$ asymptotically stable if there is a positive real β such that:

$$\forall \|x(t)\| \quad with \quad t \leq t_0, \exists N(x(t)), \quad \text{such that} \quad \forall t \geq t_0, \|x(t)\| \leq N t^{-\beta}.$$

We begin our analysis by the linear time invariant (LTI) systems.

3.1.1 Stability of fractional LTI systems

Stability of linear fractional order systems, which is of main interest in control theory, has been thoroughly investigated where necessary and sufficient conditions have been derived. In 1996, Matignon [48], have been introduced the stability properties of n-dimensiona linear fractional order systems from a point of view of control. In [23], Deng et al. studied the stability of n-dimensional linear fractional differential equation with time delays. An interesting difference between stable integer-order system and a stable fractional-order system is that the last one may have roots in right half of the complex plane.

Theorem 3.7.
Consider the N-dimensional linear differential system with fractional commensurate order α

$$ {}_0^C D_t^\alpha X = AX, \tag{3.3} $$

where A is an arbitrary constant $N \times N$ matrix.
a) The system (3.3) is asymptotically stable if and only if $|\arg(spec(A))| > \alpha\pi/2$. In this case the components of the state decay towards 0 like $t^{-\alpha}$.
b)The system (3.3) is stable if and only if $|\arg(spec(A))| \geq \alpha\pi/2$ and all eigenvalues with $|\arg(\lambda)| = \alpha\pi/2$ have a geometric multiplicity that coincides with their algebraic multiplicity.

The fact that the components of $x(t)$ slowly decay towards 0 following $t^{-\alpha}$ leads to fractional systems, sometimes, being called long memory systems.
In the limit case $\alpha \to 1$ we recover the well known classical result [19], that the eigenvalues must have negative real parts in case (a) and non-positive real parts and a full set of eigenvectors if the real parts are zero for case (b).

Proof.
If the matrix A is diagonalisable then according to theorem (2.14) and remark(2.15) the general solution is given by

$$ X(t) = \sum_{k=1}^{n} C_k u^{(k)} E_\alpha(\lambda_k t^\alpha), \tag{3.4} $$

and by theorem (1.4) its Laplace transform is

$$ X(s) = \sum_{k=1}^{n} C_k u^{(k)} \frac{s^{\alpha-1}}{s^\alpha - \lambda_k}. \tag{3.5} $$

If A is not diagonalisable then according to theorem (2.17), the general solution can be given by a linear combination of a fundamental solutions given by

$$ X^{(j,l)}(t) = \sum_{i=0}^{l-1} v^{(j,i+1)} t^{(l-1-i)\alpha} E_\alpha^{(l-1-i)}(\lambda_j t^\alpha), \quad l = 1, ..., k_j \;\; j = 1, ..., m. \tag{3.6} $$

where k_j is the multiplicity of eigenvalue λ_j and $\sum_{j=1}^{m} k_j = n$.
Taking into account the relation (1.9) and applying Laplace transform to both sides of (3.6) yields

$$X^{(j,l)}(s) = \sum_{i=0}^{l-1} v^{(j,i+1)} \frac{(l-i-1)!s^{\alpha-1}}{(s^\alpha - \lambda_j)^{l-i}}, \quad l = 1, ..., k_j \ \ j = 1, ..., m. \tag{3.7}$$

Now, if all eigenvalues lie in the region $|\arg(\lambda^{\frac{1}{\alpha}})| > \frac{\pi}{2}$; (i.e $|\arg(\lambda)| > \alpha\pi/2$), then using (3.5), (3.7) and the final value theorem (1.5) we get

$$\lim_{t\to\infty} X(t) = \lim_{s\to 0} sX(s) = 0.$$

If there is some eigenvalues lie in the region $|\arg(\lambda)| < \alpha\pi/2$, then from theorem(3.18) we have

$$\lim_{t\to\infty} |E_\alpha(\lambda t^\alpha)| = \infty$$

Thus from (3.4) and (3.6), $X(t)$ is unbounded.

Therefore, the system (3.3) is asymptotically stable if and only if all eigenvalues lie in the region $|\arg(\lambda)| > \alpha\pi/2$. □

Next we consider the stability of incommensurate rational order system [23].

Corollary 3.8.
Suppose that $\alpha_1 \neq \alpha_2 \neq ... \neq \alpha_n$ *and all* α_i*'s are rational numbers between* 0 *and* 1*, and suppose that* m *is the lowest common multiple of the denominators* u_i *of* α_i*,* $(i = 1, ..., n)$ *where* $\alpha_i = \frac{v_i}{u_i}$*,* $v_i, u_i \in \boldsymbol{Z}^+$ *for* $i = 1, ..., n$*, and setting* $\gamma = \frac{1}{m}$ *then system (3.3) is asymptotically stable if:*

$$|\arg(\lambda)| > \gamma\frac{\pi}{2} \tag{3.8}$$

for all roots λ *of the following characteristic equation*

$$det(diag([\lambda^{m\alpha_1}, ..., \lambda^{m\alpha_n}]) - A) = 0. \tag{3.9}$$

The characteristic equation of (3.3) is of fractional powers of s, this corollary tells that in case of rational orders the characteristics equation can be transformed to an integer-order polynomial equation.

Proof.
The application of the Laplace transform to both sides of (3.3) gives the equation

$$(diag([s^{\alpha_1}, ..., s^{\alpha_n}]) - A)X(s) = (s^{\alpha_1 - 1}x_1(0), ..., s^{\alpha_n - 1}x_n(0))^T, \tag{3.10}$$

multiplying s on both sides of (3.10) gives

$$(diag([s^{\alpha_1}, ..., s^{\alpha_n}]) - A)sX(s) = (s^{\alpha_1}x_1(0), ..., s^{\alpha_n}x_n(0))^T, \tag{3.11}$$

which does not have an unique solution $sX(s)$ only when

$$det(diag([s^{\alpha_1}, ..., s^{\alpha_n}]) - A) = 0. \tag{3.12}$$

Denoting $s = \lambda^{\frac{1}{\gamma}} = \lambda^m$ and subtracting in (3.12) yields the equation (3.9). If all roots of the equation (3.12) lie in open left half complex plane, $Re(s) < 0$ (i.e $|arg(s)| > \frac{\pi}{2}$ wich imply $|arg(\lambda)| > \gamma\frac{\pi}{2}$), then we consider (3.11) in $Re(s) \geq 0$. In this restricted area, (3.11) has a unique solution $sX(s) = (sX_1(s), ..., sX_n(s))$. So, we have

$$\lim_{s \to 0, Re(s) \geq 0} sX_i(s) = 0, i = 1, ..., n.$$

Using the final-value theorem of Laplace transform, we get

$$\lim_{t \to \infty} x_i(t) = \lim_{s \to 0, Re(s) \geq 0} sX_i(s) = 0, i = 1, ..., n.$$

This complete the proof. □

Theorem 3.3, remain valid [56], in the case $1 < \alpha < 2$.

3.1.2 Stability of fractional nonlinear systems

Let consider the commensurate fractional-order nonlinear autonomous system given by

$${}_0^C D_t^{\alpha} x = f(x) \tag{3.13}$$

where $x \in \mathbb{R}^n$, f is defined on a suitable subset $U \subset \mathbb{R}^n$. According to stability theorem defined in [71] and [2], an equilibrium point $\tilde{x}$ of system (3.13) is locally

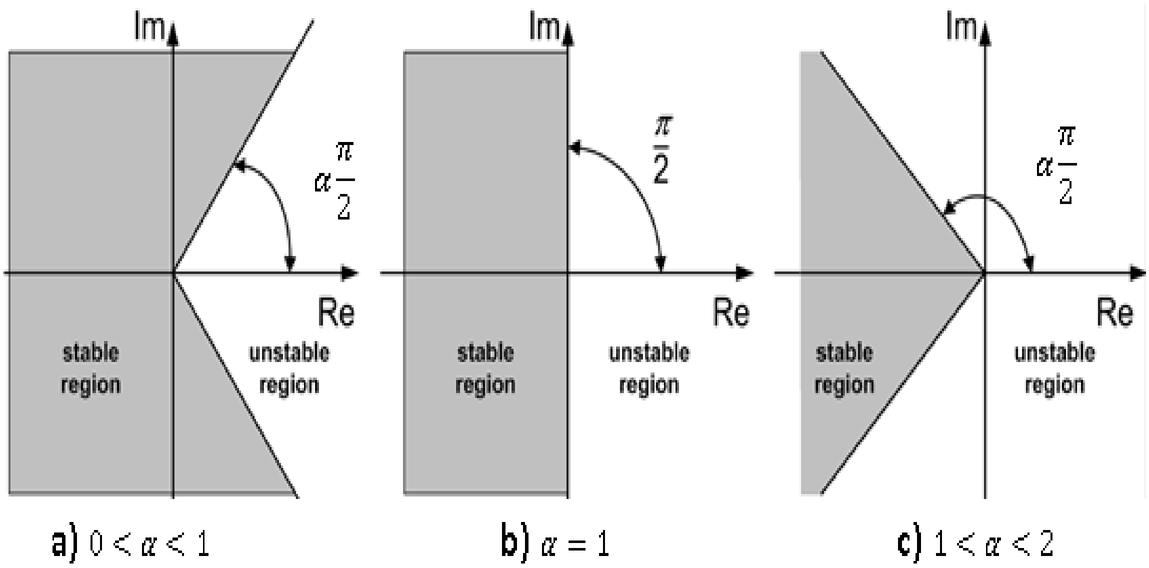

FIGURE 3.1: Stability region for fractional-order systems

asymptotically stable for a given α in $(0, 2)$ if all the eigenvalues $\lambda_i, (i = 1, 2, ..., n)$ of the Jacobian matrix $J = \frac{\partial f}{\partial x}|_{x=\tilde{x}}$ satisfy the condition

$$|\arg(\lambda_i)| > \alpha\frac{\pi}{2}, \quad i = 1, 2, ..., n. \tag{3.14}$$

Remark 3.9.
The given theoretical results make clear that the stability condition for fractional-order systems differs from the well-known condition for integer order systems. In particular, the left half-plane (stable region) for integer-order systems maps into the angular sector $|\arg(spec(J))| > \alpha\pi/2$ in the case of fractional-order systems, indicating that the stable region becomes larger and larger when the value of fractional-order α is decreased

Fig.3.1 shows stable and unstable regions of the complex plane, for $0 < \alpha < 2$.
Now, let consider the incommensurate fractional order system $\alpha_1 \neq \alpha_2 \neq ... \neq \alpha_n$ and suppose that m is the LCM of the denominators u_i of α_i, $(i = 1, ..., n)$ where $\alpha_i = \frac{v_i}{u_i}$, $v_i, u_i \in \mathbf{Z}^+$ for $i = 1, ..., n$, then the system (3.13) is asymptotically stable if:

$$|arg(\lambda)| > \frac{\pi}{2m},$$

for all roots λ of the following equation

$$det(diag([\lambda^{m\alpha_1}, ..., \lambda^{m\alpha_n}]) - J) = 0.$$

3.1.3 Some Routh-Hurwitz conditions for fractional systems

Routh-Hurwitz criterion is a powerful tool used for the stability analysis of some parameter dynamical systems, because it provides an opportunity to study the stability of such parameter system without the need to set its control parameters, therefore we can identify the stability region in the parameter space, this technique is extensively used in the area of control and synchronization. Some Routh-Hurwitz stability conditions are generalized to the fractional order case in [7], and largely used in field of control and synchronization [3, 49]. Consider the commensurate system

$$ {}_{0}^{C}D_{t}^{\alpha}x = f(x, \mu), \tag{3.15} $$

where $x \in \mathbb{R}^n$ is the state space vector, $\mu \in \mathbb{R}^m$ is the parameter vector and f is defined on a suitable subset $U \subset \mathbb{R}^n \times \mathbb{R}^m$. An interesting question arises when analysing the condition (3.14), namely, what are the conditions on μ, that all the roots of the polynomial equation

$$ P(\lambda) = \lambda^n + a_1(\mu)\lambda^{n-1} + ... + a_{n-1}(\mu)\lambda + a_n(\mu) = 0, \tag{3.16} $$

satisfy (3.14) where all the coefficients in (3.16) are real?

For $\alpha = 1$ the answer is given by the classical Routh-Hurwitz criterion [52] that is

$a_1 > 0$,

$$ \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} > 0 , $$

$$ \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0, $$

...

$$\begin{vmatrix} a_1 & 1 & 0 & \dots & & 0 \\ a_3 & a_2 & a_1 & 1 & & 0...0 \\ a_5 & a_4 & a_3 \dots & & 1 & 0...0 \\ . & . & . & . & & . \\ 0 & 0 & 0 & \dots & & a_n \end{vmatrix} > 0.$$

For $\alpha \in (0,1)$ the classical Routh-Hurwitz conditions are sufficient but not necessary, therefore we need a new version of this criterion that will be adopted in the last case.

