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LES METHODES NUMERIQUES POUR LES EQUATIONS AUX DERIVEES PARTIELLES AVEC DES CONDITIONS AUX LIMITES NON LOCALES

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Introduction

There are three important steps in the computational modelling of any physical process. First one problem definition, the second one is model, and the third one is computer simulation.

The first natural step is to define an idealisation of our problem of interest in terms of relevent quantities which we would like to measure. In defining this idealisation we expect to obtain a well-posed problem this is one that has a unique solution for a given set of parameters. It might not always be possible to guarantee the fidelity of the idealisation since, in some instences the physical process is not totally understood. An example the complex environment within a nuclear reactor where obtaining measurments is too difficult.

The second step of the modelling process is to represent our idealisation of the physical reality by a mathematical model, the governing equations of the problem. These are available for many physical phenomenon. For example the equations of elasticity in structural mechanics govern the deformation of a solid object due to applied external forces. These are complex equations that are very dificult to solve both analytically and numerically. To overcome this problem we need to introduce simplifying assumptions to reduce the complexity of the mathematical model and make it at hand to either exact or numerical solution. After the selection of an approximate, together with suitable boundary and initial conditions, we can proceed to its solution mathematical model.

Over the last few years, various processes in the natural sciences and engineering lead to the non classical parabolic initial/boundary value problems which involve non-local integral terms over the spatial domain. The integral terms may appear in the boundary conditions in which case the boundary condition is called non-local, or in the governing partial differential equation itself, which is then often referred to as a partial integro-differential equation, or in both. Non-local boundary value problems were first used by [51,57]. The presence of an integral term in a boundary condition can complicate the application of a standard numerical techniques such as finite difference method, finite element methods,spectral techniques, boundary integral equation scheme, etc. It is important to convert the non local boundary-value problems to more desirable form, to make them more applicable the problems of practical interest. In many cases it is a hard task. The use of quadrature approximations in these problems is not an easy task. The accuracy of the quadrature must be compatible with the discretization of the differential equation. The sparsity of coefficient matrices of systems of linear algebraic equations arising in the time-stepping is complicated. Due to this reason new methods are introduced to overcome these difficulties, like, Adomian decomposition method, Homotopy perturbation method and variational iteration method, etc. Up to now partial differential equations with non local boundary conditions have been one of the fastest growing aeras in various fields. Science and industry are both responsible for this growth in the last three decades.

In this thesis we will consider the numerical solution of mathematical problems which are modeled by partial differential equations with non local boundary conditions. Efficient and accurate numerical methods are introduced, analysed and used. for solving one dimensional homogeneous heat equation with nonlocal boundary conditions two steps are needed, in the first one we apply a sixth-order finite difference scheme using the method of lines semi-discretization approach to transform the model of partial differential equation into a system of first order ordinary differential equations, in the second step we solve the resulted system of first order differential equations using the technique of fourth-order Runge-Kutta method. The obtained results are more accurate than those obtained by former searchers who are dealed with this kind of problems [6]. In the next chapters we introduce the Adomian's decomposition method for solving both one dimensional , two-dimensional and three-dimensional homogeneous and non homogeneous heat equation(linear and nonlinear) [1-5]. In the last chapter we analyse and use the Homotopy perturbation method for solving linear (nolinear) equations with nonlocal boundary conditions.The obtained results are with good agreement of the exact ones.

Chapter 1

1 An overview

1.1 High order compact finite difference

Finite difference schemes can be classified into two types, explicit and implicit. Explicit schemes expresse the nodal derivatives like an explicit weighted sum of the nodal values of the function, e.g,

$$u'_i = \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2)$$

The local truncation error is

$$O(h^2) = -\frac{h^2}{6}u'''(\theta), \theta \in (x_{i-1}, x_{i+1})$$

And

$$u_i" = \frac{u_{i+1} + u_{i-1} - 2u_i}{h^2} + O(h^2)$$

The local truncation error is

$$O(h^2) = -\frac{h^2}{12}u^{(4)}(\zeta),\ \zeta \in (x_{i-1}, x_{i+1})$$

Where h denotes the step size of equally spaced mesh of the domaine of u.

Assuming that $u^{'''}$and $u^{(4)}$ are bounded, the local truncation error approaches zero at the same rate that h^2 approaches zero, when $h \to 0$. It simply said that the local truncation error is of order h^2, which is denoted by the symbols $O(h^2)$. Implicit methods increase the complexity of the algorithm since they require matrix inversion but are still relatively uncomplicated. Better approximations can be obtained by increasing the order of the truncation error of the finite difference scheme.This is commonly accomplished by including more points in the stencil of the numerical scheme.

As an example, consider an explicit centred finite difference formula with a five point stencil approximating the first derivative

$$u'_i \approx \frac{u_{i-2} - 8u_{i-1} + 8u_{i+1} - u_{i+2}}{12h} \tag{1}$$

which has a local truncation error of $O(h^4)$ given by

$$\delta_i = -\frac{1}{30} h^4 u^{(5)}(\eta), \eta \in (x_{i-2}, x_{i+2}) \tag{2}$$

The smaller truncation error is more advantageous, but it require a larger stencil.

A disatvantage for this approach is the need to include more equation for grid points near and at the boundaries. Also, for highier-order implicit schemes, the inversion of the matrices with the increased number of non-zero diagonals may be too costly. An alternative is to not enlarge the stencil, but involve values of the derivative at some nodes where the function is already evaluated.

considering the finite difference approximation of the first derivative proposed in 1966 by Collatz [56], which approximates the derivative values at

three grid points with known function values over the same three grid points

$$\frac{1}{4} u'_{i-1} + u'_i + \frac{1}{4} u'_{i+1} \approx \frac{3}{4h}(u_{i+1} - u_{i-1}) \tag{3}$$

This scheme is of order four as it will be proven later, this new scheme has a local truncation error of $O(h^4)$, similar to (1.1). However, if (1.1) is used over a discretized domaine, four additional formulas are needed at the two points on both ends where the stencil protrudes the domaine. On the contrary, scheme (1.3) only requires additional formulas at each of the end points. Assuming that at least one boundary condition is known, only one additional formula may be needed.Thus the proposed implicit scheme (1.3) gives a distinct advantage over the explicit equation (1.1).

The developement and application of the above implicit finite difference formula (1.3) to solve initial boundary value problems modled by partial differential equations is more recent appearence.

As mentioned above, some partucular formulas are reported by Collatz [56] pp. 538. However, their implementation as difference schemes approximating partial differential equations began in the early 1970s for some fluid mechanics problems. Since that time, several distinct classes of compact schemes have been developed. The two most common are the upwind and the centred schemes see, Lele [38]. In recent years, due to the appearence of faster and more powerful computing machine , compact schemes are proving more advantageous. The current emphasis of these higher-order methods has been to the field of fluid mechanics as well as other areas of aero-acoustics and electro-magnetics... In the last ten years, much work has been done with compact schemes by many authors.

This work discusses the formulation of two different approaches the first one sixth-order compact finite difference scheme and Fourth-Order Runge-Kutta Algorithm.

1.2 Consistency, Stability and Convergence

Finite difference schemes approximating partial differential equations are analyzed according to three important propreties:consistency,stability and convergence. To introduce these concepts, consjder the following boundary value problem

$$u_{xx} = q(x), 0 < x < 1 \tag{4}$$

$$u(0) = a \ , u(1) = b \tag{5}$$

Where q is contuneous and bounded function in its interval. Introducing the differential operator L

We rewrite the problem (1.4) and (1.5) in an operator form we have

$$\begin{aligned} Lu &= \{u_{xx}, 0 < x < 1 \\ u(0) &= a, u(1) = b \end{aligned} \tag{6}$$

To obtain a numerical approximation of this problem, a grid formed by points in the domain of the function u must be defined. By selecting h_x as uniform step sizes along the x-axis, the grid points

$$x = ih_x, i = 0, 1, 2, ..N \tag{7}$$

Using central difference approximations for u_{xx} on the grid points, equation (1.4) is approximated by a finite difference equation. As a consequence, the continuous problem (1.4), (1.5) is replaced by a new discret problem given by

$$\frac{U_{i+1} + U_{i-1} - 2U_i}{h_x^2} = q(x_i), \ i = 0, 1, 2, ..., N \tag{8}$$

$$U_0 = a, U_{N+1} = b \tag{9}$$

The discrete problem (1.8)-(1.9) can also be written in the form

$$AU = Q \tag{10}$$

Where U is the vector of unknowns $U = [U_1, U_2, ..., U_N]$ and

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & . & . & . & .. & . \\ 1 & -2 & 1 & . & . & . & . \\ . & 1 & -2 & 1 & . & . & . \\ . & . & 1 & -2 & 1 & . & . \\ . & . & . & 1 & -2 & 1 & . \\ . & . & . & . & 1 & -2 & 1 \\ . & . & . & . & . & 1 & -2 \end{bmatrix}, \qquad Q = \begin{bmatrix} q(x_1) - \frac{a}{h^2} \\ q(x_2) \\ . \\ . \\ . \\ q(x_{N-1}) \\ q(x_N) - \frac{b}{h^2} \end{bmatrix} \tag{11}$$

Solving this system we obtain the approximate solution U

Where

$$U = \begin{bmatrix} U_1 \\ U_2 \\ . \\ . \\ . \\ . \\ U_N \end{bmatrix}$$

Thus, the centred difference is a second oreder accurate approximation to u'' at x_i , $i = 1, 2....N$ and the local truncation error which is denoted by replacing U_i by the exact solution $u(x_i)$ in the difference equation (1,8) as follows

$$\delta_i = [u"(x_i) + \frac{h^2}{12}u''''(x_i) + O(h^4)] - q(x_i) \tag{12}$$

Or

$$\delta_i = \frac{h^2}{12}u''''(x_i) + O(h^4)$$

Where u'''' is a function independent of h , and $\delta_i = O(h^2)$ as $h \to 0$, definning δ as follows

$$\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ . \\ . \\ . \\ . \\ \delta_N \end{bmatrix}$$

Then

$$\delta = Au - Q$$

Or

$$Au = Q + \delta \tag{13}$$

Hence the global error is given by the vector

$$E = u - U$$

subtructting (1,13) from (1,10) we obtain

$$AE = -\delta \tag{14}$$

Or

$$E = \begin{bmatrix} E_1 = & u_1 - U_1 \\ E_2 = & u_2 - U_2 \\ . & . \\ . \quad = & . \\ . & . \\ . & . \\ E_N = & u_N - U_N \end{bmatrix}$$

Rewritting (1,14) in the following form

$$A^h E^h = -\delta^h \tag{15}$$

The superscrit h means that we are acting on a grid equally spaced with a mesh step h. solving the above system we have

$$E^h = -(A^h)^{-1}\delta^h$$

and taking norms gives

$$\left\|E^h\right\| = \left\|(A^h)^{-1}\delta^h\right\| \leq \left\|(A^h)^{-1}\right\| \left\|\delta^h\right\| \tag{16}$$

Supposing that there is a constant C independent of h such that

$$\left\|(A^h)^{-1}\right\| \leq C \tag{17}$$

Then

$$\left\|E^h\right\| \leq C \left\|\delta_h\right\| \tag{18}$$

Where the norms used are

$$\|E\|_\infty = \max_{1\leq i\leq N} |E_i| = \max_{1\leq i\leq N} |u(x_i) - U_i|$$

Or

$$\|E\|_1 = h\sum_{i=1}^{N}|E_i|$$

And so on.