Definition 3.10.
The discriminant $D(P)$ of a polynomial $P(\lambda)$ is defined by

$$D(P) = (-1)^{n(n-1)/2} R(P, P'),$$

where P' is the derivative of P and $R(P, P')$ is the $(2n-1) \times (2n-1)$ resultant of $P(\lambda)$ and its derivative $P'(\lambda)$, given as follows

$$R(P, P') = \begin{vmatrix} 1 & a_1 & \dots & a_n & 0 & \dots & 0 \\ 0 & 1 & a_1 & \dots & a_n & 0... & 0 \\ . & . & . & . & . & . & . \\ 0 & \dots & 0 & 1 & a_1 & \dots & a_n \\ n & (n-1)a_1 & \dots & a_{n-1} & 0 & \dots & 0 \\ 0 & n & (n-1)a_1 & \dots & a_{n-1} & 0... & 0 \\ . & . & . & . & . & . & . \\ 0 & \dots & 0 & n & (n-1)a_1 & \dots & a_{n-1} \end{vmatrix}$$

For $n = 3$, we have

$$D(p) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3(a_1)^3 - 4(a_2)^3 - 27(a_3)^2.$$

Noting that if $D(P) > 0$ (< 0), there is an even (odd) number of pairs of complex roots for the equation $P(\lambda) = 0$.
For $n = 3$, $D(P) > 0$ implies that all the roots are real, and $D(P) < 0$ implies that there is only one real root and one pair of complex conjugate roots.

Proposition 3.11.
1) For $n = 1$, the condition for (3.14) is $a_1 > 0$.
2) For $n = 2$:

-If $D(p) \geq 0$, the condition for (3.14) is $a_1 > 0$ and $a_2 > 0$.
-If $D(p) < 0$, the condition for (3.14) is $\left|\tan^{-1}\left(\frac{\sqrt{4a_2-(a_1)^2}}{a_1}\right)\right| > \alpha\frac{\pi}{2}$.
3) For $n = 3$:
- When $D(p) > 0$, the necessary and sufficient conditions of (3.14) are the classical Routh-Hurwitz conditions given by $a_1 > 0$, $a_3 > 0$ and $a_1a_2 > a_3$.
- When $D(p) < 0$, we distinct the three following cases
a) If $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $\alpha < \frac{2}{3}$ then (3.14) is satisfied.
b) If $a_1 < 0$, $a_2 < 0$ and $\alpha > \frac{2}{3}$ then all roots of $P(\lambda) = 0$ satisfies $|\arg(\lambda)| < \alpha\frac{\pi}{2}$.
c) If $a_1 > 0$, $a_2 > 0$ and $a_1a_2 = a_3$ then (3.14) is satisfied for all $\alpha \in [0, 1)$.
4) For general $n > 1$, the necessary and sufficient condition for (3.14) is

$$\int_0^{\infty} \frac{dz}{P(z)}\bigg|_{C_2} + \int_{-\infty}^{0} \frac{dz}{P(z)}\bigg|_{C_1} = 0,$$

where C_1 is the curve

$$z = x(1 - i\tan\alpha\pi/2),$$

and C_2 is the curve

$$z = x(1 + i\tan\alpha\pi/2).$$

Proof.
1)For $n = 1$, we have $P(\lambda) = \lambda + a_1$ which posses a single real root $\lambda = -a_1$ then (3.14) is satisfied if and only if $a_1 > 0$.

2)For $n = 2$, we have

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2,$$

its roots are given by

$$\lambda_\pm = \frac{-a_1 \pm \sqrt{(a_1)^2 - 4a_2}}{2}.$$

-If $D(p) \geq 0$, $\lambda_\pm$ are real and (3.14) will be converted to classical Routh-Hurwitz conditions, namely $a_1 > 0$, $a_2 > 0$.

-If $D(p) < 0$, $\lambda_\pm$ are complex conjugates and the condition (3.14) is equivalent to

$$\left|\tan^{-1}\left(\frac{\sqrt{4a_2 - (a_1)^2}}{a_1}\right)\right| > \alpha\frac{\pi}{2}.$$

3)For $n = 3$, we have:

-When $D(p) > 0$ then all the roots of $P(\lambda) = 0$ are real. Thus, the classical Routh-Hurwitz conditions are equivalents to (3.14).

-When $D(p) < 0$, the roots of $P(\lambda) = 0$ are one real $\lambda_0 = -b$ and a complex conjugate pair $\lambda_\pm = \beta \pm i\gamma$. Hence,

$$P(\lambda) = (\lambda + b)(\lambda - \beta - i\gamma)(\lambda - \beta + i\gamma),$$

and its coefficients are

$$a_1 = b - 2\beta,\ a_2 = \beta^2 + \gamma^2 - 2b\beta,\ a_3 = b(\beta^2 + \gamma^2).$$

a)

$$a_3 > 0 \text{ imply } b > 0,$$

$$a_1 > 0 \text{ imply } b > 2\beta,$$

and

$$a_2 > 0 \text{ imply } \frac{\beta^2}{cos^2(\theta)} > 2b\beta > 4\beta^2$$

thus

$$\theta > \frac{\pi}{3}$$

where $\theta = |\arg(\lambda)|$, so if $\alpha < \frac{2}{3}$ then (3.14) is satisfied.

b)

$$a_1 < 0 \text{ imply } b < 2\beta$$

and

$$a_2 < 0 \text{ imply } \frac{\beta^2}{cos^2(\theta)} < 2b\beta < 4\beta^2$$

thus

$$\theta < \frac{\pi}{3}$$

so if $\alpha > \frac{2}{3}$ then

$$|\arg(\lambda)| < \alpha\frac{\pi}{2}.$$

c)

$$a_1a_2 = a_3 \text{ imply } \beta(\beta^2+\gamma^2)+b^2\beta = 2b\beta^2$$

thus

$$\beta = 0 \; or \; \beta^2+\gamma^2+b^2 = 2b\beta.$$

The last equality is not valid if both $a_1 > 0$ and $a_2 > 0$ thus

$$\min_{\lambda}|\arg(\lambda)| = \frac{\pi}{2}.$$

Therefore (3.14) is satisfied for all $\alpha \in [0,1)$.

4) For general $n > 1$ if $P(z)$ has no roots in the region

$$|\arg(\lambda)| < \alpha\frac{\pi}{2},$$

then the function $\frac{1}{P(z)}$ will be analytic in this region. Using Cauchy theorem

$$\oint_C f(z)dz = 0,$$

for all $f(z)$ analytic within and on C, and the fact that $P(z)$ is polynomial of degree > 1 this completes the proof □

Corollary 3.12.
For general $n > 1$, a necessary condition for (3.14) is $a_n > 0$.

Proof.
For general $n > 1$, we have

$$P(\lambda) = [\prod_i(\lambda+b_i)][\prod_j(\lambda^2-2\beta_j\lambda+\beta_j^2+\gamma_j^2)]$$

then

$$a_n = [\prod_i b_i][\prod_j(\beta_j^2+\gamma_j^2)].$$

So if $a_n \leq 0$, there exists at last i_0 such that $(b_{i_0} \leq 0)$. Hence, there exists at last a positive real rout $(-b_{i_0} \geq 0)$ of $P(\lambda) = 0$. Thus, $\min_\lambda|\arg(\lambda)| = 0$. Therefore, for all $\alpha \in [0,1)$ (3.14) is not satisfied. Hence, the necessary condition for (3.14) is $a_n > 0$.

□

3.1.4 Lyapunov direct method for fractional system

Lyapunov direct method is used for studying both local and global stability of the corresponding systems. In this section we discus the extension of Lyapunov direct method for fractional-order nonlinear systems which leads to the Mittag-Leffler stability [20, 45].

Theorem 3.13.
Let $x = 0$ be an equilibrium point for the system

$$ {}_0^C D_t^{\alpha} x(t) = f(t, x), \quad \alpha \in [0, 1), \tag{3.17}$$

and $\boldsymbol{D} \subset \mathbb{R}^n$ be a domain containing the origin.

Let $V(t, x(t)) : [0, \infty) \times \boldsymbol{D} \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x such that

$$ \alpha_1 \|x\|^a \leq V(t, x(t)) \leq \alpha_2 \|x\|^{ab}, \tag{3.18}$$

$$ {}_0^C D_t^{\beta} V(t, x(t)) \leq -\alpha_3 \|x\|^{ab}, \tag{3.19}$$

where $t \geq 0, x \in \boldsymbol{D}, \beta \in [0, 1), \alpha_1, \alpha_2, \alpha_3, a$ and b are arbitrary positive constants. Then, $x = 0$ is Mittag-Leffler stable.
If the assumptions hold globally on $\mathbb{R}^n$. Then, $x = 0$ is globally Mittag-Leffler stable.

The following theorem gives a generalized fractional Lyapunov direct method.

Theorem 3.14.
Let $x = 0$ be an equilibrium point for the system (3.17) and $\boldsymbol{D} \subset \mathbb{R}^n$ be a domain containing the origin.

Let $V(t, x(t)) : [0, \infty) \times \boldsymbol{D} \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x such that

$$\alpha_1 \|x\|^a \leq V(t, x(t)) \leq \alpha_2 \, {}_0^C D_t^{-\eta} \|x\|^{ab}, \tag{3.20}$$

$${}_0^C D_t^{\beta} V(t, x(t)) \leq -\alpha_3 \|x\|^{ab}, \tag{3.21}$$

where $t \geq 0, x \in \boldsymbol{D}, \beta \in [0, 1), \eta \neq \beta, \eta > 0, |\beta - \eta| < 1, \alpha_1, \alpha_2, \alpha_3, a$ and b are arbitrary positive constants. Then, $x = 0$ is asymptotically stable.

Now we apply the class-K functions to the analysis of fractional Lyapunov direct method.

Definition 3.15.
A continuous function $\alpha : [0, t) \to [0, \infty)$ is said to belong to class-K if it is strictly increasing and $\alpha(0) = 0$.

Lemma 3.16. *(Fractional comparison principle)*
Assume that ${}_0^C D_t^{\alpha} x(t) \geq {}_0^C D_t^{\alpha} y(t)$ and $x(0) = y(0)$, for $\alpha \in (0, 1)$. So $x(t) \geq y(t)$.

Theorem 3.17.
Let $x = 0$ be an equilibrium point for the system (3.17). Assume that there exists a Lyapunov function $V(t, x(t))$. and a class-K functions $\alpha_i(i = 1, 2, 3$ satisfying

$$\alpha_1(\|x\|) \leq V(t, x(t)) \leq \alpha_2(\|x\|), \tag{3.22}$$

$${}_0^C D_t^{\beta} V(t, x(t)) \leq -\alpha_3(\|x\|), \tag{3.23}$$

where $\beta \in [0, 1)$. Then $x = 0$ is asymptotically stable.

Example 3.1.
Let consider the fractional system

$${}_0^C D_t^{\alpha} |x(t)| = -|x(t)|, \tag{3.24}$$

where $\alpha \in (0, 1)$. We choose the Lipschitz function $V(t, x) = |x|$ as a Lyapunov candidate and $\alpha_1 = \alpha_2 = a = b = 1, \alpha_3 = -1$. Then, $\alpha_1 |x(t)|^a \leq V(t, x) \leq \alpha_2 |x(t)|^{ab}$ and ${}_0^C D_t^{\alpha} V(t, x) \leq -|x(t)|$. Applying theorem (3.13) gives the Mittag-Leffler stability of the equilibrium point $x = 0$.

3.2 Periodic solutions

Recently, much attention has been focused on the existence of periodic solutions in fractional-order systems [69, 73, 74, 78, 83]. The aim of this section is to highlight on one of the basic differences between fractional order and integer order systems. It is analytically shown that a time invariant fractional order system contrary to its integer order counterpart cannot generate exactly periodic signals. As a result, a limit cycle cannot be expected in the solution of these systems.

3.2.1 Fractional-order derivatives of periodic functions

Suppose that $x(t)$ is a non-constant periodic function with a specific period T, i.e.

$$x(t+T) = x(t), \quad for\ all\ t \geq 0. \tag{3.25}$$

Taking the derivative of both sides of (3.25) we obtain

$$\frac{dx}{dt}(t+T) = \frac{dx}{dt}(t), \quad for\ all\ t \geq 0. \tag{3.26}$$

Hence, the derivative of a non-constant periodic function $x(t)$ with period T is a periodic function with the same period T. Now we ask the following reasonable question. is there a similar result for the fractional derivative of a non-constant periodic function? A negative answer for this question is claimed in [69].

Theorem 3.18.
Suppose that $x(t)$ is a non-constant periodic function with a specific period T and m-times differentiable. The fractional-order derivative function ${}_0D_t^{\alpha}x(t)$ (symbol ${}_0D_t^{\alpha}$ denote the Riemann-Liouville, Grünwald-Litnikov or Caputo fractional-order derivative operator) where $0 < \alpha \notin \boldsymbol{N}$ and m is the first integer greater than α, cannot be a periodic function with period T.

Proof.
The proof of this theorem can be found in [69] □

Example 3.2.
Consider the function $x(t) = sin(t)$, the Laplace transform of fractional-order derivative of $x(t)$ is given as

$$L(\ {}_0D_t^{\alpha}x(t)) = s^{\alpha}X(s) = \frac{s^{\alpha}}{1+s^2} \tag{3.27}$$

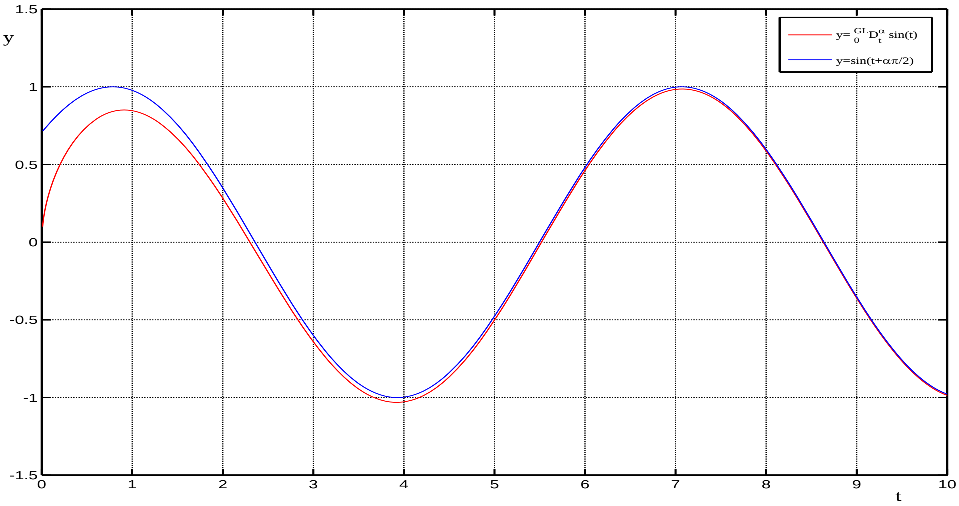

FIGURE 3.2: Graphical representation of the fractional derivative $_0D_t^{0.5}sin(t)$ and the function $sin(t + 0.5\frac{\pi}{2})$

where $0 < \alpha < 1$. *The inverse Laplace transform of (3.27) is obtained as*

$$_0D_t^{\alpha}x(t) = t^{1-\alpha}E_{2,2-\alpha}(-t^2). \tag{3.28}$$

For $\alpha = 1$ *the function* $t^{1-\alpha}E_{2,2-\alpha}(-t^2)$ *is a non-constant periodic function and for* $0 < \alpha < 1$ *this function is not periodic, but it asymptotic converges to the periodic function* $sin(t + \alpha\pi/2)$ *as shown in Figure 3.2.*

3.2.2 Non-existence of periodic solutions in a class of fractional-order systems

Given a fractional-order time-invariant system based on the Caputo derivative and a vector of continuous functions f in the form

$$^C_0D_t^{\alpha}x(t) = f(x), \quad \alpha \in (0, 1) \tag{3.29}$$

a non-constant solution

$$x(t) = (x_1(t), x_2(t), ..., x_n(t))^T,$$

of the system (3.29) is said to be a periodic solution if there exists a constant $T > 0$ such that:

$$x(t+T) = x(t), \tag{3.30}$$

for all $t \geq 0$. The minimum of such T is called period of this solution. The periodic orbit or cycle is the image of the interval $[0, T]$ under

$$x(t) = (x_1(t), x_2(t), ..., x_n(t))^T,$$

in the state space $\mathbb{R}^n$ [79].

The main outcome of this section is summarized in the following theorem:

Theorem 3.19.
The fractional-order time-invariant system (3.29) defined via the Caputo derivative cannot have any non-constant smooth periodic solution.

This result has been extended in [69], by analytically proving that fractional-order system (3.29) based on Grünwald-Letnikov derivative or Riemann-Liouville derivative cannot generate exact periodic solutions.

Proof.
Suppose that $\tilde{x}(t)$ is a solution for differential equation (3.29). If $\tilde{x}(t)$ is a non-constant periodic function with periodic T, then

$$f(\tilde{x}(t)) = f(\tilde{x}(t+T)), \tag{3.31}$$

for all $t > 0$. From (3.29) and (3.31) we get

$${}_0D_t^\alpha \tilde{x}(t+T) = {}_0 D_t^\alpha \tilde{x}(t),$$

for all $t > 0$, which is in contradiction with theorem (3.18). Consequently, $\tilde{x}(t)$, as a solution of differential equation (3.29), cannot be a non-constant periodic function. □

Example 3.3.
Consider the marginally stable LTI system

$$\begin{cases} {}_0^C D_t^\alpha x = k\ cos(\alpha\frac{\pi}{2})x + k\ sin(\alpha\frac{\pi}{2})y \\ {}_0^C D_t^\alpha y(t) = -k\ sin(\alpha\frac{\pi}{2})x + k\ cos(\alpha\frac{\pi}{2})y. \end{cases} \tag{3.32}$$

System (3.32) can be written in matricial form as follows

$$ {}_0^C D_t^\alpha X = AX \tag{3.33}$$

where

$$A = \begin{pmatrix} k\ cos(\alpha\frac{\pi}{2}) & k\ sin(\alpha\frac{\pi}{2}) \\ -k\ sin(\alpha\frac{\pi}{2}) & k\ cos(\alpha\frac{\pi}{2}) \end{pmatrix},$$

and

$$X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix A has two complex conjugate eigenvalues

$$\lambda_{1,2} = k(\ cos(\alpha\frac{\pi}{2}) \pm isin(\alpha\frac{\pi}{2})),$$

and its corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix},$$

and

$$v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Hence, the general solution of (3.32) is given by

$$\begin{aligned} X(t) = {} & \frac{1}{2}\left[c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]\left[E_\alpha(\lambda_1 t^\alpha) + E_\alpha(\lambda_2 t^\alpha)\right] \\ & -\frac{1}{2i}\left[c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} - c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]\left[E_\alpha(\lambda_1 t^\alpha) - E_\alpha(\lambda_2 t^\alpha)\right], \end{aligned} \tag{3.34}$$

where c_1, c_2 are arbitrary real numbers.

For special case $\alpha = 1$ the solution can be rewritten as

$$X(t) = \begin{pmatrix} c_1(cos(kt) - sin(kt)) \\ c_2(cos(kt) + sin(kt)) \end{pmatrix}. \tag{3.35}$$

If $(x_0, y_0) \neq (0, 0)$*, then all solutions of system (3.32) are periodic of period* $\frac{2\pi}{k}$*. But, for case* $0 < \alpha < 1$*, non-zero solutions of (3.32) are not periodic, although they converge to periodic signals [74].*

Remark 3.20.
The fractional order system with lower terminal of $a = \pm\infty$ could have periodic solutions [83].

3.3 Bifurcation and chaos in fractional systems

Chaotic systems have been a focal point of renewed interest for many researchers in the past few decades. Such non-linear systems can occur in various natural and man-made systems, and are known to have great sensitivity to initial conditions. Thus, two trajectories starting at arbitrarily nearby initial conditions in such systems could evolve in drastically different fashions, and soon become totally uncorrelated. At first glance, chaotic time trajectories look very much like noise. In fact, chaotic signals and noise have similar broad-band frequency spectrum characteristics. However, there is a fundamental difference between noise and chaos, which is determinism. Chaos can be classified as deterministic but unpredictable. Whereas noise is neither deterministic nor predictable. This unpredictability of chaotic time signals has been utilized for secure communication applications [6]. Basically, the useful signal is encapsulated in a chaotic envelope (produced by a chaotic oscillator) at the transmitter end, and is transmitted over the communication channel as a chaotic signal. At the receiver end, the information-bearing signal is recovered using various techniques [8]. It has been shown that fractional-order systems, as generalizations of many well-known systems, can also behave chaotically, such as the fractional order systems of Lorenz [34], Chua [35], Chen [44], Rössler [43], Coullet [65], modified Van der Pol-Duffing [50] and Liu [77].
It has been shown that, chaos in fractional order autonomous systems can occur for orders less than three and this cannot happen in their integer order counterparts according to the Poincaré-Bendixon theorem [38].
Various scenarios of transition to chaos have been detected in fractional order systems. A well know one of them is the period doubling to chaos which is initialized in general by a Hopf bifurcation [3, 68].

3.3.1 Hopf bifurcation

Different from integer order systems, there exist less theoretical tools to study dynamics of fractional-order systems. Recall that Hopf bifurcation in integer order systems can be investigated in detail by means of normal form theory and center manifold theorem [36], while similar tools have not yet developed for fractional systems. So detailed results about fractional Hopf bifurcation are few. Only through stability theory of equilibrium points and numerical simulations, we have analyzed Hopf bifurcation of 3-dimensional fractional-order systems in [3].
Let consider the following three-dimensional fractional-order commensurate system:

$$D^q x = f(\beta, x) \tag{3.36}$$

where $q \in]0,2[$, $x \in \mathbb{R}^3$, and suppose that E is an equilibrium point of this system. In the integer case (when $q = 1$) the stability of E is related to the sign of $Re(\lambda_i), i = 1,2,3$ where λ_i are the eigenvalues of the jacobian matrix $\frac{\partial f}{\partial x}|_E$.

If $Re(\lambda_i) < 0$ for all $i = 1,2,3$ then E is locally asymptotically stable. If there exist i such that $Re(\lambda_i) > 0$ then E is unstable.

The conditions of system (3.36) with $q = 1$, to undergo a Hopf bifurcation at the equilibrium point E when $\beta = \beta^*$, are

- The jacobian matrix has two complex-conjugate eigenvalues $\lambda_{1,2}(\beta) = \theta(\beta) \pm i\omega(\beta)$, and one real $\lambda_3(\beta)$ (this can be expressed by $D(P_E(\beta)) < 0$),
- $\theta(\beta^*) = 0$, and $\lambda_3(\beta^*) \neq 0$,
- $\omega(\beta^*) \neq 0$,
- $\left.\frac{d\theta}{d\beta}\right|_{\beta=\beta^*} \neq 0$.

But in the fractional case the stability of E is related to the sign of

$$m_i(q,\beta) = q\frac{\pi}{2} - |\arg(\lambda_i(\beta))|, \quad i = 1,2,3.$$

If $m_i(q,\beta) < 0$ for all $i = 1,2,3$, then E is locally asymptotically stable.
If there exist i such that $m_i(q,\beta) > 0$, then E is unstable.
So the function $m_i(q,\beta)$ has a similar effect as the real part of eigenvalue in integer systems, therefore we extend the Hopf bifurcation conditions to the fractional systems by replacing $Re(\lambda_i)$ with $m_i(q,\beta)$ as follows

- $D(P_E(\beta)) < 0$,
- $m_{1,2}(q, \beta^*) = 0$, and $\lambda_3(\beta^*) \neq 0$,
- $\left.\frac{\partial m}{\partial \beta}\right|_{\beta=\beta^*} \neq 0$.

Remark 3.21.
The limit cycle which appear through a Hopf bifurcation is not a solution for a fractional system but it attracts a nearby solutions.