Definition 1 Suppose a finite difference for a linear boundary value problem gives a linear system of the form

$$A^hU^h = Q^h$$

Where h is the mesh width. We say the method is stable if $(A^h)^{-1}$ exists and there exist a positive number h_1 and a constant$C$, independent of h such that

$$\left\|(A^h)^{-1}\right\| \leq C, for\ all\ \ h \prec h_1 \tag{19}$$

Definition 2 The difference scheme (1,8) is consistent with the continuous problem (1,4) if $\left\|\delta^h\right\| \to 0$ as $h \to 0$. Moreover if the inequality

$$\left\|\delta^h\right\| \leq c_1 h^k \tag{20}$$

holds for some positive constants $c_1\ and\ k$, then it is said that the difference scheme (1,8) is of order h^k consistent with the continuous problem(1,4)

The concepte of convergence is now presented.

Theorem 1 If the difference scheme (1,8) is stable and is also consistent with the continuous problem (1,4) then, the discrete solution U_h of (1,8)converges to the solution u of (1,4) and satisfies

$$\left\|E^h\right\| \leq \left\|(A^h)^{-1}\right\| \left\|\delta^h\right\| \leq C\left\|\delta^h\right\| \to 0\ as\ h \to 0 \tag{21}$$

Where C is independent of h. The proof results from (1,19) and (1,20), hence, the discret solution U^h of (1,8) converges to the continuous one u of (1,4) with order of $O(h^2)$.

1.3 The Compact Scheme

Considering high-order approximation for the first derivative of a function u using implicit schemes as defined in [56], pp.538-539.For this we suppose a function u of one variable defined on the real line $\mathbb{R}$. A uniform partition formed by discrete points $x_i,\ i = 0, 1, 2, ...$,is defined in $\mathbb{R}$. An implicit numerical approximation of the first derivative u' at the grid points can be given by

$$\alpha u'_{i-1} + u'_i + \alpha u'_{i+1} = \frac{a}{2h}(u_{i+1} - u_{i-1}) \tag{22}$$

where α and a are arbitrairy constants. In fact equation (1,22) represents a family of numerical approximations for u'.

Theorem 2 If u is an $n+1$ times differentiable function on $\mathbb{R}$ $(n \geq 4)$ and $x_i,\ i = 0, 1, 2, ...,$ is a uniform partition of $\mathbb{R}$ with step size h, then the implicit finite difference scheme equation (1,22) defines a one parameter family of numerical approximations for u' with second order formal accuracy. Fourth-order maximum formal accuracy is obtained for $a = \frac{3}{2}$ *and* $\alpha = \frac{1}{4}$ with local truncation error

$$\delta_i = \frac{1}{120} h^4 u^{(5)}(\zeta), \zeta \in (x_{i-1}, x_{i+1}) \tag{23}$$

Proof Grouping all terms of equation (1,22) on the left-hand side, expanding the functions u and its first derivative u'at each node according to their Taylor expansions, and substituting them into (1,22) leads to

$$\begin{aligned} &\alpha u'_{i-1} + u'_i + \alpha u'_{i+1} - \frac{a}{2h}(u_{i+1} - u_{i-1}) \\ = &2\alpha(u'_i + \frac{h^2}{2!}u'''_i + \frac{h^4}{4!}u_i^{(5)} + ...) + u'_i \\ &-\frac{a}{2h}[(u_i + hu'_i + \frac{h^2}{2!}u''_i + \frac{h^3}{3!}u'''_i + \frac{h^4}{4!}u_i^{(4)} + ...) \\ &-(u_i - hu'_i + \frac{h^2}{2!}u''_i - \frac{h^3}{3!}u'''_i + \frac{h^4}{4!}u_i^{(4)} - ...)] \end{aligned} \tag{24}$$

combining like terms gives

$$\begin{aligned} &\alpha u'_{i-1} + u'_i + \alpha u'_{i+1} - \frac{a}{2h}(u_{i+1} - u_{i-1}) \\ = &(2\alpha + 1 - a)u'_i + (2\frac{\alpha}{2!} - \frac{a}{3!})h^2 u'''_i \\ &+(2\frac{\alpha}{4!} - \frac{a}{5!})h^4 u_i^{(5)} + ... \end{aligned}$$

By setting $(2\alpha + 1 - a) = 0$, the first term is eliminated and equation (22) becomes a one parameter family of second order schemes, that, is the constant a is uniquely determined by the parameter α as $a = 2\alpha + 1$.The truncation error is given by

$$\delta_i = (2\frac{\alpha}{2!} - \frac{a}{3!})h^2 u'''_i = \frac{4\alpha - 1}{6} h^2 u'''_i$$

In addition, if the second coefficient term in (1,24) is forced to zero, that is, $\frac{2\alpha}{2!} - \frac{a}{3!} = 0$, then both constants α and a are uniquely determined. These values are $\alpha = \frac{1}{4}$ and $a = \frac{3}{2}$ which are the constants defining (3). The local truncation error is

$$\delta_i = (2\frac{\alpha}{4!} - \frac{a}{5!})h^4 u_i^{(5)} = \frac{1}{120}h^4 u^{(5)}(\zeta)$$

which proves that the implicit compact scheme

$$\frac{1}{4}u'_{i-1} + u'_i + \frac{1}{4}u'_{i+1} = \frac{3}{4h}(u_{i+1} - u_{i-1})$$

has the same formal fourth-order accuracy as the five-point explicit centered finite difference scheme. An important advantage of the scheme (1,22) is that its stencil only consists of three points instead of five as in the explicit centered counterpart. The formal order of accuracy for the implicit scheme (1,22) can be easily increased by enlarging its stencil, maintainig a tridiagonal matrix for the unknown derivative values. For this purpose , consider the scheme

$$\alpha u'_{i-1} + u'_i + \alpha u'_{i+1} = \frac{a}{2h}(u_{i+1} - u_{i-1}) + \frac{b}{4h}(u_{i+2} - u_{i-2}) \tag{25}$$

The analogous of theorem 2 for this new scheme can be formulated as follows here.

Theorem 3 If u is $n+1$ times differentiable function on $\mathbb{R}$ $(n \geq 6)$ and $x_i,\ i = 0, 1, 2, ...,$ is a uniform partition of $\mathbb{R}$ with step size h_x, then the implicit finite difference equation (1,24) defines a one parameter family of numerical approximations for u' with four-order formal accuray. A sixth-order maximum formal accuracy is obtained for $a = \frac{14}{9}, \alpha = \frac{1}{3}, and\ b = \frac{1}{9}$ with local truncation error

$$\delta_i = \frac{1}{1200}h^6 u^{(7)}(\xi), \xi \in (x_{i-2}, x_{i+2}) \tag{26}$$

Proof Grouping all terms of equation (1,25)) on the left-hand side, expanding the functions u and its first derivative u' at each node according to their Taylor expansions, substituting them into (1,25), and combining like terms leads to

$$\begin{aligned}
&\alpha u'_{i-1} + u'_i + \alpha u'_{i+1} - \frac{a}{2h}(u_{i+1} - u_{i-1}) - \frac{b}{4h}(u_{i+2} - u_{i-2}) \\
&= (2\alpha + 1 - a - b)u'_i + (2\frac{\alpha}{2!} - \frac{a}{3!} - \frac{2^2 b}{3!})h^2 u'''_i \\
&+(2\frac{\alpha}{4!} - \frac{a}{5!} - \frac{2^4 b}{5!})h^4 u_i^{(5)} + (2\frac{\alpha}{6!} - \frac{a}{7!} - \frac{2^6 b}{7!})h^6 u_i^{(7)}
\end{aligned}$$

By setting $2\alpha + 1 - a - b = 0$ and $2\frac{\alpha}{2!} - \frac{a}{3!} - \frac{2^2 b}{3!} = 0$, the first two terms in the right-hand side are eliminated and the equation (1,25) becomes a one parameter family of fourth-order schemes.That is, the constants a and b are uniquely determined by the parameter α. The truncation error is given by the $O(h^4)$ term in the right-hand side. If this third term is forced to zero, then all constants α, a and b are uniquely detrmined. These values are $\alpha = \frac{1}{3}, a = \frac{14}{9}, and\, b = \frac{1}{9}$. As a consequence, the following sixth-order compact scheme approximation for the first derivative is obtained

$$\frac{1}{3}u'_{i-1} + u'_i + \frac{1}{3}u'_{i+1} = \frac{7}{9}h(u_{i+1} - u_{i-1}) + \frac{1}{36h}(u_{i+2} - u_{i-2}) \quad (27)$$

The local truncation error for this particular scheme is

$$\delta_i = -\frac{1}{1260}h^6 u^{(7)}(\varsigma), \varsigma \in (x_{i-2}, x_{i+2})$$

Note that the presence of factorial terms yields the coefficients of the truncation error that is very small. This may in fact result in an even higher order of formal accuracy for the given scheme than is suggested by $O(h^6)$. The previous sixth-order scheme (1,27) usually written as

$$\frac{1}{3}u'_{i-1} + u'_i + \frac{1}{3}u'_{i+1} = \frac{7}{9h}(u_{i+1} - u_{i-1}) + \frac{1}{36h}(u_{i+2} - u_{i-2}) \quad (28)$$

The following will also adopt the same convention. Similar procedures as those used in proving the above theorems, can be followed to derive other compact schemes. When dealing with the boundary value problems, the complete compact differencing scheme consits of two differents types of formulas. The interior formula, which is the heart of the compact scheme, approximates derivative values at all but the boundary and near boundary points. To approximate derivative values at these points, one-sided difference schemes that mimic the implicit nature and the formal order of accuracy of the interior scheme may be used. The number of points excluded by the interior scheme depends on the stencil.