3.3.2 A necessary condition to have chaos in fractional-order systems

A saddle point in a 3-D nonlinear integer order system, is an equilibrium point on which the equivalent linearized model, has at least, one eigenvalue in the stable region and one in the unstable region. A saddle point is of index 1 if one of the eigenvalues is in the unstable region and others are in the stable region. A saddle point is of index 2 if two eigenvalues are in the unstable region and one is in the stable region. In chaotic systems, it is found that scrolls are generated only around the saddle points of index 2. The saddle points of index 1 are responsible only for connecting the scrolls [12, 18, 41, 66]. In the 3-D commensurate fractional order systems like their ordinary counterparts, the saddle points of index 2 play a key role in generation of scrolls [21, 22]. Assume that a 3-D chaotic system

$$\dot{x} = f(x),$$

displays a chaotic attractor. For every scroll existing in the chaotic attractor, this system has a saddle point of index 2 encircled by its respective scroll. Suppose that Ω is the set of equilibrium points of the system surrounded by scrolls. The corresponding fractional system

$$D^{\alpha} x = f(x),$$

possesses the same equilibriums points. Hence, a necessary condition for fractional order system to exhibit the chaotic attractor similar to its integer order counterpart is instability of the equilibrium points in Ω. Otherwise, one of these equilibrium points becomes asymptotically stable and attracts the nearby trajectories. According to (3.14), this necessary condition is mathematically equivalent to [75]

$$\alpha\frac{\pi}{2} - \min_i(|\arg(\lambda_i)|) \geq 0.$$

However, referring to 3-D integer-order systems, recent findings have shown that in general case the local instability of the equilibrium points cannot be considered as a necessary condition to generate chaos. For example, in [82], a simple 3-D autonomous system displays a chaotic attractor located around two stable node-type of foci as its only equilibrium points. Additionally, in [76], it has been reported a 3-D autonomous chaotic system that has only one equilibrium and furthermore, this equilibrium is a stable node-focus. these recent findings make clear that in general case a necessary condition to generate chaos is the global instability of the equilibrium points. In order to confirm the existence or no-existence of chaotic behaviores in a fractional-order system, two useful tools are a valuable, namely the bifurcation diagram and the Lyapunov exponents.

3.3.3 Lyapunov exponents

Lyapunov exponents were first introduced by Lyapunov in order to study the stability of non-stationary solutions of ordinary differential equations. These exponents provide a meaningful way to categorize steady-state behavior of dynamical systems, determine instability in the system, classify invariant sets, and approximate the dimension of strange attractors or other non-trivial invariant sets. Lyapunov exponents

$$\lambda_i \ (i = 1, ..., n),$$

are the average exponential rates of divergence or convergence of nearby orbits in the state space. The signs of Lyapunov exponents indicate the stability property of the dynamic system. For example, when all Lyapunov exponents are negative, trajectories from all directions in the state space converge to the equilibrium point. In this case, the system is exponentially stable about the equilibrium point and the attractor of the system is a fixed point. If one exponent is zero while others are negative, trajectories converge from all and the attractor in the state space is a one-dimensional curve. If the trajectory is further bounded and forms a closed loop, the system performs a periodic motion and has a stable limit cycle. Two zero Lyapunov exponents mean that the attractor is a two-dimensional torus in the state space, indicating quasi-periodic motion. If at least one Lyapunov exponent is positive, two initially nearby trajectories separate at an exponential rate and the system is chaotic. The computation of Lyapunov characteristic exponents

(LCE) for nonlinear dynamical systems is a fundamental problem for understanding the dynamical behaviour of nonlinear systems, and can be classified on two set (analytical methods based on the mathematical model and numerical methods based on an observed time series). Many researches works have been devoted to this end, including [9–11, 31, 39, 81, 84].

Although an autonomous fractional system cannot define a dynamical system in the sense of semigroup because of the memory property determined by the fractional derivative, we can't use directly classical analytical methods for computation of Lyapunov exponents in fractional systems based on the knowledge of Jacobian matrix, but we can still estimating Lyapunov exponents from time series data after performing a phase-space reconstruction. A time series is a sequence of observations which are ordered in time. Since a single experimental time series is affected by all of the relevant dynamical variables, it contains a relatively complete historical record of the dynamics. The procedure of calculating Lyapunov exponents from a time series can be summarized in the following steps [81, 84]:

1. Reconstructing the dynamics in a finite-dimensional space. Choose an embedding dimension d_E and construct a d_E-dimensional orbit representing the time evolution of the system by the time-lag method. This means that we define

$$y_i = (x_i, x_{i+T_{lag}}, ..., x_{i+(d_E-1)T_{lag}}), \tag{3.37}$$

   where T_{lag} is the time lag. Equation (3.37) provides the fiducial trajectory for the analysis of Lyapunov exponents.

2. Determining the neighbors y_j of y_i , i.e., the point of the orbit which are contained in a shell of suitable radius r, and centered at y_i

$$r_{\min} \leq \|y_j - y_i\| \leq r. \tag{3.38}$$

3. Determining the $d_E \times d_E$ matrix J_i which describes how the time evolution sends small vectors around y_i to small vectors around y_{i+1}. The matrix J_i is

obtained by looking for neighbors y_j of y_i, and imposing

$$J_i(y_j - y_i) \approx y_{j+1} - y_{i+1}. \tag{3.39}$$

The elements of J_i are obtained by a least-squares method then we obtain a sequence of matrices $J_1, J_2, J_3....$

4. Using QR decomposition, one determines successively orthogonal matrices $Q_{(j)}$ and upper triangular matrices $R_{(j)}$ with positive diagonal elements such that $Q_{(0)}$ is the unit matrix and

$$\begin{aligned} J_1 Q_{(0)} &= Q_{(1)} R_{(1)}, \\ J_2 Q_{(1)} &= Q_{(2)} R_{(2)}, \\ &\dots \\ J_{j+1} Q_{(j)} &= Q_{(j+1)} R_{(j+1)}. \end{aligned}$$

   This decomposition is unique except in the case of zero diagonal elements. Then Lyapunov exponents λ_K^i are given by

$$\lambda_K^i = \frac{1}{TK} \sum_{j=0}^{K-1} \ln R_{(j)ii}, \tag{3.40}$$

   where K is the available number of matrices, T is sampling time step, and $i = 1, 2, ..., d_E$.

5. Repeating Step 2 through Step 4 along the fiducial trajectory until the convergent Lyapunov exponents are achieved.

Another approach for estimating Lyapunov exponents in fractional-order systems recently introduced is the semi-analytical method in [16].

3.3.4 The 0-1 test for validating chaos

An efficient binary test for chaos called the $0-1$ test has been recently proposed and applied to fractional systems in [13, 14]. The idea underlying the test is to construct a random walk-type process from the data and then to examine how the variance of the random walk scales with time. Specifically, consider a set of discrete data, sampled at times $n = 1, 2, 3, ...$, representing a one-dimensional observable data set obtained from the system dynamics. Consider the real valued function $p(n)$, as defined in [14]. On the basis of the function $p(n)$, define the mean square displacement $M(n)$. In particular:

- If the behavior of $p(n)$ is Brownian (i.e., the underlying dynamics is chaotic), then $M(n)$ grows linearly in time.
- If the behavior of $p(n)$ is bounded (i.e., the underlying dynamics is non-chaotic), then $M(n)$ is bounded.

Thus it should be examined whether the asymptotic growth rate

$$K = \lim_{n \to \infty} \frac{logM(n)}{logn},$$

approaches 0 or 1.

When K is close to 0, the motion is classified as regular (i.e. periodic or quasi-periodic); when K is close to 1, the motion is classified as chaotic.

Chapter 4

Applications

This chapter is devoted to the application of the tools previously presented, as a results three papers are published in a certain international journals.
In the first paper [1] titled "A new chaotic attractor from hybrid optical bistable system" we postulate a new three-dimensional autonomous chaotic system, where the single quadratic nonlinearity in the original hybrid optical bistable system is replaced by a single cubic non linearity; the new system can generates two 1-scroll chaotic attractors simultaneously or one 2-scroll chaotic attractor. The chaotic behaviors are validated by means of Bifurcation diagram with an associated Poincaré map and the Lyapunov exponent spectrum.
The second paper [3] presents and analyzes the fractional-order modified hybrid optical system, furthermore fractional Hopf bifurcation conditions are postulated. It has been demonstrated that chaos, as well as the other usual nonlinear dynamic phenomena, occur in this systems with mathematical order less than three. The Largest Lyapunov exponents and the bifurcation diagrams shows the period-doubling bifurcation and the transformation from periodic to chaotic motion through the fractional-order and confirms the justness of the proposed fractional Hopf bifurcation conditions (in this system).
The fact that financial variables possess long memories makes fractional modelling appropriate for dynamical behaviors in financial systems. Chaotic phenomenon makes prediction impossible in the financial world then the deletion of this phenomenon from fractional financial system is very useful, the main contribution of the last paper [2] is to this end. Nonlinear feedback control scheme has been extended to control fractional financial system. The results are proved analytically by applying the Lyapunov linearization method and stability condition for fractional system. Numerically the unstable fixed points have been successively

stabilized for different values of fractional-order, moreover unstable periodic orbits has been stabilized.

4.1 A new chaotic attractor from hybrid optical bistable system

Nonlinear Dyn (2012) 67:457–463
DOI 10.1007/s11071-011-9994-5

ORIGINAL PAPER

A new chaotic attractor from hybrid optical bistable system

Mohammed-Salah Abdelouahab · **Nasr-eddine Hamri**

Received: 25 August 2010 / Accepted: 21 February 2011 / Published online: 17 March 2011

Abstract In this work, a new three-dimensional autonomous chaotic system has been introduced by modifying a hybrid optical system. The single quadratic nonlinearity is replaced by a single cubic nonlinearity; the new system can display two 1-scroll chaotic attractors simultaneously or one 2-scroll chaotic attractor. The bifurcation diagram is obtained and Lyapunov spectrum is calculated for the proposed system. The results show that the new system exhibits rich complexity features such as stable, periodic, and chaotic dynamics.

Keywords New chaotic attractor · Hopf bifurcation · Lyapunov spectrum · Bifurcation diagram

1 Introduction

Chaos has been found to be very useful and has great potential in many technological disciplines such as in computer sciences, power systems protection, biomedical systems analysis, flow dynamics and liquid mixing, encryption, and communications.

The first chaotic attractor in a three-dimensional autonomous system was discovered by Lorenz in 1963, while studying atmospheric convection [1]; this system has seven terms on the right-hand side, two of which are nonlinear (xz and xy). In 1976, Rössler found a three-dimensional quadratic autonomous chaotic system [2], which also has seven terms on the right-hand side, but with only one quadratic nonlinearity (xz). Obviously, the Rössler system has a simpler algebraic structure as compared to the Lorenz system. In 1979, Rössler proposed another even simpler (algebraic) system [3], which has only six terms with a single quadratic nonlinearity (y^2). Some attention has been focused on effectively creating chaos via simple physical systems such as electronic circuits and switching piecewise-linear controllers. In 1983, Chua has introduced a simple electronic circuit that exhibits chaotic behavior, which can be accurately modeled by means of a system of three nonlinear ordinary differential equations [4].

In 1984, Flüggen and Mitschke reported the observation of chaos in a hybrid optical bistable device and studied its realization as an electronic circuit; see Fig. 1. The structure of the system is described by a third-order differential equation with a quadratic nonlinearity; this equation can be transformed on a three-dimensional autonomous system which has only six terms with a single quadratic nonlinearity (x^2). This system can display only a 1-scroll attractor [5] and [6]. Many other works in literature focused on introducing new chaotic systems [7–17]. In this paper, we introduce a new three-dimensional autonomous chaotic system by modifying a hybrid optical system; a sin-

M.-S. Abdelouahab (✉) · N.-e. Hamri
Department of Science and Technology, University Center of Mila, Mila 43000, Algeria
e-mail: medsalah3@yahoo.fr

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4.2 Hopf bifurcation and chaos in fractional-order modified hybrid optical system

Nonlinear Dyn (2012) 69:275–284
DOI 10.1007/s11071-011-0263-4

ORIGINAL PAPER

Hopf bifurcation and chaos in fractional-order modified hybrid optical system

Mohammed-Salah Abdelouahab · Nasr-Eddine Hamri · Junwei Wang

Received: 27 July 2011 / Accepted: 1 November 2011 / Published online: 23 November 2011

Abstract In this paper, a chaotic fractional-order modified hybrid optical system is presented. Some basic dynamical properties are further investigated by means of Poincaré mapping, parameter phase portraits, and the largest Lyapunov exponents. Fractional Hopf bifurcation conditions are proposed; it is found that Hopf bifurcation occurs on the proposed system when the fractional-order varies and passes a sequence of critical values. The chaotic motion is validated by the positive Lyapunov exponent. Finally, some numerical simulations are also carried out to illustrate our results.

Keywords Fractional system · Stability · Hopf bifurcation · Chaos

1 Introduction

The idea of fractional calculus has been known since the development of the regular calculus, and it means a generalization of integration and differentiation to arbitrary order. It has been found that many systems in interdisciplinary fields can be described by the fractional differential equations, such as viscoelastic systems, dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, and quantum evolution of complex systems [1–5].