1.4 Interior scheme

The compact scheme for the first derivative at interior points (1,22) and (1,25) are particular cases of the more general well-known schemes defined, as fllows

$$\sum_{k=-L}^{L} \beta_k U'_{i+k} = \frac{1}{h}\sum_{l=-M}^{M} \alpha_l U_{i+l}, \beta_0 = 1, \beta_k = \beta_{-k} \quad (29)$$

By expading the summations, the schemes are shown as

$$\beta_{-L}U'_{i-L} + \beta_{-L+1}U'_{i-L+1} + ... + \beta_{-1}U'_{i-1} + U'_i + .. + \beta_L U'_{i+L}$$
$$= \frac{1}{h}(\alpha_{-M}U_{i-M} + \alpha_{-M+1}U_{i-m+1} + .. + \alpha_{-1}U_{i-1} + \alpha_0 U_i + \alpha_1 U_{i+1} + .. + \alpha_M U_{i+M}.$$

The left-hand side there are 2L+1 derivative values and the right-hand side has a 2M+1 node stencil. To avoid the computational complexity when uing the implicit schemes, we should restricte $L \leq 2$. The formula (1,29) for the first derivative reduces to

$$\gamma U'_{i-3} + \beta U'_{i-2} + \alpha U'_{i-1} + U'_i + \alpha U'_{i+1} + \beta U'_{i+2} + \gamma U'_{i+3} = \tag{30}$$
$$= \frac{a}{2h}(U_{i+1} - U_{i-1}) + \frac{b}{4h}(U_{i+2} - U_{i-2}) + \frac{c}{6h}(U_{i+3} - U_{i-3}) + \frac{d}{8h}(U_{i+4} - U_{i-4})$$

Further study of compact schemes will be reduced to the $L = 2$ case where $\gamma = 0$. The second derivative scheme is

$$\beta U''_{i-2} + \alpha U''_{i-1} + U''_i + \alpha U''_{i+1} + \beta U''_{i+2} \tag{31}$$
$$= \frac{a}{h^2}(U_{i+1} - 2U_i + U_{i-1}) + \frac{b}{4h^2}(U_{i+2} - 2U_i + U_{i-2}) + \frac{c}{9h^2}(U_{i+3} - 2U_i + U_{i-3})$$

Similarely a third derivative centered compact scheme is given by

$$\beta U'''_{i-2} + \alpha U'''_{i-1} + U'''_i + \alpha U'''_{i+1}\beta U'''_{i+2} \tag{32}$$
$$\frac{a}{2h^3}(U_{i+2} - 2U_{i+1} + 2U_{i-1} - U_{i-2}) + \frac{b}{8h^3}(U_{i+3} - 3U_{i+1} + 3U_{i-1} - U_{i-3})$$

Finally, a fourth derivative compact scheme can be written as

$$\beta U^{(4)}_{i-2} + \alpha U^{(4)}_{i-1} + U^{(4)}_i + \alpha U^{(4)}_{i+1} + \beta U^{(4)}_{i+2} = \frac{a}{h^4}(U_{i+2} - 4U_{i+1} + 6U_i - 4U_{i-1} + U_{i+2}) \tag{33}$$

The formal order of accuracy of the compact schemes can be obtained by expanding each term in above equations in Taylor series about x_i and then matching the Taylor series coefficients for the terms in the scheme as performed in theorems 2 and 3. Here the derivation of centred compact differencing extended in general form to schemes with pentadiagonal implicit matrix up to 9 grid points stencils. First, consider the Taylor expansions for the left-hand side terms in (1,30) with $\gamma = 0$.

$$U'_{i-2} = U'_i - 2hU''_i + \frac{2^2h^2}{2!}U'''_i - ..\frac{2^9h^9}{9!}U^{(10)}_i + \frac{2^{10}h^{10}}{10!}U^{(11)}_i + R_{11}(x) \tag{34}$$

$$U'_{i-1} = U'_i - hU''_i - \frac{h_x^2}{2!}U''_i - ... - \frac{h^9}{9!}U^{(10)}_i + \frac{h^{10)}}{10!}U^{(11)}_i + R_{11}(x) \tag{35}$$

$$U'_i = U'_i$$

$$U'_{i+1} = U'_i + hU''_i + \frac{h^2}{2!}U''_i + ... + \frac{h^9}{9!}U_i^{(10)} + \frac{h^{10}}{10!}U_i^{(11)} + R_{11}(x)$$

$$U'_{i+2} = U'_i + 2hU''_i + \frac{2^2h^2}{2!}U'''_i + .. + \frac{2^9h^9}{2^9}U_i^{(10)} + \frac{2^{10}h^{10}}{2^{10}}U_i^{(11)} + R_{11}(x)$$

We do the same thing for expanding the righ-hand side

$$U_{i-4} = U_i - 4hU'_i + \frac{4^2h^2}{2!}U''_i - .. - \frac{4^9h^9}{9!}U_i^{(10)} - R_{11}(x)$$

$$U_{i-3} = U_i - 3hU'_i + \frac{3^2h^2}{2!}U''_i \_ ... - \frac{3^9h^9}{9!}U_i^{(9)} + \frac{3^{10}h^{10}}{10!} - R_{11}(x)$$

$$U_{i-2} = U_i - 2hU'_i + \frac{2^2h^2}{2!}U''_i - .. - \frac{2^9h^9}{9!}U_i^{(9)} + \frac{2^{10}h^{10}}{10!}U_i^{(10)} - R_{11}(x)$$

$$U_{i-1} = U_i - h_xU'_i + \frac{h^2}{2!}U''_i - .. - \frac{h^9}{9!}U_i^{(9)} + \frac{h^{10}}{10!}U_i^{(10)} + R_{11}(x)$$

$$U_{i+1} = U_i + hU'_i + \frac{h^2}{2!}U''_i + .. + \frac{h^9}{9!}U_i^{(9)} + \frac{h^{10}}{10!}U_i^{(10)} + R_{11}(x)$$

$$U_{i+2} = U_i + 2hU'_i + \frac{2^2h^2}{2!}U''_i + .. + \frac{2^9h^9}{9!}U_i^{(9)} + \frac{2^{10}h^{10}}{10!}U_i^{(10)} + R_{11}(x)$$

$$U_{i+3} = U_i + 3hU'_i + \frac{3^2h^2}{2!}U''_i + .. + \frac{3^9h^9}{9!}U_i^{(9)} + \frac{3^{10}h^{10}}{10!}U_i^{(10)} + R_{11}(x)$$

$$U_{i+4} = U_i + 4hU'_i + \frac{4^2h^2}{2!}U''_i + .. + \frac{4^9h^9}{9!}U_i^{(9)} + \frac{4^{10}h^{10}}{10!}U_i^{(10)} + R_{11}(x)$$

Doing the same like in theorems (2) and (3) the last expansions are substituted into (2.9) we gather all like terms we obtain

$$(1+2\alpha+2\beta)U'_i + \frac{2}{2!}(\alpha+2^2\beta)h^2U'''_i + \frac{2}{4!}(\alpha+2^4\beta)h^4U_i^{(5)} + \frac{2}{6!}(\alpha+2^6\beta)h^6U_i^{(7)} +$$

$$+\frac{2}{8!}(\alpha+2^8\beta)h^8U_i^{(9)} + \frac{2}{10!}(\alpha+2^{10}\beta)h^9U_i^{(11)} + ... + . =$$

$$(a+b+c+d)U'_i + \frac{1}{3!}(a+2^2b+3^2c+4^2d)h^2U'''_i + \frac{1}{5!}(a+2^4b+3^4c+4^4d)h^4U_i^{(5)} +$$

$$+\frac{1}{7!}(a+2^6b+3^6c+4^6d)h^6U_i^{(7)} + \frac{1}{9!}(a+2^8b+3^8c+4^8d)h^8U_i^{(9)} + ..$$

$$+\frac{1}{11!}(a+2^{10}b+3^{10}c+4^{10}d)h^{10}U_i^{(11)} + ...$$

Equating coefficients with the same power of h we get the follownig system of six equations

$$a+b+c+d = 2(\alpha+\beta)+1 \tag{36}$$

$$a + 2^2 b + 3^2 c + 4^2 d = 2\frac{3!}{2!}(\alpha + 2^2\beta) \tag{37}$$

$$a + 2^4 b + 3^4 c + 4^4 d = 2\frac{5!}{4!}(\alpha + 2^4\beta) \tag{38}$$

$$a + 2^6 b + 3^6 c + 4^6 d = 2\frac{7!}{6!}(\alpha + 2^6\beta) \tag{39}$$

$$a + 2^6 b + 3^6 c + 4^6 d = 2\frac{7!}{6!}(\alpha + 2^6\beta) \tag{40}$$

$$a + 2^8 b + 3^8 c + 4^8 d = 2\frac{11!}{10!}(\alpha + 2^8\beta) \tag{41}$$

$$a + 2^{10} b + 3^{10} c + 4^{10} d = 2\frac{1!}{10!}(\alpha + 2^{10}\beta) \tag{42}$$

Solving this system we have

Scheme	γ	β	α	a	b	c	d	*order*
T_4			$\frac{1}{4}$	$\frac{3}{2}$				4
T_6			$\frac{1}{3}$	$\frac{14}{9}$	$\frac{1}{9}$			6
T_8			$\frac{3}{8}$	$\frac{25}{16}$	$\frac{1}{5}$	$\frac{-1}{80}$		8
T_{10}			$\frac{2}{5}$	$\frac{39}{25}$	$\frac{4}{15}$	$\frac{1}{35}$	$\frac{1}{525}$	10

Table 1. Coefficients of interior schemes for first derivative

Scheme	β	α	a	b	c	d	*order*
T_4		$\frac{1}{10}$	$\frac{6}{5}$				4
T_6		$\frac{2}{11}$	$\frac{12}{11}$	$\frac{3}{11}$			6
T_8		$\frac{9}{38}$	$\frac{147}{152}$	$\frac{51}{95}$	$\frac{-23}{700}$		8
T_{10}		$\frac{8}{29}$	$\frac{1126}{1305}$	$\frac{988}{1305}$	$\frac{-74}{1015}$	$\frac{43}{9135}$	10

Table2 : Coefficientsofinteriorschemes forsecondderivative

1.5 Adomian decompostion method

Usually numerical methods are based on discretization techniques, and only approximate values of the solution are obtained and only for some values of time and space. With Adomian decomposition method, the solution is obtained by a series expanssion of the so called Adomian's polynomials, not requiring discretization of the variables, and, therefore, not being affected by errors associated to discretization. Also this method does not require linearization or perturbation and, consequently, does not change the actual solution of the problem. As well, Adomian's decomposition method is very competent on finding an approximate or exact solution for linear and non linear problems, not required in many cases, large computer memory.To introduce the Adomian decomposition method, consider the initial boundary value problem

$$u_t - \nabla(a(x)\nabla u) = G(x,t,u), x \in \Omega \in R, t \in]0,T] \tag{43}$$

With the initial condition

$$u(x,0) = u_0(x), x \in \Omega \tag{44}$$

Where $G(x,t,u)$ is non linear vector function, some assumptions are taken for the data a, G, and u_0 in order to assure the existence and uniqueness of the solution of the system (1,45).

The principal algorithm of the Adomian decomposition method applied to a general non linear equation is in the form

$$L(u) + R(u) + N(u) = q \tag{45}$$

The linear term are decomposed into $L+R$, while the non linear terms are represented by $N(u)$. L is taken as the highest order derivative, and R is the remainder of the linear operator. L^{-1} is regarded as the inverse operator of L and is defined by a definete integration from 0 to t, i.e.,

$$L^{-1}(.) = \int_0^t (.)dt \tag{46}$$

If L a second-order operator

$$L^{-1}(.) = \int_0^t \int_0^t (.)dt\ dt \tag{47}$$

$$L^{-1}L(u) = u(x,t) - u(x,0) - tu_t(x,0) \tag{48}$$

Operating on both sides of equation (1,47) with the inverse operator L^{-1} yields

$$L^{-1}(L(u)) = L^{-1}(q) - L^{-1}(R(u)) - L^{-1}(N(u)) \tag{49}$$

Or

$$u(x,t) = u(x,0) + tu_t(x,0) + L^{-1}(q) - L^{-1}(R(u)) - L^{-1}(N(u)) \tag{50}$$

The decomposition method represents the solution of equation (1,32) as a series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{51}$$

The nonlinear operator, $N(u)$, is decomposed as follows

$$N(u)) = \sum_{n=0}^{\infty} A_n \tag{52}$$

Substituting (1,33) and (1,34) into (1,32) we have

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0 - L^{-1}(R(\sum_{n=0}^{\infty} u_n)) - L^{-1}(\sum_{n=0}^{\infty} A_n) \tag{53}$$

Where

$$u_0 = u(x,0) + tu_t(x,0) + L^{-1}(q) \tag{54}$$

Consequently, it can be written as

$$u_1 = -L^{-1}(R(u_0)) - L^{-1}(A_0) \tag{55}$$

$$u_2 = -L^{-1}(R(u_1)) - L^{-1}(A_1)$$

$$..$$

$$u_{n+1} = -L^{-1}(R(u_n)) - L^{-1}(A_n)$$

Where A_n are Adomian's polynomials of $u_0,\ u_1,\ u_2, ...$, and are obtained from the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[G(\sum_{i=0}^{\infty} \lambda^i u_i)]_{\lambda=0}, n = 0, 1, 2, ... \tag{56}$$

Equation (1,38) gives

$$A_0 = g(u_0)$$

$$A_1 = u_1 \frac{d}{du_0} g(u_0)$$

$$A_2 = u_2 \frac{d}{du_0} g(u_0) + \frac{u_1^2}{2!} \frac{d^2}{du_0^2} g(u_0)$$

$$A_3 = u_3 \frac{d}{du_0} g(u_0) + u_1 u_2 \frac{d^2}{du_0^2} g(u_0) + \frac{u_1^3}{3!} \frac{d^3}{du_0^3} g(u_0) \tag{57}$$

..

..

..

The accuracy level of the approximation of $u(x,t)$ can be enhanced by computing components as far as we like. The n-term approximant

$$lim_{n \to \infty} S_n = u(x,t), where \tag{58}$$

$$S_n = \sum_{k=0}^{n-1} u_k(x,t), k \geq 0$$

can be used to approximate the solution.

Convergence of the solution

We consider the following hypotheses

$$(H_1) \qquad (T(u) - T(v), v - u) \geq k \|u - v\|^2, k > 0, u, v \in H \tag{59}$$

(H_2) Whatever may be $M > 0$, there exists a constant $C(M) > 0$ such that for $u, v \in H$ with $\|u\| \leq M$, we have

$$(T(u) - T(v), w) \leq CM \|u - v\| \|w\| \ \ for\ every\ w \in H$$

Where H is a Hilbert space.

Theorem. If N is lipschitzian function in H, the Adomian method applied to the following nonlinear heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u)$$

Where $g(u)$ is the nonlinear terms converges.

Proof. We consider the above equation, then we set

$$L(u) = \frac{\partial u}{\partial t},\ R(u) = -\frac{\partial^2 u}{\partial x^2},\ N(u) = -g(u)$$

We have

$$L(u) = \frac{\partial u}{\partial t} = -T(u) = \frac{\partial^2 u}{\partial x^2} + g(u)$$

This operator is hemicontinuous. We can the convergence hypothesis (H_1) :ie.

There exists a constant $k > 0$, such that for $u, v \in H$ we have

$$(T(u) - T(v), u - v) \geq k \left\| u - v \right\|^2 ,$$

$$T(u) - T(v) = -\frac{\partial^2}{\partial x^2}(u - v) - (g(u) - g(v)),$$

$$(T(u) - T(v), u - v) = (-\frac{\partial^2}{\partial x^2}(u - v), u - v) - (g(u) - g(v)),$$

But there exists a real $\delta > 0$ such that

$$(-\frac{\partial^2}{\partial x^2}(u - v), u - v) \geq \delta \left\| u - v \right\|^2$$

Because

$$\frac{\partial^2}{\partial x^2}$$

Is a differential operator in H. In addition ,

$$(g(u) - g(v), u - v) \leq \alpha \left\| u - v \right\|^2$$

Where $\alpha > 0$ is the lipschitzian constant and therefore

$$(T(u) - T(v), u - v) \geq (\delta - \alpha) \left\| u - v \right\|^2 ,$$

And taking $k = \delta - \alpha$, then we obtain hypothesis (H_1), we can now prove the hypothesis (H_2), i.e.

$$\forall M > 0, \ni C(M) > 0 \textit{ such that } \ \|u\| \leq \|M\| \ , \|v\| \leq M \Longrightarrow$$
$$(T(u) - T(v), w) \leq C(M) \|u - v\| \|w\| , \ \forall w \in H$$

Thus we obtain

$$(T(u) - T(v), w) \leq \|u - v\| \|w\| + \alpha \|u - v\| \|w\| \leq C(M) \|u - v\| \|w\|$$

Where $C(M) = 1 + \alpha$.Hence, the hypothesis (H_2) is satisfied.

1.6 The homtopy perturbation method

The explicit solutions of differential equations are obtained by making use of reliable algorithm like homotopy perturbation method (HPM). In recent years, much attention has been given to the study of HPM ,He [13-17], and [23, 27, 29] for solving a wide range of problems whose mathematical models are governed by differential equations or system of differential equations. HPM deform a difficult problem into an infinite set of problems which are easier to solve without any need to transform nonlinear terms.The speed of convergence of the method is based on a rapidly convergent series with easily computable components. Numerical results show that the homotopy perturbation method is easy to implement and accurate when applied to solve Partial differential equations. to illustrate the basic ideas of this method, we consider the following equation:

$$L[u(x,t)] + N[u(x,t)] = q(x,t),\ \ (x,t) \in \Omega \tag{60}$$

Subject to the boundary condition

$$B(u, \frac{\partial u}{\partial \eta}) = 0,\ \ (x,t) \in \Omega$$

And initial condition

$$u(x,0) = u_0 \tag{61}$$

Where L is a linear operator, N a nonlinear operator and $q(x,t)$ is the source term, B is a boundary operator and Γ is the boundary of the domain Ω. we define a convex homotopy $H(u,p)$ by

$$H(u,p) = (1-p)[L(v) - L(u_0)] + p[L(u) + N(u) - q(x,t)] = 0 \tag{62}$$

We have

$$H(u,0) = L(v) - L(u_0) \tag{63}$$

$$H(u,1) = L(u) + N(u) - q(x,t) \tag{64}$$

This shows that $H(u,p)$ continuously traces an implicitly defined curve from a starting point $H(v_0,0)$ to a solution function $H(f,1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem

$$L(v) - L(u_0) = 0$$

Continuously deforms the original problem

$$L(u) + N(u) - q(x,t) = 0$$

The embedding parameter $p \in [0,1]$ can be considered as an expanding parameter [13-14,17]. We can assume that the solution of equation (44) can be written as a power series in p, as following

$$v = v_0 + v_1 + v_2 + v_3 + ... \tag{65}$$

The comparison of like powers of p give solutions of various orders and the best approximation is $u = lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + ...$

It constitutes a main objective of this thesis to perform a numerical analysis of heat equation with nonlocal boundary conditions in both cases one-dimensional, two-dimensional and three-dimensional and wave equation , studying its solutions and behaviour by different numerical methods, namely High- order finite diffirence method, Adomian's decomposition method and homotopy perturbation method. In the second chapter we used the sixth-order finite difference scheme for solving a one dimensional diffusion equation with an integral boundary condition, the obtained results are of order $O(h_x^6 + h_t^4)$ [6]. in the last six chapters we used the Adomian's decompsition method and homotopy perturbation method, the obtained results are all exact [1-5].