Optics is a field in which the use of conventional calculus plays a major role, and it is of interest to see how fractional calculus may offer useful mathematical tools in this field. For example; fractionalization of Gaussian beams is given in [6], fractionalization of the Fourier transform and its applications has been already studied by several researchers [7–9], a fractional variational optical flow model is introduced in [10], and a new class of nondiffracting fractional vortex beams that connect Bessel beams of successive order in a smooth transition is introduced in [11]. On the other hand, memory effect has been observed in optical systems [12, 13]; this fact makes fractional modeling appropriate for dynamic behaviors in optical systems. Based on the above motivations, one might be tempted to introduce the fractional-order version of the modified hybrid optical system presented in our previous work [14].

There are several definitions of fractional derivatives [15–18].

In this paper, we use the Caputo-type fractional derivative defined in [15] by

M.-S. Abdelouahab (✉) · N.-E. Hamri
Institute of Science and Technology, University Center of Mila, Mila 43000, Algeria
e-mail: medsalah3@yahoo.fr

J. Wang
Cisco School of Informatics, Guangdong University of Foreign Studies, Guangzhou 510006, P.R. China

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4.3 Chaos control of a fractional-order financial system

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 270646, 18 pages
doi:10.1155/2010/270646

Research Article

Chaos Control of a Fractional-Order Financial System

Mohammed Salah Abd-Elouahab,[1] Nasr-Eddine Hamri,[1] and Junwei Wang[2]

[1] *Department of Science and Technology, University Center of Mila, Mila 43000, Algeria*
[2] *Cisco School of Informatics, Guangdong University of Foreign Studies, Guangzhou 510006, China*

Correspondence should be addressed to Mohammed Salah Abd-elouahab, medsalah3@yahoo.fr

Received 4 November 2009; Revised 29 April 2010; Accepted 2 June 2010

Academic Editor: Carlo Cattani

Fractional-order financial system introduced by W.-C. Chen (2008) displays chaotic motions at order less than 3. In this paper we have extended the nonlinear feedback control in ODE systems to fractional-order systems, in order to eliminate the chaotic behavior. The results are proved analytically by applying the Lyapunov linearization method and stability condition for fractional system. Moreover numerical simulations are shown to verify the effectiveness of the proposed control scheme.

1. Introduction

Nonlinear chaotic systems have attracted more attention of researchers in various fields of natural sciences. This is because these systems are rich in dynamics, and possess great sensitivity to initial conditions. Since the chaotic phenomenon in economics was first found in 1985, great impact has been imposed on the prominent western economics at present, because the chaotic phenomenon's occurring in the economic system means that the macroeconomic operation has in itself the inherent indefiniteness. Although the government can adopt such macrocontrol measures as the financial policies or the monetary policies to interfere, the effectiveness of the interference is very limited. The instability and complexity make the precise economic prediction greatly limited, and the reasonable prediction behavior has become complicated as well. In the fields of finance, stocks, and social economics, because of the interaction between nonlinear factors, with all kinds of economic problems being more and more complicated and with the evolution process from low dimensions to high dimensions, the diversity and complexity have manifested themselves in the internal structure of the system and there exists extremely complicated phenomenon and external characteristics in such a kind of system. So it has become more and more important to study the control of the complicated continuous economic system, and stabilize the instable periodic or stationary solutions, in order to make the precise economic prediction possible [1, 2].

Great interest has been paid to the application of fractional calculus in physics, engineering systems, and even financial analysis [3, 4]. The fact that financial variables possess long memories makes fractional modelling appropriate for dynamic behaviors in financial systems. Moreover, the control and synchronization of fractional-order dynamic systems is also performed by various researchers [5–10]. Fractional-order financial system proposed by Chen in [11] displays many interesting dynamic behaviors, such as fixed points, periodic motions, and chaotic motions. It has been found that chaos exists in this system with orders less than 3, period doubling, and intermittency routes to chaos were found. In this paper, we propose to eliminate the chaotic behaviors from this system, by extending the non-linear feedback control in ODE systems to fractional-order systems. This paper is organized as follows. In Section 2, we present the financial system and its fractional version. In Section 3 general approach to feedback control scheme is given, and then we have extended this control scheme to fractional-order financial system, numerical results are shown. Finally, in Section 4 concluding comments are given.

2. Financial System

2.1. Integer-Order Financial System

Recently, the studies in [1, 2] have reported a dynamic model of finance, composed of three first-order differential equations. The model describes the time-variation of three state variables: the interest rate x, the investment demand y, and the price index z. The factors that influence the changes of x mainly come from two aspects: firstly, it is the contradiction from the investment market, (the surplus between investment and savings); secondly, it is the structure adjustment from goods prices. The changing rate of y is in proportion with the rate of investment, and in proportion by inversion with the cost of investment and the interest rate. The changes of z, on one hand, are controlled by the contradiction between supply and demand of the commercial market, and on the other hand, are influenced by the inflation rate. Here we suppose that the amount of supplies and demands of commercials is constant in a certain period of time, and that the amount of supplies and demands of commercials is in proportion by inversion with the prices. However, the changes of the inflation rate can in fact be represented by the changes of the real interest rate and the inflation rate equals the nominal interest rate subtracts the real interest rate. The original model has nine independent parameters to be adjusted, so it needs to be further simplified. Therefore, by choosing the appropriate coordinate system and setting an appropriate dimension to every state variable, we can get the following more simplified model with only three most important parameters:

$$\begin{aligned}\dot{x} &= z + (y - a)x,\\ \dot{y} &= 1 - by - x^2,\\ \dot{z} &= -x - cz,\end{aligned} \tag{2.1}$$

where $a \geq 0$ is the saving amount, $b \geq 0$ is the cost per investment, and $c \geq 0$ is the elasticity of demand of commercial markets. It is obvious that all three constants, a, b, and c, are nonnegative, For more detail about the study of the local topological structure and bifurcation of this system; see [1, 2]. We assume that a is control parameter and $b = 0.1,\ c = 1$.

2.1.1. Analysing the System

(i) If $a \geq 9$, system (2.1) has one fixed point:

$$p_1 = (0, 10, 0). \tag{2.2}$$

(ii) If $a < 9$, system (2.1) has three fixed points:

$$p_1 = (0, 10, 0), \qquad p_{2,3} = \left(\mp\sqrt{\frac{9-a}{10}}, a+1, \pm\sqrt{\frac{9-a}{10}} \right). \tag{2.3}$$

To study the stability of equilibrium points we apply the Lyapunov's first (indirect) method [12] so we have the following theorem.

Theorem 2.1. *Let $x = x^*$ be an equilibrium point of a nonlinear system:*

$$\dot{x} = f(x), \tag{2.4}$$

where $f : D \to \mathbb{R}^n$ is continuously differentiable and $D \subset \mathbb{R}^n$ is the neighborhood of the equilibrium point x^*. Let λ_i denote the eigenvalues of the Jacobian matrix $A = \partial f/\partial x|_{x^*}$ then the following are considered.

(i) *If* $\operatorname{Re}\lambda_i < 0$ *for all* i, *then* $x = x^*$ *is asymptotically stable.*

(ii) *If* $\operatorname{Re}\lambda_i > 0$ *for one or more* i, *then* $x = x^*$ *is unstable.*

(iii) *If* $\operatorname{Re}\lambda_i \leq 0$ *for all* i *and at least one* $\operatorname{Re}\lambda_j = 0$, *then* $x = x^*$ *may be either stable, asymptotically stable, or unstable.*

Since A is only defined at x^*, stability determined by the indirect method is restricted to infinitesimal neighborhoods of x^*.

To study the signs of the real parts of eigenvalues, we have the following famous criterion [13].

Criterion 1 (Routh-Hurwitz). Given the polynomial $P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$, where the coefficients $a_i,\ i = 1, 2, \ldots, n$, are real constants, define the n Hurwitz matrices

$$
\begin{aligned}
H_1 &= (a_1),\\
H_2 &= \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}\\
&\vdots\\
H_n &= \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{pmatrix},
\end{aligned}
\tag{2.5}
$$

where $a_i = 0$ if $i > n$.

All of roots of the polynomial have negative real part if and only if the determinants of all Hurwitz matrices are positive: $\det H_i > 0,\ i = 1, 2, \ldots, n$.

Routh-Hurwitz criteria for $n = 3$ are $a_1 > 0,\ a_3 > 0$ and $a_1a_2 - a_3 > 0$.

Stability of p_1

The Jacobian matrix of system (2.1) at the equilibrium point p_1 is

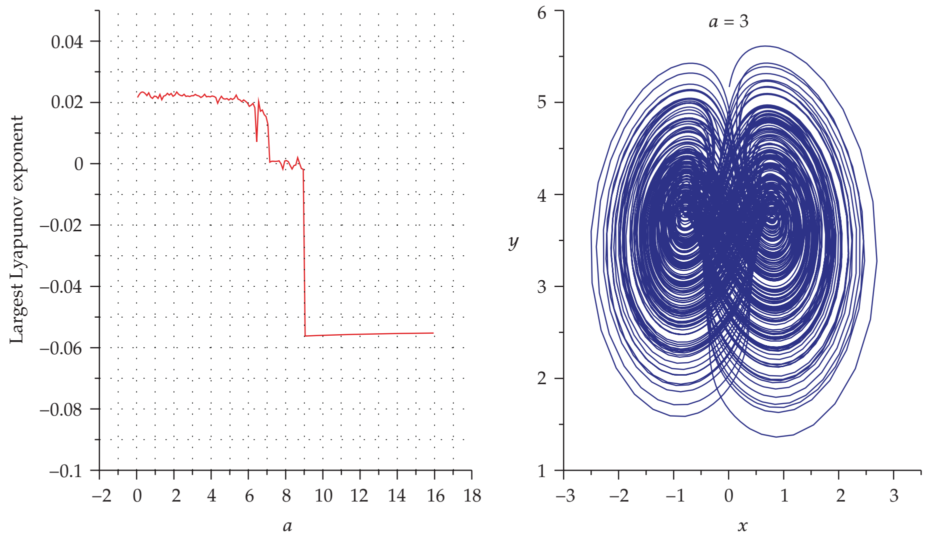
$$
J_{p_1} = \begin{pmatrix} 10 - a & 0 & 1 \\ 0 & -\dfrac{1}{10} & 0 \\ -1 & 0 & -1 \end{pmatrix}, \tag{2.6}
$$

its characteristic polynomial is

$$
P(\lambda) = \lambda^3 + \left(a - \frac{89}{10}\right)\lambda^2 + \left(\frac{11a - 99}{10}\right)\lambda + \left(\frac{a - 9}{10}\right). \tag{2.7}
$$

By applying the Routh-Hurwitz criterion we find that the real parts of these eigenvalues are all negative if and only if

$$
\begin{aligned}
a - \frac{89}{10} &> 0,\\
a - 9 &> 0,\\
\left(a - \frac{89}{10}\right)\left(\frac{11a - 99}{10}\right) - \left(\frac{a - 9}{10}\right) &> 0.
\end{aligned}
\tag{2.8}
$$



(a) (b)

Figure 1: (a) Largest Lyapunov exponent according to a. (b) Chaotic attractor for $a = 3$.

Then it follows that $a > 9$, and thus p_1 is locally asymptotically stable if and only if $a > 9$.

Stability of $p_{2,3}$

The Jacobian matrix of system (2.1) at the equilibrium points $p_{2,3}$ is

$$J_{p_{2,3}} = \begin{pmatrix} 1 & \pm\sqrt{\dfrac{9-a}{10}} & 1 \\ \mp 2\sqrt{\dfrac{9-a}{10}} & -0.1 & 0 \\ -1 & 0 & -1 \end{pmatrix}, \tag{2.9}$$

and its characteristic polynomial is

$$\tilde{p}(\lambda) = \lambda^3 + \frac{1}{10}\lambda^2 + \left(-\frac{1}{5}a + \frac{18}{10}\right)\lambda + \left(-\frac{1}{5}a + \frac{18}{10}\right). \tag{2.10}$$

The real parts of these eigenvalues are all negative if and only if

$$\begin{gathered} -\frac{1}{5}a + \frac{18}{10} > 0, \\ \frac{1}{10}\left(-\frac{1}{5}a + \frac{18}{10}\right) - \left(-\frac{1}{5}a + \frac{18}{10}\right) > 0. \end{gathered} \tag{2.11}$$

Then it follows that

$$a < 9, \quad a > 9. \tag{2.12}$$

So $p_{2,3}$ are unstable for every value of a.

In order to detect the chaos we calculate the largest Lyapunov exponent $\lambda_{\max}$ using the scheme proposed by Wolf et al. [14]. The initial states are taken as $x(0) = 2, y(0) = 3, z(0) = 2$, Figure 1(a) displays the evolution of $\lambda_{\max}$ according to a and Figure 1(b) displays chaotic attractor for $a = 3$. System (2.1) displays chaotic behavior in the windows $0 < a < 7$ ($\lambda_{\max} > 0$), periodic behavior in $7 \leq a \leq 9$ ($\lambda_{\max} \approx 0$) and stationary behavior for $a > 9$ ($\lambda_{\max} < 0$).