Chapter 2

2 A one-dimensional diffusion equation with an integral condition

In this chapter, we first introduce the compact sixth-order finite difference formula then we adjust compact finite difference formula for the following heat equation with non local boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, (x,t) \in]0,1[\times]0,T[\tag{66}$$

$$u(x,0) = f(x), 0 < x < 1 \tag{67}$$

$$u_x(1,t) = g(t), 0 < t < T \tag{68}$$

$$\int_0^b u(x,t)dx = m(t) \tag{69}$$

where $f(x)$, $g(t)$, b and $m(t)$ are known. This problem describe certain chemicals absorbing light at various frequencies. The intensity of such light on photoelectric cell gives us an electric signal which is proportional to the total amount of chemical present in the volume through which the light passes. Let $u(x,t)$ denote the chemical concentration which is diffusing in a straight glasse tube with x measured in the direction of axis of the tube. Then the electric signal produced by a light beam passing through the tube at right angles between $x = 0$ and $x = b$ is proportional to $\int_0^b u(x,t)dx$. This integral represents the total mass of chemical in $0 \leq x \leq b$ at time t [18]. For such diffusion processes, the integral condition (4) aries naturally and can be used as supplementary information in the determination of unknown concentration $u(x,t)$.

J.cannon and J.vander hoek [49] studied the existence and uniqueness propreties of this problem.

A.B gumel [30] has proposed numerical scheme of order $O(h_x^2 + h_t^2)$ L_0_Stable parallel Algorithm for solving this problem.

Later, M.Akram and pasha [18] have proposed a more accurate algorithm of order $O(h_x^3 + h_t^3)$. We propose a more accurate scheme of order $O(h_x^6 + h_t^4)$. The numerical experiments show that the proposed sixth-order schemes are unconditionally stable and more accurate than that in [18], furthermore for the choice

$$h_x = \tfrac{1}{8} \ and \ \ h_t = \tfrac{1}{1000}$$

The approximate solution coincides with the exact one at more than half of grid points discretization.

2.1 SIXTH-ORDER COMPACT FINITE DIFFERENCE FORMULA

Compact formula is a special finite difference method which uses the values of the function and its derivatives only a

three consecutive points.

First keeping time continuous, we carry out a spatial discretization of $\frac{\partial^2 U}{\partial x^2}$, we divide the interval $[0, 1]$ using a uniform grid $0 = x_0 < x_1 < x_2 < ...... < x_N$

with a mesh size $h_x = x_{i+1} - x_i = \frac{1}{N}, i = 0, 1, 2, ..., N-1, N$.

2.2 STANDARD COMPACT FINITE DIFFERENCE

The standard sixth-order compact finite difference formula for second derivative is

$$\frac{h_x^2}{12}(U''_{x^2 i-1} + 10U''_{x^2 i} + U''_{x^2_{i+1}}) = U_{i-1} - 2U_i + U_{i+1}) \tag{70}$$

where $U_i = U(x_i, t)$ and the coefficients can be determined in the following way.

1- Write the compact finite difference formula in general form

$$h_x^2(a_{-1}U^{"}_{i-1} + a_0U^{"}_i + a_1U^{"}_{i+1}) = b_{-1}U_{i-1} + b_0U_i + b_1U_{i+1} \tag{71}$$

where $a_{-1}, a_0, a_1, b_{-1}, b_0$ and b_1 are parameters to be determined.

2- Expand botIh sides of the equation (2.6) using Taylor series at the point x_i

with respect to the discretization parameter h_x.

3- We obtain six equations by setting the coefficients h_x^j, $j = 0, 1, ..., 5$

equal zero.solve the six equations for the six unknown parameters. The obtained accuracy is $O(h_x^6)$ for formula (2.6)

2.2- Write equation (2.1) in a discret point form

$$\frac{\partial u(x_i, t)}{\partial t} = \frac{\partial^2 u(x_i, t)}{\partial x_i^2}, i = 1, ..., N-1 \tag{72}$$

Equation (2.5) is valid only for $i = 2, 3, ..., N-2$ to attain the same accuracy at $i = 1$

and $i = N - 1$ special formula must be developed.
When $i = 1$ we use the formula

$$\frac{h_x^2}{12}(14U_1^{"} - 5U_2^{"} + 4U_3^{"} - U_4^{"}) = U_0 - 2U_1 + U_2 \tag{73}$$

From Simpson integration Rule we have

$$\int_0^b u(x,t)dx \approx \frac{h_x}{3}(u_0 + 4u_1 + u_2) = m(t),$$

b has been chosen as a grid point, and when i=N-1 we use the formula

$$\frac{h_x^2}{12}(\frac{-127}{30}U_{N-4}^{"} + \frac{86}{5}U_{N-3}^{"} - \frac{257}{10}U_{N-2}^{"} + \frac{461}{15}U_{N-1}^{"}) = U_{N-2} - U_{N-1} + hU_N' \tag{74}$$

We use U to stand for the approximation value of u throughout this chapter.

All Formula are $O(h_x^6)$ or written in Matrix Form

$$AU" = MU + H \tag{75}$$

Where

$$A = \frac{h_x^2}{12}\begin{bmatrix} 14 & -5 & 4 & -1 & 0 & 0 & . & . & . & 0 \\ 1 & 10 & 1 & 0 & 0 & . & . & . & . & 0 \\ 0 & 1 & 10 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 10 & 1 & 0 & 0 & . & . & 0 \\ 0 & 0 & 0 & 1 & 10 & 1 & 0 & 0 & . & 0 \\ 0 & . & . & 0 & 1 & 10 & 1 & 0 & . & 0 \\ 0 & . & . & . & 0 & 1 & 10 & 1 & 0 & 0 \\ 0 & . & . & . & . & 0 & 1 & 10 & 1 & 0 \\ 0 & . & . & . & . & . & 0 & 1 & 10 & 0 \\ 0 & . & . & . & . & 0 & \frac{-127}{30} & \frac{86}{5} & \frac{-257}{10} & \frac{461}{15} \end{bmatrix}$$

$$M = \begin{bmatrix} -6 & 0 & 0 & . & . & . & . & . & . & 0 \\ 1 & -2 & 1 & 0 & 0 & . & . & . & . & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & . & . & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & . & . & 0 \\ 0 & . & . & 0 & 1 & -2 & 1 & 0 & . & 0 \\ 0 & . & . & . & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & . & . & . & . & 0 & 1 & -2 & 1 & 0 \\ 0 & . & . & . & . & . & 0 & 1 & -2 & 1 \\ 0 & . & . & . & . & . & . & 0 & 1 & -1 \end{bmatrix}, H = \begin{bmatrix} \frac{3}{h}m(t) \\ 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ 0 \\ hU_N' \end{bmatrix}.$$

Finally, we obtain

$$U" = A^{-1}M\ U + A^{-1}H$$

Putting $A^{-1}M = B$ and $A^{-1}H = R(t)$

$$U" = B\ U(t) + R(t\ ) \qquad (76)$$

Substituting in (2.7) we get a system of ordinary differential equations

$$\frac{dU}{dt} = B\ U(t) + R(t) \qquad (77)$$

with the initial condition

$$U(0) = f(x)$$

Putting $f(t, U) = B\ U(t) + R(t)$, we obtain the following equation

$$\frac{dU}{dt} = f(t, U) \qquad (78)$$

We solve this equation using fourth-order Runge-Kutta Method as followning

$k_1 = f(t_0, U_0)$
$k_2 = f(t_0 + \frac{1}{2}h_t, \frac{K_1}{2}h_tU_0)$
$k_3 = f(t_0 + \frac{1}{2}h_t, \frac{K_2}{2}h_tU_0)$
$k_4 = f(t_0 + h_t, k_3h_t + U_0)$

$$U_{N+1} = U_N + \frac{1}{6}h_t(k_1 + 2k_2 + 2k_3 + k_4) \qquad (79)$$

2.3 COMPUTATIONAL RESULTS

In order to test the sixth-order compact finite difference scheme, we consider the problem. Consider the heat

equation with

$f(x) = 0.5x^2$
$g(t) = 1$
$m(t) = 0.75t + \frac{1}{6}(0.75)^3$

which is easily seen to have exact solution $u(x, t) = 0.5x^2 + t$.

Using Runge-Kutta method, the problem is solved for

$$h_x = \frac{1}{8}, h_t = \frac{1}{1000}, h_x = \frac{1}{10}, h_t = \frac{1}{1000}$$

The results of approximate solution are tabulated in Tables 1 and 2

$h_x = \frac{1}{8}, h_t = \frac{1}{1000}$

x	exact solution	approximate solution	absolute error
$\frac{1}{8}$	8.8125×10^{-3}	8.3865×10^{-2}	7.5053×10^{-2}
$\frac{1}{4}$	0.03225	2.8182×10^{-2}	4.068×10^{-3}
$\frac{3}{8}$	7.1313×10^{-2}	7.1477×10^{-2}	1.64×10^{-4}
$\frac{1}{2}$	0.126	0.126	0.0
$\frac{5}{8}$	0.19631	0.19631	0.0
$\frac{3}{4}$	0.28225	0.28225	0.0
$\frac{7}{8}$	0.38381	0.38381	0.0

Table 1

$h_x = \frac{1}{10}, h_t = \frac{1}{1000}$

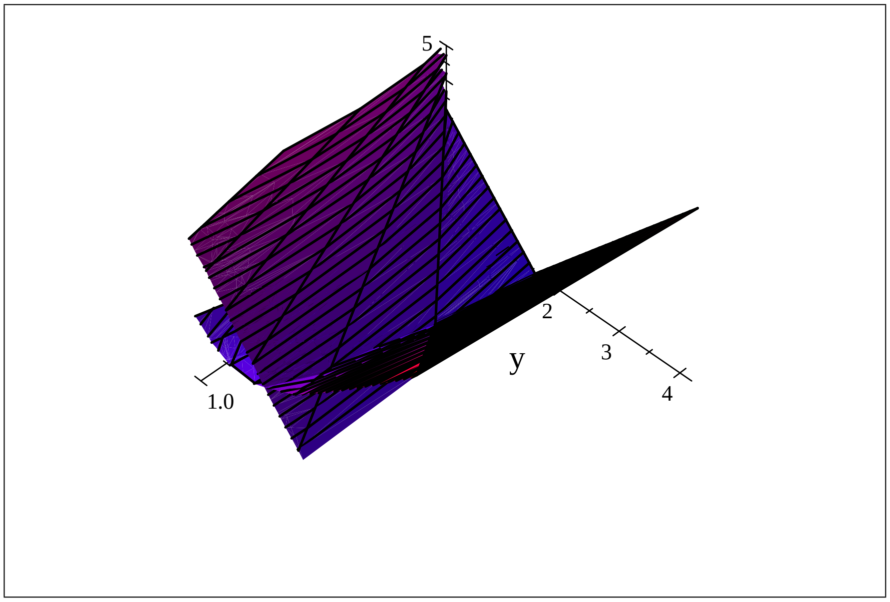
x	exact solution	approximate solution	absolute error
$\frac{1}{10}$	0.006	0.14449	0.13849
$\frac{1}{5}$	0.021	1.7083×10^{-2}	3.917×10^{-3}
$\frac{3}{10}$	0.046	4.5849×10^{-2}	1.51×10^{-4}
$\frac{2}{5}$	0.081	8.10446×10^{-2}	4.46×10^{-5}
$\frac{1}{2}$	0.126	0.126	0.0
$\frac{3}{5}$	0.181	0.181	0.0
$\frac{7}{10}$	0.246	0.246	0.0
$\frac{4}{5}$	0.321	0.321	0.0
$\frac{9}{10}$	0.406	0.406	0.0

Table 2

Conclusion

It is observed that the results obtained using compact sixth-order finite difference scheme are unconditionally stable and highly accurate and more efficient if compared to those obtained by Akram and Pasha [18]. the method developed is sixth-order accurate in space and fourth-order in time with very high speed fourth-order Runge-Kutta Algorithm. It should to be noted that, only one iterate was needed to obtain the results shown in both tables 1 and 2.

$u_{ex} = 0.5x + t$



Variation of $u_{ex} = 0.5x + t$ for different values of x and t

2.4 One-dimensional nonhomogeneous heat equation with nonlocal boundary conditions

Statement of the Problem

In this chapter, we consider the non homogeneous heat equation in one dimension with the non local boundary conditions. There has recently been much attention to the search for better and more accurate solution methods for determining a solution, approximate or exact to this type of problems. Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t), 0 < x < 1, 0 < t \leq T \tag{80}$$

Subject to the given initial condition

$$u(x,0) = f(x), 0 \leq x \leq 1 \tag{81}$$

And the non local boundary conditions

$$u(0,t) = \int_0^1 \phi(x,t)u(x,t)dx + g_{(t)_1}, 0 < t \leq T \tag{82}$$

$$u(1,t) = \int_0^1 \psi(x,t)u(x,t)dx + g(t)_2, 0 < t \leq T \tag{83}$$

where f,g_1, g_2, ϕ, ψ *and* q are known functions and are suffeciently smooth, T is given constant. Many authors as [6] , [9] [18-22] ,[30-31] and [24-26], have suggested traditional techniques for solving this type of problems in M. A. Rahman [8], has proposed a fourth-order numerical finite difference scheme for the solution of this problem. We propose a new technique for solving the given problem, this technique is based on Adomian's series solution method. This method provides us an exact solution which is much better result than that in [8]

.

2.5 Adomian Decomposition Method

To introduce the Adomian decomposition method, consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(k(x)\frac{\partial u}{\partial x}) + G(x,t,u), x \in \Omega \subset R, t \in (0,T] \tag{84}$$

With the initial condition

$$u(x,0) = f(x), x \in \Omega \tag{85}$$

And the nonlocal boundary conditions

$$u(0,t) = \int_0^1 \varphi(x,t)u(x;t)dx + g(t)_1, 0 < t \leq T \qquad (86)$$

$$u(1,t) = \int_0^1 \psi(x,t)u(x,t)dx + g(t)_2, 0 < t \leq T \qquad (87)$$

Where $G(x,t;u)$ is nonlinear function, assuming that sufficiently smooth in order to assure the existence and uniqueness of the solution to the equation (2,1)

In this section, we outline the steps to obtain a solution of (3,1)-(3,4) using Adomian decomposition method, which is initiated by G.Adomian[36], [40],and [47]. To begin it is convenient to rewrite the problem in the standard form

$$L_t(u) = L_{xx}(u) + q(x,t) \qquad (88)$$

Where the differential operators L_t and L_{xx} are given by

$L_t(.) = \frac{\partial}{\partial t}(.)$ and $L_{xx} = \frac{\partial^2}{\partial x^2}$.

Assuming that the inverse operator L_t^{-1} exists and it is defined as

$$L_t^{-1} = \int_0^t (.)dt \qquad (89)$$

Applying inverse operator L_t^{-1} on both sides of $(3,5)$ and using the initial condition yields

$$L_t^{-1}(L_t(u)) = L_t^{-1}(L_{xx}(u)) + L_t^{-1}(q(x,t))$$

Or

$$u(x,t) = f(x) + L_t^{-1}(L_{xx}(u)) + L_t^{-1}(q(x,t))$$

Now we decompose the unkown function $u(x,t)$ by a sum of components defined by the series [36]

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) \qquad (90)$$

Where u_0 is identified as $u(x,0)$, the components $u_k(x,t)$ are obtained by the recursive formula

$$\sum_{k=0}^{\infty} u_k(x,t) = f(x) + L_t^{-1}\{L_{xx}(\sum_{k=0}^{\infty} u_k(x,t))\} + L_t^{-1}(q(x,t))$$

Or

$$u_0(x,t) = f(x) + L_t^{-1}(q(x,t)) \qquad (91)$$

$$u_{k+1}(x,t) = L_t^{-1}(L_{xx}(u_k(x,t))), k \geq 0 \quad (92)$$

We note that the recursive relationship is constructed on the basis that the zeroth component $u_0(x,t)$ is defined by all terms that arise from the initial condition and from integrating the source term,the remaining components $u_k(x,t), k \geq 1$, can be completly determined such that each term is computed by using the previous term. Accordingly, considering few terms only the relations (3,8) and (3,9) give

$$u_0 = f(x) + L_t^{-1}(x,t)$$

$$u_1 = L_t^{-1}(L_{xx}(u_0))$$

$$u_2 = L_t^{-1}(L_{xx}(u_1))$$

and so on. As a result, the components u_0 ,$u_{1,}u_2$,... are identified and the series solution thus entirely determined. However, in many cases the exact solution in a closed form may be obtained as we can see in our examples.

2.6 Numerical Examples

$EXAMPLE\, 1\; we\; consider\; the\; problem\; (3,1)\; with,$

$$f(x) = x^2, 0 < x < 1,\;\; g_1(t) = \frac{1}{4(t+1)^2}, 0 < t < 1 \quad (93)$$

$$g_2(t) = \frac{3}{4(t+1)^2}, 0 < t < 1,\;\; \phi(x,t) = x, 0 < x < 1 \quad (94)$$

$\psi(x,t) = x,\; 0 < x < 1,\; q(x,t) = \frac{-2(x^2+t+1)}{(t+1)^3},\; 0 < x < 1,\; 0 < t \leq 1$

which has exact solution $u(x,t) = (\frac{x}{t+1})^2$, we rewrite the given problem in an operator form

$$L_t(u(x,t)) = L_{xx}(u(x,t)) + q(x,t) \quad (95)$$

where

$L_t(.) = \frac{\partial}{\partial t}(.)$,$L_{xx} = \frac{\partial^2}{\partial x^2}(.)$,$L_t^{-1} = \int_0^t(.)dt$

Applying the inverse operator L_t^{-1}on both sides of (3,12) we have

$$u(x,t) = u(x,0) + L_t^{-1}(L_{xx}(u(x,t))) + L_t^{-1}(q(x,t)) \quad (96)$$

Now the recursive formula is

$$u_0 = f(x) + L_t^{-1}(q(x,t)$$

Or

$$u_0 = x^2 + L_t^{-1}\{\frac{-2(x^2+t+1)}{(t+1)^3}\}$$

And

$$u_{k+1}(x,t) = L_t^{-1}(L_{xx}(u_k(x,t))), k \geq 0 \tag{97}$$

Using the recursive relation we compute the components as follows

$$u_0 = x^2 + \int_0^t \frac{-2(x^2+t+1)dt}{(t+1)^3} = \frac{x^2}{(t+1)^2} + \frac{2}{t+1} - 2 \tag{98}$$

$$u_1 = L_t^{-1}(L_{xx}(u_0)) = \int_0^t \frac{2dt}{(t+1)^2} = \frac{-2}{t+1} \ +2$$

$$u_k = 0, k \geq 2 \tag{99}$$

The solution in the series form is given by

$$u(x,t) = \sum_{k=0}^{\infty} u(x,t)$$

Or

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_k(x,t), \quad k \geq 2$$

Hence, the solution of (3,1) with (3,10) and (3,11) is given as

$$u(x,t) = \frac{x^2}{(t+1)^2}$$

which is the exact solution.