2.2. Fractional-Order Financial System

Chen has introduced in [11] the generalization of system (2.1) for fractional incommensurate-order model which takes the form

$$\begin{aligned} D^{q_1}x &= z + (y - a)x, \\ D^{q_2}y &= 1 - by - x^2, \\ D^{q_3}z &= -x - cz. \end{aligned} \tag{2.13}$$

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order but there are several definitions of fractional derivatives.

In this paper, we use the Caputo-type fractional derivative defined in [15] by:

$$\begin{aligned} D^q f(t) &= \frac{1}{\Gamma(n-q)} \int_0^t (t-\tau)^{n-q-1} f^{(n)}(\tau) d\tau \\ &= j^{n-q}\left(\frac{d^n}{dt^n} f(t)\right), \end{aligned} \tag{2.14}$$

where $n = [q]$ is the value of q rounded up to the nearest integer, Γ is the gamma function and j^α is the Riemann-Liouville integral operator defined by

$$j^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{2.15}$$

For the numerical solutions of system (2.13) we use the Adams-Bashforth-Moulton predictor-corrector scheme [16].

We assume that q ($q_1 = q_2 = q_3 = q$) is the control parameter, and $c = 1, \ b = 0.1, a = 3$. Fractional system (2.13) has the same fixed points $p_{1,2,3}$ as integer system (2.1), but for the stability analysis we have this theorem introduced in [17, 18].

Theorem 2.2. *The fractional linear autonomous system*

$$\begin{matrix} D^{\alpha}X = AX \\ X(0) = X_0 \end{matrix} \quad X \in \mathbf{R}^n,\ 0 < \alpha < 2,\ A \in \mathbf{R}^n \times \mathbf{R}^n, \tag{2.16}$$

is locally asymptotically stable if and only if

$$\min_i \left|\arg(\lambda_i)\right| > \alpha\frac{\pi}{2}, \quad i = 1, 2, \ldots, n. \tag{2.17}$$

Proposition 2.3. *Let $x = x^*$ be an equilibrium point of a fractional nonlinear system*

$$D^{\alpha}x = f(x), \quad 0 < \alpha < 2. \tag{2.18}$$

If the eigenvalues of the Jacobian matrix $A = \partial f/\partial x|_{x^*}$ satisfy

$$\min_i \left|\arg(\lambda_i)\right| > \alpha\frac{\pi}{2}, \quad i = 1, 2, \ldots, n, \tag{2.19}$$

then the system is locally asymptotically stable at the equilibrium point x^*.

Proof. Let $x = x^* + \delta x$. Substituting in (2.18), we find

$$D^{\alpha}(x^* + \delta x) = f(x^* + \delta x). \tag{2.20}$$

so

$$D^{\alpha}(\delta x) = f(x^*) + A\delta x + \bigcirc\left(\|\delta x\|^2\right). \tag{2.21}$$

Since $f(x^*) = 0$ (x^* is the equilibrium point of system (2.18)) and $\lim_{\|\delta x\|\to 0}(\bigcirc(\|\delta x\|^2)/\|\delta x\|) = 0$, then

$$D^{\alpha}\delta x \approx A\delta x. \tag{2.22}$$

Taking into account Theorem 2.2, we deduce that If the eigenvalues of the matrix A satisfy

$$\min_i \left|\arg(\lambda_i)\right| > \alpha\frac{\pi}{2}, \quad i = 1, 2, \ldots, n, \tag{2.23}$$

then x^* is locally asymptotically stable.

This completes the proof. □

Stability of p_1

The Jacobian matrix of system (2.13) at the equilibrium point p_1 is

$$J_{p_1} = \begin{pmatrix} 7 & 0 & 1 \\ 0 & -\dfrac{1}{10} & 0 \\ -1 & 0 & -1 \end{pmatrix}, \tag{2.24}$$

and its characteristic polynomial is

$$P(\lambda) = \lambda^3 - \frac{59}{10}\lambda^2 - \frac{66}{10}\lambda - \frac{6}{10}. \tag{2.25}$$

its eigenvalues are $\lambda_1 \approx -0.87298$, $\lambda_2 = -1/10$, $\lambda_3 \approx 6.8730$, we note that λ_3 is real positive then $|\arg(\lambda_3)| = 0 < q(\pi/2)$, for all $q \in]0, 2[$, so p_1 is unstable for all $q \in]0, 2[$.

Stability of $p_{2,3}$

The Jacobian matrix of system (2.13) at the equilibrium point $p_{2,3}$ is

$$J_{p_{2,3}} = \begin{pmatrix} 1 & \pm\sqrt{\dfrac{3}{5}} & 1 \\ \mp 2\sqrt{\dfrac{3}{5}} & -\dfrac{1}{10} & 0 \\ -1 & 0 & -1 \end{pmatrix}, \tag{2.26}$$

its characteristic polynomial is

$$\tilde{p}(\lambda) = \lambda^3 + \frac{1}{10}\lambda^2 + \frac{6}{5}\lambda + \frac{6}{5}, \tag{2.27}$$

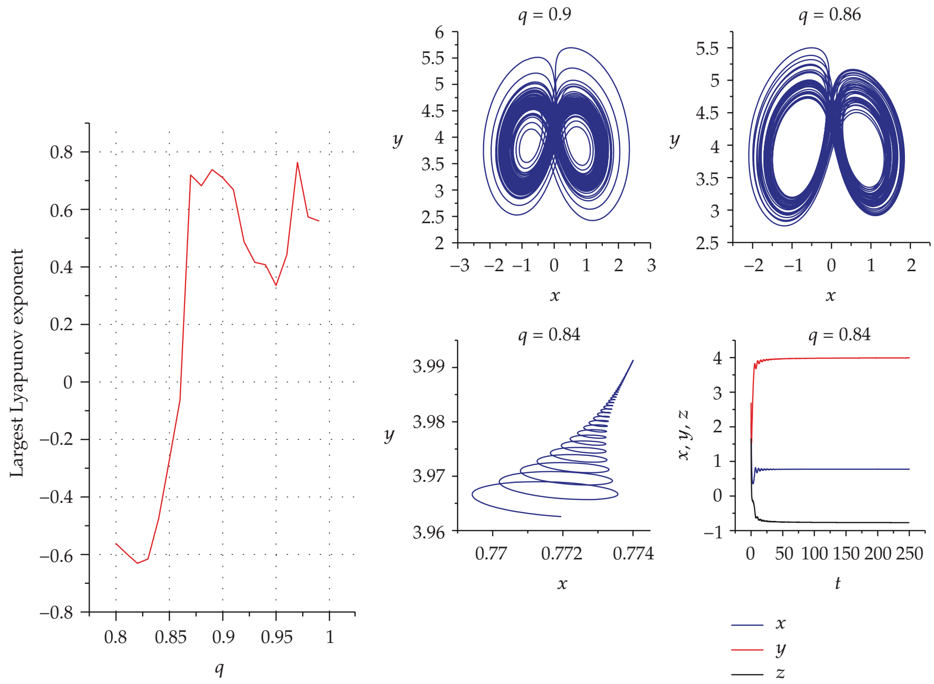
and its eigenvalues are $\lambda_1 \approx 0.31278 + 1.2474i$, $\lambda_2 \approx 0.31278 - 1.2474i$, and $\lambda_3 \approx -0.72556$, we have

$$|\arg(\lambda_{1,2})| \approx 1.3251, \qquad |\arg(\lambda_3)| = \pi, \tag{2.28}$$

so $\min_i |\arg(\lambda_i)| \approx 1.3251$, then the critical value of q is

$$q_c = \frac{2\min_i |\arg(\lambda_i)|}{\pi} \approx 0.8436, \tag{2.29}$$

(i) If $q < 0.8436$, then $p_{2,3}$ are locally asymptotically stable.



(a) (b)

Figure 2: (a) Largest Lyapunov exponent according to q. (b) Phase diagram for some values of q.

(ii) If $q > 0.8436$, then $p_{2,3}$ are unstable.

In order to detect the chaos, we calculate the largest Lyapunov exponent $\lambda_{\max}$.

The initial states are taken as $x(0) = 2$, $y(0) = 3$, $z(0) = 2$, Figure 2(a) shows the evolution of $\lambda_{\max}$ according to q. System (2.13) exhibits chaotic behaviors for $q \geq 0.86$.

3. Feedback Control

3.1. Integer Case

A general approach to control a nonlinear dynamical system via feedback control can be formulated as follows:

$$\dot{x}(t) = f(x, u, t), \tag{3.1}$$

where $x(t)$ is the system state vector, and $u(t)$ the control input vector. Given a reference signal $\tilde{x}(t)$, the problem is to design a controller in the state feedback form:

$$u(t) = g(x, t), \tag{3.2}$$

where g is vector-valued function, so that the controlled system

$$\dot{x}(t) = f\left(x, g(x,t), t\right) \tag{3.3}$$

can be driven by the feedback control $g(x,t)$ to achieve the goal of target tracking so we must have

$$\lim_{t\to t_f} \|x(t) - \tilde{x}(t)\| = 0. \tag{3.4}$$

Proposition 3.1. *Let us consider the nonlinear system*

$$\dot{e} = F(e,t), \tag{3.5}$$

where $e = x - \tilde{x}$, $\tilde{x}(t)$ *is a periodic orbit (or fixed point) of the given system* (3.1) *with* $u = 0$, *and* $F(e,t) = f(x, g(x,t), t) - f(\tilde{x}, 0, t)$.

If 0 *is a fixed point of system* (3.5) *and all eigenvalues of the jacobian matrix* $A = \partial F/\partial x|_0$ *have negative real parts then the trajectory* $x(t)$ *of system* (3.3) *converge to* $\tilde{x}(t)$

Proof. Since $\tilde{x}(t)$ is a periodic orbit (or fixed point) of the given system (3.1) with $u = 0$, so it satisfies

$$\dot{\tilde{x}}(t) = f(\tilde{x}, 0, t), \tag{3.6}$$

a subtraction of (3.6) from (3.1) gives

$$\dot{x}(t) - \dot{\tilde{x}}(t) = f\left(x, g(x,t), t\right) - f(\tilde{x}, 0, t), \tag{3.7}$$

so

$$\dot{e} = F(e,t). \tag{3.8}$$

Since all eigenvalues of the jacobian matrix A have negative real parts, it follows from Theorem 2.1 that 0 is asymptotically stable, so we have $\lim_{t\to+\infty}\|e(t)\| = 0$ then $\lim_{t\to+\infty}\|x(t)- \tilde{x}(t)\| = 0$, finally $x(t) \underset{t\to t_f}{\to} \tilde{x}(t)$. □

3.2. Fractional Case

Let us consider the fractional system

$$D^{\alpha}x(t) = f(x,u,t). \tag{3.9}$$

We proceed as in the integer case. the controlled system can be written as

$$D^{\alpha}x(t) = f\left(x, g(x,t), t\right). \tag{3.10}$$

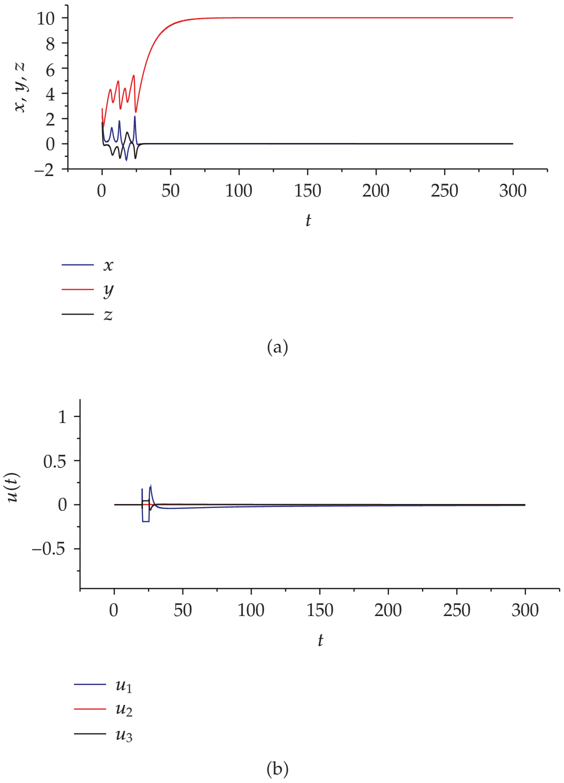


Figure 3: (a) Stabilizing the equilibrium point p_1 for $q = 0.9$. (b) Evolution of the perturbation $u(t)$.

Let $\tilde{x}(t)$ be a periodic orbit (or fixed point) of the given system (3.9) with $u = 0$, then we obtain the system error

$$D^{\alpha} e(t) = F(e, t) \tag{3.11}$$

Proposition 3.2. *If* 0 *is a fixed point of system* (3.11) *and the eigenvalues of the jacobian matrix* $A = \partial F / \partial x|_0$ *satisfies the condition*

$$\min_i \left| \arg(\lambda_i) \right| > \alpha \frac{\pi}{2}, \quad i = 1, 2, \ldots, n \tag{3.12}$$

then the trajectory $x(t)$ *of system* (3.10) *converge to* $\tilde{x}(t)$.

Proof. It follows directly from Proposition 2.3. □

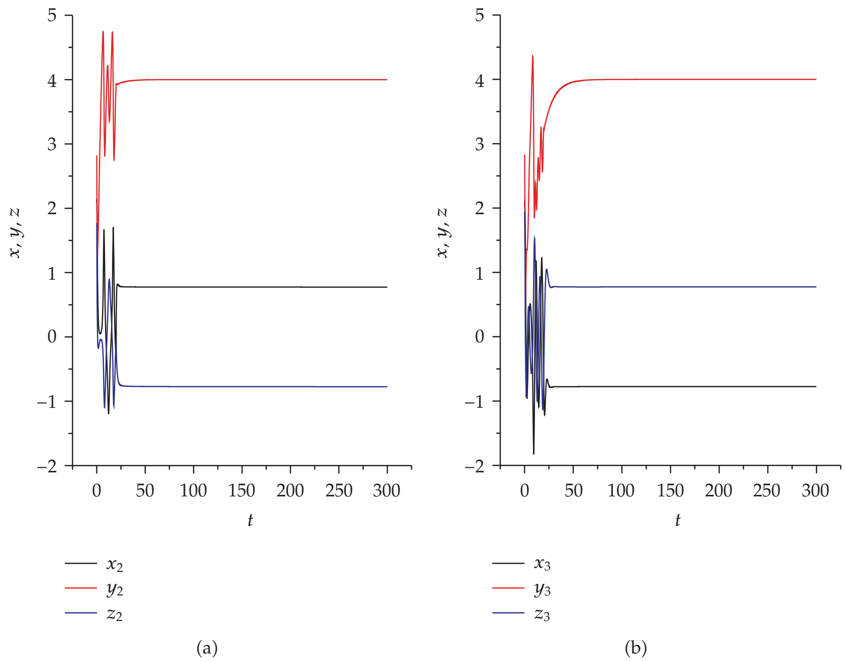


Figure 4: (a) Stabilizing the equilibrium point p_2 for $q = 0.95$. (b) Stabilizing the equilibrium point p_3 for $q = 1.4$.

3.3. Application to the Fractional Financial System

Let us consider the fractional financial system (2.13), we propose to stabilize unstable periodic orbit (or fixed point) $(\tilde{x}, \tilde{y}, \tilde{z})$, the controlled system is as follows:

$$\begin{aligned} D^{q_1}x &= z + (y - a)x + u_1(t), \\ D^{q_2}y &= 1 - by - x^2 + u_2(t), \\ D^{q_3}z &= -x - cz + u_3(t). \end{aligned} \tag{3.13}$$

Since $(\tilde{x}, \tilde{y}, \tilde{z})$ is solution of (2.13), then we have:

$$\begin{aligned} D^{q_1}\tilde{x} &= \tilde{z} + (\tilde{y} - a)\tilde{x}, \\ D^{q_2}\tilde{y} &= 1 - b\tilde{y} - \tilde{x}^2, \\ D^{q_3}\tilde{z} &= -\tilde{x} - c\tilde{z}. \end{aligned} \tag{3.14}$$

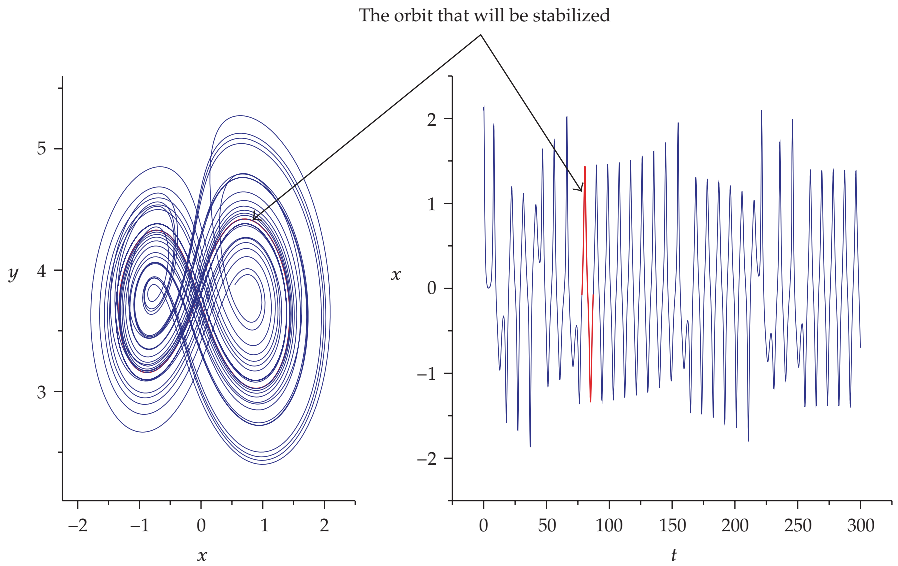


Figure 5: Selecting an unstable periodic orbit in the chaotic attractor of period $T = 9$ for $q = 0.97$.

Subtracting (3.14) from (3.13) with notation, $e_1 = x - \tilde{x},\ e_2 = y - \tilde{y},\ e_3 = z - \tilde{z}$, we obtain the system error:

$$\begin{aligned} D^{q_1} e_1 &= e_3 - a e_1 + xy - \tilde{x}\tilde{y} + u_1(t), \\ D^{q_2} e_2 &= -b e_2 - e_1(x + \tilde{x}) + u_2(t), \\ D^{q_3} e_3 &= -e_1 - c e_3 + u_3(t). \end{aligned} \tag{3.15}$$

We define the control functions as follow:

$$\begin{aligned} u_1(t) &= -(xy - \tilde{x}\tilde{y}), \\ u_2(t) &= e_1(x + \tilde{x}), \\ u_3(t) &= e_1. \end{aligned} \tag{3.16}$$

So the system error (3.15) becomes

$$\begin{aligned} D^{q_1} e_1 &= e_3 - a e_1, \\ D^{q_2} e_2 &= -b e_2, \\ D^{q_3} e_3 &= -c e_3. \end{aligned} \tag{3.17}$$

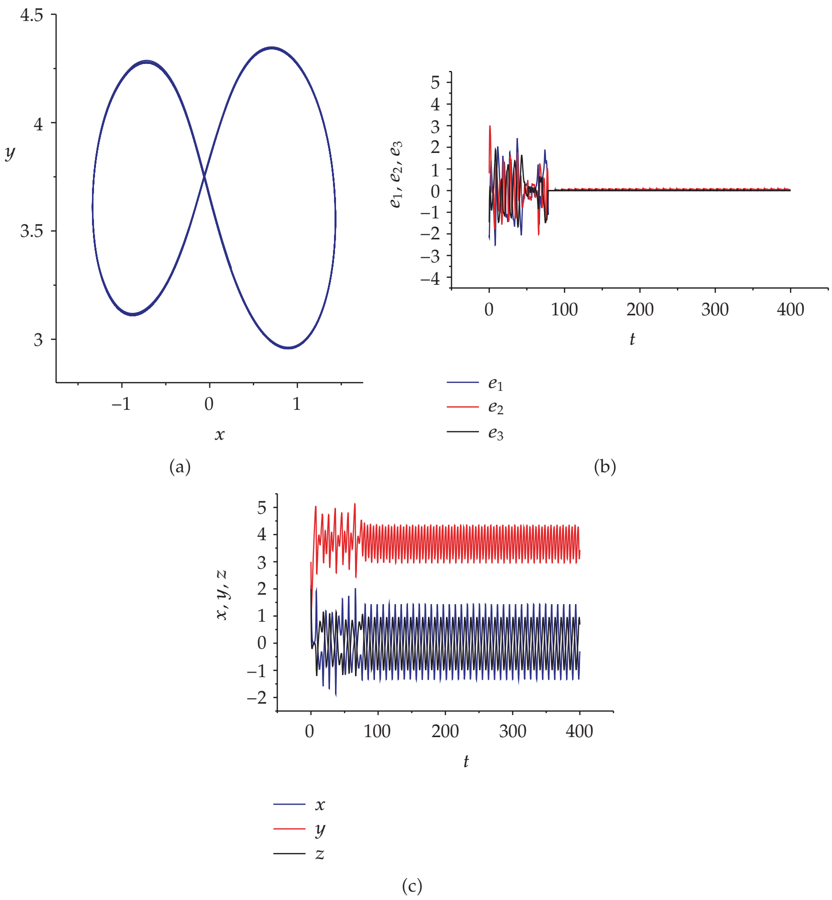


Figure 6: Stabilizing unstable periodic orbit of period $T = 9$ for $q = 0.97$.

The Jacobian matrix is $\begin{bmatrix} -a & 0 & 1 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{bmatrix}$ and its characteristic polynomial is:

$$p(x) = x^3 + (a + b + c)x^2 + (ab + c(a + b))x + abc \tag{3.18}$$

so we have the eigenvalues $\lambda_1 = -a,\ \lambda_2 = -b,\ \lambda_3 = -c$. Since all eigenvalues are real negatives one has $\arg(\lambda_i) = \pi$, therefore $|\arg(\lambda_i)| > q(\pi/2)$, for all q satisfies $0 < q < 2$, it follows from Proposition 3.2 that the trajectory $x(t)$ of system (3.13) converges to $\tilde{x}(t)$ and the control is completed.

3.4. Simulation Results

In this section we give numerical results which prove the performance of the proposed scheme. As mentioned in Section 2.3 we have implemented the improved Adams-Bashforth-Moulton algorithm for numerical simulation.

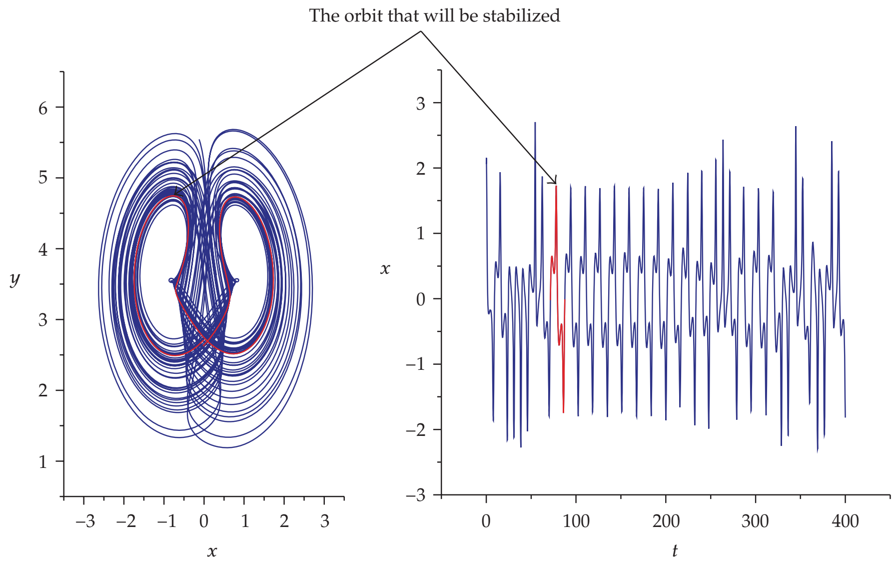


Figure 7: Selecting an unstable periodic orbit in the chaotic attractor of period $T = 16.05$ for $q = 1.1$.

The initial states are taken as $x(0) = 2,\ y(0) = 3,\ z(0) = 2$.

3.4.1. Stabilizing the Unstable Fixed Points

The control can be started at any time according to our needs, so we choose to activate the control when $t \geq 20$, in order to make a comparison between the behavior before activation of control and after it.

For $q = 0.9$ unstable point p_1 has been stabilized, as shown in Figure 3(a), note that $u_1(t) = -(x(t)y(t) - 0 \times 10) = -x(t)y(t)$, so the control is activated when $t \geq 20$ and $|x(t)y(t)| \leq 0.2$ (more precisely $t = 22.5$) in order to make the perturbation $u_1(t)$ smaller. firstly the evolution of $x(t), y(t), z(t)$ is chaotic, then when the control is started at $t = 22.5$ we see that p_1 is rapidly stabilized.