EXAMPLE 2 *In this example we consider*

$$q(x,t) = 0 \tag{100}$$

$$u(x,0) = f(x) = 0.5x^2, 0 < x < 1 \tag{101}$$

$$u_x(1,t) = g(t) = 1, 0 < t < T$$

$$\int_0^b u(x,t)dx = m(t) = 0.75t + \frac{1}{6}(0.75)^3 \tag{102}$$

where b is belongs to $]0,1[$. We rewrite the given problem in an operator form as

$$L_t(u(x,t)) = L_{xx}(u(x,t)) + q(x,t) \tag{103}$$

where

$L_t(.) = \frac{\partial}{\partial t}(.), L_{xx} = \frac{\partial^2}{\partial x^2}, L_t^{-1} = \int_0^t (.)dt.$

Applying the inverse operator L_t^{-1} on both sides of (3,20), we have

$$u(x,t) = u(x,0) + L_t^{-1}(L_{xx}(u(x,t))) + L_t^{-1}(q(x,t) \tag{104}$$

Now the recursive formula is

$$u_0(x,t) = f(x) + L_t^{-1}(0)$$

Or

$$u_0(x,t) = 0.5x^2 + L_t^{-1}(0)$$

And

$$u_{k+1}(x,t) = L_t^{-1}(L_{xx}(u_k(x,t))), k \geq 0$$

Using the recursive relation we compute the components as follows

$$u_0(x,t) = 0.5x^2 \tag{105}$$

$$u_1(x,t) = L_t^{-1}(L_{xx}(u_0(x,t))$$

Or

$$u_1(x,t) = \int_0^t dt = t \tag{106}$$

$$u_k(x,t) = 0, k \geq 2 \tag{107}$$

Thus, the solution in series form is given by

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

Or

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_k(x,t), k \geq 2$$

Hence, the solution of (3,1) with (16-19) is given as

$$u(x,t) = 0.5x^2 + t$$

This solution coincides with the exact one.

EXAMPLE 3 *consider the problem*

$$q(x,t) = -30x^4 + 6t^5$$

$$u(x,0) = u_0(x,t) = f(x) = x^6, 0 \leq x \leq 1$$

$$u(0,t)=\int_0^1 \phi(x,t)u(x,t)dx+g_1(t), 0\le t\le T$$

$$u(1,t)=\int_0^1 \psi(x,t)u(x,t)dx+g_2(t), 0\le t\le T$$

Where

$$\phi(x,t)=0.2,\ \psi(x,t)=0.4$$

$$g_1(t)=\frac{4}{5}t^6-\frac{1}{35},\ g_2(t)=\frac{3}{5}t^6+\frac{33}{35}$$

Applying the inverse operator L_t^{-1} on both sides of (3,1), we have

$$u(x,t)=u(x,0)+L_t^{-1}(L_{xx}(u(x,t)))+L_t^{-1}(q(x,t)) \qquad (108)$$

Now the recursive formula is

$$u_0(x,t)=f(x)+L_t^{-1}(q(x,t))$$

Or

$$u_0(x,t)=x^6+L_t^{-1}(-30x^4+6t^5)$$

And

$$u_{k+1}(x,t)=L_t^{-1}(L_{xx}(u_k(x,t))), k\ge 0$$

Computing the components u_0, u_1, u_2 and u_3

$$u_0(x,t)=x^6+L_t^{-1}(-30x^4+6t^5)=x^6+\int_0^t(-30x^4+6t^5)dt=x^6-30x^4t+t^6$$

$$u_1=L_t^{-1}(L_{xx}(u_0))=L_t^{-1}(30x^4-360x^2t)=\int_0^t(30x^4-360x^2t)dt=30x^4t-180x^2t^2$$

$$u_2=L_t^{-1}(L_{xx}(u_1))=L_t^{-1}(360x^2t-360t^2)=\int_0^t(360x^2t-360t^2)dt=180x^2t^2-120t^3$$

$$u_3=L_t^{-1}(L_{xx}(u_2))=L_t^{-1}(360t^2)=\int_0^t 360t^2dt=120t^3$$

$$u_k=0, k\ge 4$$

Finally, we obtain the solution

$$u(x,t)=u_0+u_1+u_2+u_3$$

Or

$$u(x,t)=x^6+t^6$$

This solution coincides with the exact one.

Example 4

Consider solving the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (\pi^2+1)e^t sin(\pi x), x \in (0,1),\ \ t \in (0,1)$$

$$u(0,t) = u(1,t) = 0 \tag{109}$$

$$u(x,0) = sin(\pi x)$$

The exact solution of this equation is

$$u(x,t) = e^t sin(\pi x)$$

to solve this problem we write it in an operator form as

$$L_t u(x,t) = L_{xx}(u(x,t)) + (\pi^2+1)e^t sin(\pi x) \tag{110}$$

Operating L_t^{-1} on both sides of (3,27) and imposing the initial condition we obtain

$$u(x,t) = u(x,0) + L_t^{-1}(L_{xx}(u(x,t))) + L_t^{-1}((\pi^2+1)e^t sin(\pi x)) \tag{111}$$

Where L_t^{-1} is a one fold integral operator, which means that

$$L_t^{-1} = \int_0^t (.)dt$$

The Adomian decomposition method assumes a series solution for $u(x,t)$ given by an infinite sum of components

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) \tag{112}$$

The components $u_k(x,t)$ are computed recursively as follows

$$u_0(x,t) = sin(\pi x) + L_t^{-1}((\pi^2+1)e^t sin(\pi x)) \tag{113}$$

And

$$u_{k+1}(x,t) = L_t^{-1}(L_{xx}(u_k(x,t)), k \geq 0 \tag{114}$$

To find the solution , one solves the above recursive relations respectively, we obtain

$$u_0(x,t) = sin(\pi x) + \int_0^t (\pi^2+1)e^t sin(\pi x)dt$$

$$u_0(x,t) = -\pi^2 sin(\pi x) + (\pi^2+1)e^t sin(\pi x)$$

$$u_1(x,t) = L_t^{-1}(L_{xx}(u_0(x,t)) = \int_0^t (\pi^4 sin(\pi x) - (\pi^2+1)\pi^2 e^t sin(\pi x))dt$$

$$u_1(x,t) = (\pi^2+1)\pi^2 sin(\pi x) + \pi^4 sin(\pi x)t - \pi^2(\pi^2+1)sin(\pi x)e^t$$

$$u_2(x,t) = L_t^{-1}(L_{xx}(u_1(x,t)) = \int_0^t (-\pi^6 sin(\pi x))t - \pi^4(\pi^2+1)sin(\pi x) + \pi^4(\pi^2+1)sin(\pi x)e^t)dt$$

$$u_2(x,t) = -\pi^4(\pi^2+1)sin(\pi x) - \pi^4(\pi^2+1)(sin(\pi x))t - (\pi^6 sin(\pi x))\frac{t^2}{2!} + (\pi^4(\pi^2+1)sin(\pi x))e^t$$

$$u_3(x,t) = L_t^{-1}(L_{xx}(u_2(x,t)) = \int_0^t (\pi^6(\pi^2+1)sin(\pi x) + \pi^6(\pi^2+1)sin(\pi x)t + \pi^8 sin(\pi x)\frac{t^2}{2!} - \pi^6(\pi^2+1)e^t$$

$$u_3(x,t) = \pi^6(\pi^2+1)sin(\pi x) + \pi^6(\pi^2+1)sin(\pi x)t + \pi^6(\pi^2+1)sin(\pi x)\frac{t^2}{2!} + \pi^8 sin(\pi x)\frac{t^3}{3!} - \pi^6(\pi^2+1)sin($$

$$u_4(x,t) = L_t^{-1}(L_{xx}(u_3(x,t)) = \int_0^t L_{xx}(u_3(x,t))dt$$

$$u_4(x,t) = -\pi^8(\pi^2+1)sin(\pi x) - \pi^8(\pi^2+1)(sin\pi x)t - \pi^8(\pi^2+1)sin\pi x\frac{t^2}{2!} - \pi^8(\pi^2+1)sin\pi x\frac{t^3}{3!} - \cdot$$
$$-\pi^{10}(\pi^2+1)sin\pi x\frac{t^4}{4!} + \pi^8(\pi^2+1)(sin(\pi x))e^t$$

$$u_5(x,t) = \pi^{10}(\pi^2+1)sin(\pi x) + \pi^{10}(\pi^2+1)sin(\pi x)t + \pi^{10}(\pi^2+1)sin(\pi x)\frac{t^2}{2!} +$$
$$\pi^{10}(\pi^2+1)sin\pi x\frac{t^3}{3!} + \pi^{10}(\pi^2+1)sin\pi x\frac{t^4}{4!} + \pi^{12}(\pi^2+1)sin\pi x\frac{t^5}{5!} - \pi^{10}(\pi^2+1)sin(\pi x)e^t$$

$$u_6(x,t) = -\pi^{12}(\pi^2+1)sin(\pi x) - \pi^{12}(\pi^2+1)sin(\pi x)t - \pi^{12}(\pi^2+1)sin(\pi x)\frac{t^2}{2!} - \pi^{12}(\pi^2+1)sin\pi x\frac{t}{3}$$
$$-\pi^{12}(\pi^2+1)sin(\pi x)\frac{t^4}{4!} - \pi^{12}(\pi^2+1)sin(\pi x)\frac{t^5}{5!} - \pi^{12}(\pi^2+1)sin(\pi x)\frac{t^6}{6!} +$$
$$+\pi^{12}(\pi^2+1)sin(\pi x)e^t$$

And so on, then

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + ... = e^t sin(\pi x) \tag{115}$$

This result is in good agreement with analytic solution

.

Example 5

We consider the problem (3,1)-(3,4) with:

$$u(x,0) = f(x) = \pi^2 cos(x), 0 < x < 1 \tag{116}$$

And the boundary conditions

$$u(1,t) = g(t) = (\pi^2 + t)cos(1.0), 0 < t < 1$$

$$\int_0^b u(x,t)dt = (\pi^2 + t)sin(0.75), 0 < t < 1 \tag{117}$$

$$q(x,t) = (1 + \pi^2 + t)cos(x)$$

$b = 0.75$, from equations (3,8) and (3,9) we obtain

$$u_0 = \pi^2 cos(x) + \int_0^t (1 + \pi^2 + t)cos(x)dt = \pi^2 cos(x) + \tag{118}$$

$$+(1 + \pi^2)tcos(x) + \frac{t^2}{2!}cos(x)$$

$$u_1 = L_t^{-1}[L_{xx}(u_0)] = -\pi^2 tcos(x) - (1 + \pi^2)\frac{t^2}{2!}cos(x) - \frac{t^3}{3!}cos(x) \tag{119}$$

$$u_2 = L_t^{-1}[L_{xx}(u_1)] = \pi^2 \frac{t^2}{2!}cos(x) + (1 + \pi^2)\frac{t^3}{3!}cos(x) + \frac{t^4}{4!}cos(x) \tag{120}$$

$$u_3 = L_t^{-1}[L_{xx}(u_2)] = -\pi^2 \frac{t^3}{3!}cos(x) - (1 + \pi^2)\frac{t^4}{4!}cos(x) - \frac{t^5}{5!}cos(x) \tag{121}$$

$$u_4 = L_t^{-1}[L_{xx}(u_3)] = \pi^2 \frac{t^4}{4!}cos(x) + (1 + \pi^2)\frac{t^5}{5!}cos(x) + \frac{t^6}{6!}cos(x) \tag{122}$$

$$u_5 = L_t^{-1}[L_{xx}(u_4)] = -\pi^2 \frac{t^5}{5!}cos(x) - (1 + \pi^2)\frac{t^6}{6!}cos(x) - \frac{t^7}{7!}cos(x) \tag{123}$$

....