In Figure 3(b) we observe the evolution of the perturbation $u(t)$, when the control is started we see that $u_2(t)$ and $u_3(t)$ are very small but $u_1(t)$ is a bit larger, after that the perturbation $u(t)$ becomes close to zero rapidly.

For $q = 0.95$, the unstable point p_2 has been stabilized, as shown in Figure 4(a).

For $q = 1.4$ the fixed point p_3 was stabilized, Figure 4(b) shows the results of control.

When t is less than 20, there is a chaotic behavior, but when the control is activated at $t = 20$, the two points p_2 and p_3 are rapidly stabilized.

In the real world of finance if we want to have a good investment demand we can choose to stabilize p_1, and in this case the interest rate and price index will be near zero. During the recent financial crisis in 2009 many banks decided to reduce interest rates to nearly zero in order to control this situation.

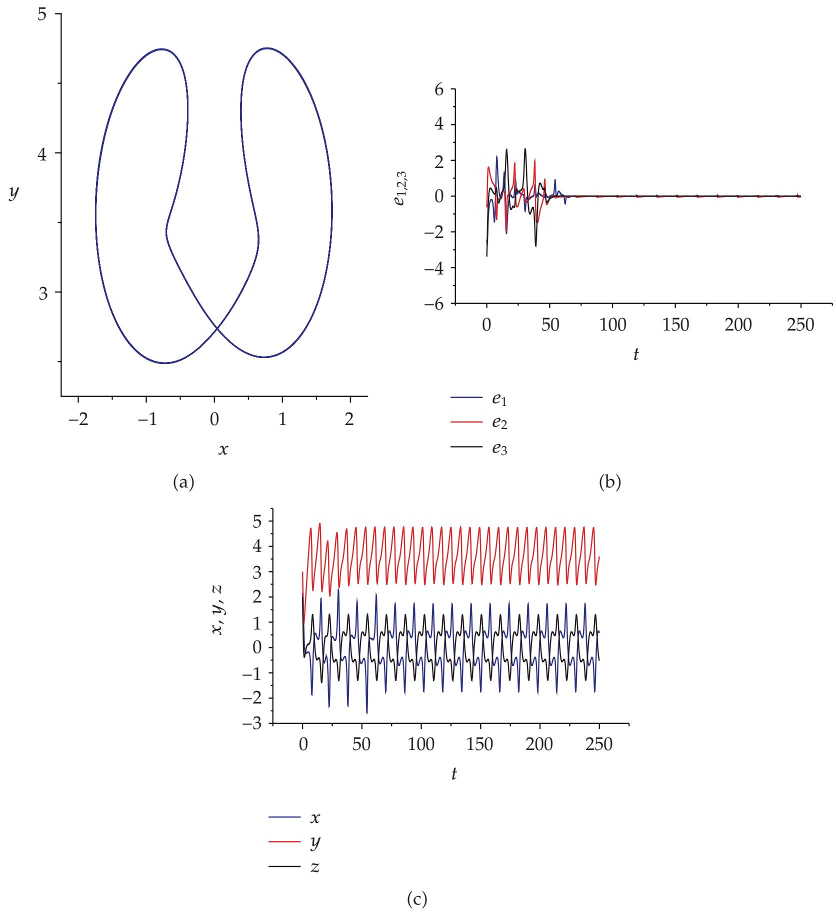


Figure 8: Stabilization an unstable periodic orbit of period $T = 16.05$ for $q = 1.1$.

3.4.2. Stabilizing Unstable Periodic Orbit

Although the unstable periodic orbits are dense in the chaotic attractor, we can choose one of them (which represent the performance of the system), by analyzing data experimental, after that we stabilize it. In this paper the close-return (CR) method [19] is used for the detection of UPO embedded in the attractor.

For $q = 0.97$ we choose an unstable periodic orbit with period $T = 9$, localized in the interval $[78.2, 87.2]$ as shown in Figure 5, then the control is started at $t = 87.2$, when the trajectory $x(t)$ begins to emerge from the unstable orbit, Figure 6 displays the results of control, if t is less then 78.2 there is chaotic behavior (the error $e(t)$ is large), after the activation of control, this chaotic behavior is replaced by a periodic behavior and we note that the error $e(t)$ becomes very close to zero.

For $q = 1.1$ we choose an unstable periodic orbit with period $T = 16.05$, localized in the interval $[71.45, 87.5]$ as shown in Figure 7, the control is started at $t = 20$, Figure 8 displays the results of control. Although the control is executing at $t = 20$, it does not give effect rapidly, and the orbit is stabilized at $t = 63$, when the control is activated the error begins to diminish, and becomes close to zero after $t = 63$.

The stabilization of the periodic orbits is very important, because it permits, on the one hand to make some predictions, and secondly, it is more realistic than the stabilization of the stationary points in the financial circle, where one cannot generally fix the interest rate and the investment demand as well as the price index, for a long period.

4. Conclusions

Chaotic phenomenon makes prediction impossible in the financial world; then the deletion of this phenomenon from fractional financial system is very useful, the main contribution of this paper is to this end.

Nonlinear feedback control scheme has been extended to control fractional financial system. The results are proved analytically by applying the Lyapunov linearization method and stability condition for fractional system. Numerically the unstable fixed points $p_{1,2,3}$ have been successively stabilized for different values of q; moreover unstable periodic orbit has stabilized. This proves the performance of the proposed scheme.

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General conclusion and perspectives

The work accomplished in the frame of this thesis has been a significant enrichment of our knowledge about some concerns of the subject of fractional-order chaotic systems. In the first three chapters we have presented some tools that are eventually used in study of fractional-order systems. In the last chapter we have exposed our contributions to the analysis of fractional-order chaotic systems where three articles are presented. The first article postulate and further studied a new chaotic system of three-dimensional autonomous equations with cubic nonlinearity, which can generate two 1-scroll chaotic attractors simultaneously, (or a one 2-scroll chaotic attractor) with three equilibria. Dynamical behaviors of this new chaotic system, including some basic dynamical properties, bifurcations, periodic windows, routes to chaos, have been analyzed both theoretically and numerically, by means of a bifurcation diagram with an associated Poincaré map and Lyapunov exponent spectrum. The second paper present and analyze the fractional-order modified hybrid optical system. It has been demonstrated that chaos, as well as the other usual nonlinear dynamic phenomena, occur in this systems with mathematical order less than three. The Largest Lyapunov exponents and the bifurcation diagrams show the period-doubling bifurcation and the transformation from periodic to chaotic motion through the fractional-order and confirms the justness of the proposed fractional Hopf bifurcation conditions (in this system). The theoretical analysis which validates conditions of Hopf bifurcation repose on the normal form and center manifold theorem, unfortunately these tools are not developed yet in fractional order systems. The last paper deals with the extension of nonlinear feedback control scheme in order to control fractional financial system. The results are proved analytically by applying the Lyapunov linearization method and stability condition for fractional system. Numerically the unstable fixed points have been successively stabilized for different values of fractional order; moreover unstable periodic orbit has been stabilized, which proves the performance of the proposed scheme. the results obtained in these papers are modestly important, referenced and cited by others authors in diverse publications. As a result of our review, some important directions of study have appeared as a natural prolonging objectives for the present work. Additional efforts are needed in both theory and application of fractional-order systems. the existence of exact periodic solution in fractional-order autonomous system, the problem of stability, the problem of calculating Lyapunov spectrum and application of fractional calculus in circuit theory especially in circuit elements with memory are still ender active study.

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Abstract

This thesis deals with fractional-order chaotic systems. The main highlight is on some basic differences between a fractional-order system and its integer order counterpart. Namely, stability conditions, existence of periodic solutions and minimal total order for which chaos can occur etc...
The finding of a new chaotic attractor from Hybrid optical bistable system is reported and dynamic of the new system is investigated in both integer and fractional-order cases. It is shown that asymptotic stability of equilibrium points of the fractional system can occur with positive real part of some corresponding eigenvalues which is not the case in integer-order systems. We have established criterion under which a fractional-order system undergoes Hopf bifurcation. The results are validated by mean of stability theory and numerical simulations. It is shown that chaos can be occurred in fractional-order system with total order less than three which is not the case in integer-order system due to the Poincaré-Bendixon theorem.
Finally, nonlinear feedback control scheme has been extended to control fractional financial system. The results are proved analytically by applying the stability condition for fractional system. Numerically the unstable fixed points have been successively stabilized for different values of fractional order; moreover some unstable periodic orbits have been stabilized.

Keywords: Fractional-order derivatives, Stability of fractional-order systems, Bifurcation, Periodic solutions, Chaos, Chaos control.

Résumé

Cette thèse porte sur les systèmes chaotiques d'ordre fractionnaire. Nous mettons en relief quelques différences de base entre un système d'ordre fractionnaire et le système d'ordre entier correspondant. A savoir, les conditions de stabilité, l'existence des solutions périodiques et l'ordre total minimal pour lequel le chaos peut se produire etc....

La découverte d'un nouvel attracteur chaotique en modifiant un système optique est rapporté et sa dynamique a été analysée dans le cas entier ainsi que dans le cas fractionnaire. Il est montré que les points d'équilibre peuvent être asymptotiquement stables même s'il existe des valeurs propres correspondantes de parties réelles positives ce qui est impossible pour les systèmes d'ordre entier. Nous avons établi des critères pour lesquels un système d'ordre fractionnaire volue vers une bifurcation de Hopf. Les résultats sont confirmés en utilisant la théorie de la stabilité et des simulations numériques. Il est montré aussi que le chaos se produit dans ce système d'ordre fractionnaire avec un ordre total inferieur à trois ce qui est impossible pour un système d'ordre entier d'après le théorème de Poincaré-Bendixon.

Finalement, la méthode de contrôle par rétroaction (feedback) non linéaire a été étendu pour contrôler un système financier d'ordre fractionnaire. Les résultats sont analytiquement prouvés en utilisant la condition de stabilité des systèmes fractionnaires. Numériquement les points fixes instables ont été stabilisés successivement pour différentes valeurs de l'ordre fractionnaire, de plus quelques orbites périodiques instables ont été stabilisées.

Mots clés : Dérivées d'ordre fractionnaire, Stabilité des systèmes d'ordre fractionnaire, Bifurcation, Solutions périodiques, Chaos, Contrôle du chaos.

ملخص

هذه الأطروحة تتناول بالدراسة موضوع الأنظمة الشواشة (طبيعة ديناميكية شبه فوضوية) ذات رتب كسرية حيث نبرز بعض الفروق الأساسية بين الأنظمة ذات رتب كسرية والأنظمة ذات رتب صحيحة الموافقة لها، من حيث شروط الاستقرار، وجود الحلول الدورية و كذا الحد الأدنى للرتبة الكلية التي يمكن أن تحدث الشواش الخ...

لقد تم في هذه الأطروحة تقديم جذاب شواش جديد وذلك بإدخال بعض التغييرات على نظام هجين بصري ثنائي الاستقرار وقد تمت دراسة الطبيعة الديناميكية للنظام الجديد في حالة الرتبة عدد صحيح وكذا في حالة الرتبة عدد كسري.

وقد تبين أنه يمكن أن يحدث استقرار لنقاط التوازن في هذا النظام ذو الرتبة الكسرية حتى بوجود بعض القيم الذاتية الموافقة ذات جزء حقيقي موجب وهذا غير ممكن بالنسبة للنظم ذات رتب صحيحة. من جهة أخرى اقترحنا في هذه الأطروحة بعض الشروط والتي بموجبها يجتاز النظام الكسري تفرع من نوع هوبف. هذه النتائج تم التصديق عليها باستخدام نظرية استقرار الأنظمة الكسرية وكذلك بالحساب العددي. لقد لوحظ أن الشواش يمكن أن يحدث في هذا النظام الكسري مع رتبة كلية أقل من ثلاثة وهذا مستحيل الحدوث في الأنظمة ذا ت رتب صحيحة حسب نظرية بوانكري بندكسن.

أخيرا، تم تمديد طريقة المراقبة بالتغذية المترددة غير الخطية للتحكم في النظام المالي ذو الرتبة الكسرية. النتائج تم إثباتها تحليليا بتطبيق نظرية الاستقرار الخاصة بالنظم الكسرية. عدديا تم التحكم في نقاط التوازن وكذا بعض المدارات غير المستقرة.

كلمات مفتاحيه: النظم ذات المشتقات الكسرية، استقرار النظم الكسرية، التفرع، الحلول الدورية الشواش، التحكم في الشواش.