And so on,

$$u_{n-2} = L_t^{-1}[L_{xx}(u_{n-3})] = (-1)^{n-2}cos(x)[\pi^2 \frac{t^{n-2}}{(n-2)!} + (1+\pi^2)\frac{t^{n-1}}{(n-1)!} + \frac{t^n}{n!}] \tag{124}$$

$$u_{n-1} = L_t^{-1}[L_{xx}(u_{n-2})] = (-1)^{n-1}cos(x)[\pi^2 \frac{t^{n-1}}{(n-1)!} + (1+\pi^2)\frac{t^n}{n!} + \frac{t^{n+1}}{(n+1)!}] \tag{125}$$

$$u_n = L_t^{-1}[L_{xx}(u_{n-1})] = (-1)^n cos(x)[\pi^2 \frac{t^n}{n!} + (1 + \pi^2)\frac{t^{n+1}}{(n+1)!} + \frac{t^{n+2}}{(n+2)!} \tag{126}$$

We sum the first n-terms, we have

$$S_n = \sum_{i=0}^{n} u_i = (\pi^2 + t)cosx + (-1)^n[\pi^2 \frac{t^{n+1}}{(n+1)!} + \frac{t^{n+2}}{(n+2)!}], 0 < t < 1 \tag{127}$$

Hence

$$u(x,t) = lim_{n\to\infty} S_n = (\pi^2 + t)cos(x) \tag{128}$$

This solution is in good agreement with the exact one.

2.7 Conclusion

In this chapter Adomian decomposition method is proposed for solving non homogeneous heat equation with nonlocal boundary conditions and initial condition. The results obtained show that the Adomian decomposition method provides us an exact solution.

Example 3

Table 1 $\quad h_x = \frac{1}{10},\ h_t = \frac{1}{100}$

Comparing absolute Error for Adomian 4 *terms*

x_i	u_{Ad}	u_{ex}	$\lvert u_{ex} - u_{Ad}\rvert$
0	-1.2000×10^{-4}	1.0×10^{-12}	1.2000×10^{-4}
0.1	-1.1900×10^{-4}	1.0×10^{-6}	1.18×10^{-4}
0.2	-5.6×10^{-5}	6.4×10^{-5}	0.00012
0.3	6.09×10^{-4}	7.29×10^{-4}	0.00012
0.4	3.976×10^{-3}	4.096×10^{-3}	0.00012
0.5	1.5505×10^{-2}	1.5625×10^{-2}	0.00012
0.6	4.6536×10^{-2}	4.6656×10^{-2}	0.00012
0.7	0.11753	0.11765	0.00012
0.8	0.26202	0.26214	0.00012
0.9	0.53132	0.53144	0.00012
1.0	0.99988	1.0	0.00012

Example 4

$$h_x = \frac{1}{10},\ h_t = \frac{1}{100}$$

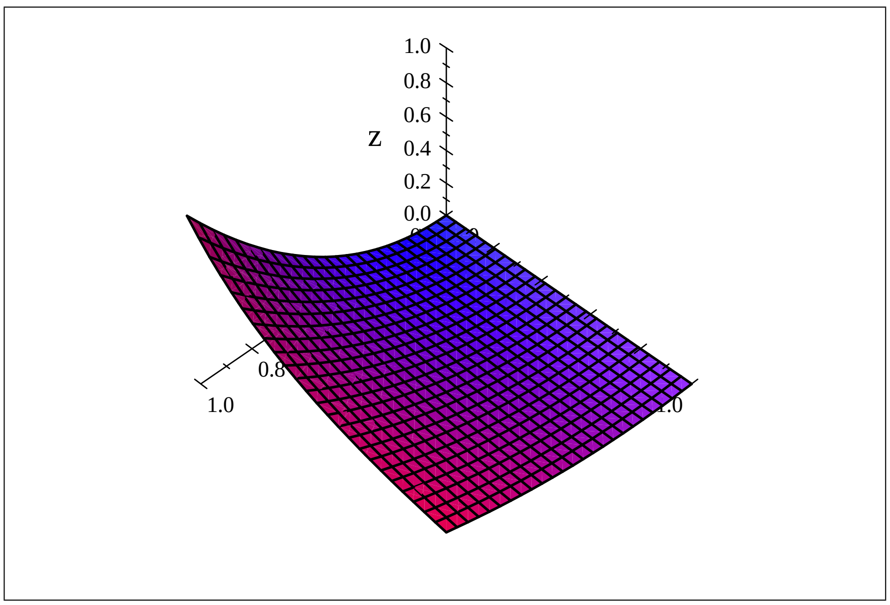
x_i	u_{ex}	u_{Ad}	$4 - Iterates$	$\lvert u_{ex} - u_{Ad} \rvert$
0.0	0.0	0.0		0.0
0.1	0.34152	0.32905		0.01247
0.2	0.6496	0.62589		0.02371
0.3	0.8941	0.86147		0.03263
0.4	1.0511	1.0127		0.0384
0.5	1.1052	1.0648		0.0404
0.6	1.0511	1.0127		0.0384
0.7	0.8941	0.86147		0.03263
0.8	0.6496	0.62589		0.02371
0.9	0.34152	0.32905		0.01247
1.0	0.0	0.0		0.0

Example 5

$h_x = \frac{1}{10},\ h_t = \frac{1}{25}$

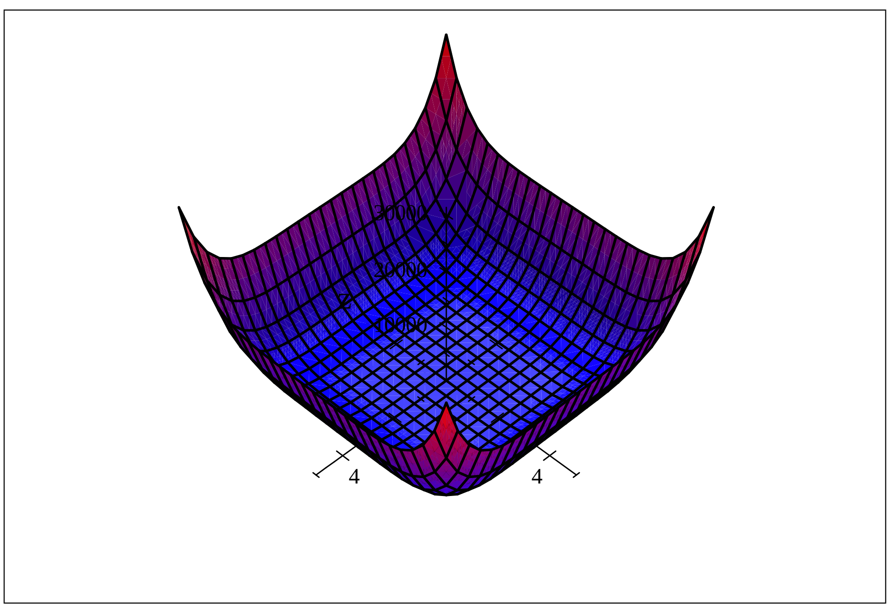
x_i	u_{ex}	$4-iterates$ u_{Ad}	$\lvert u_{ex}-u_{Ad}\rvert$
0.0	9.9096	9.9096	0.0
0.1	9.8601	9.8601	0.0
0.2	9.7121	9.7121	0.0
0.3	9.467	9.467	0.0
0.4	9.1274	9.1274	0.0
0.5	8.6965	8.6965	0.0
0.6	8.1787	8.1787	0.0
0.7	7.5793	7.5793	0.0
0.8	6.9041	6.9041	0.0
0.9	6.1599	6.1599	0.0
1.0	5.3542	5.3542	0.0

$u_{ex} = \frac{x^2}{(t+1)^2}$



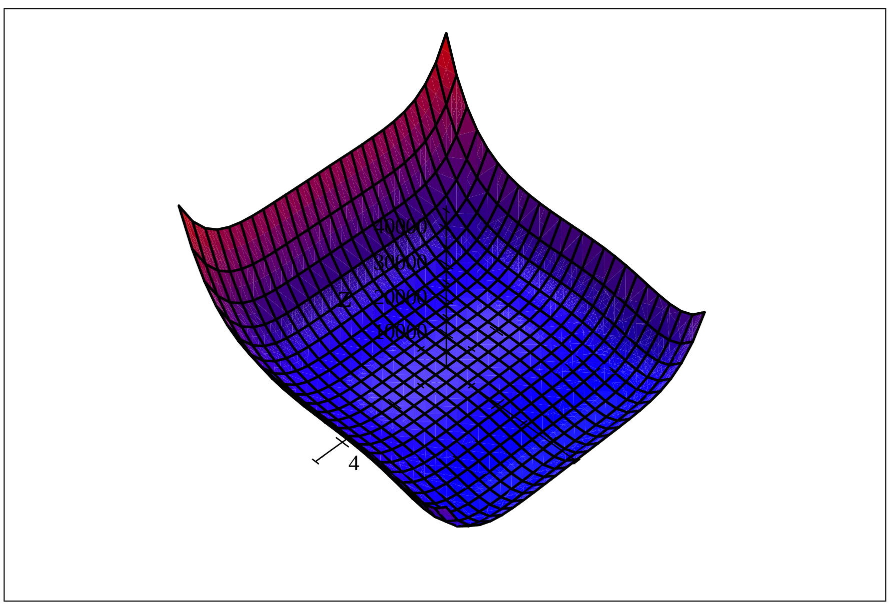
Variation of $u_{ex} = \frac{x^2}{(t+1)^2}$ for different values of x and t

$u_{ex} = x^6 + t^6$



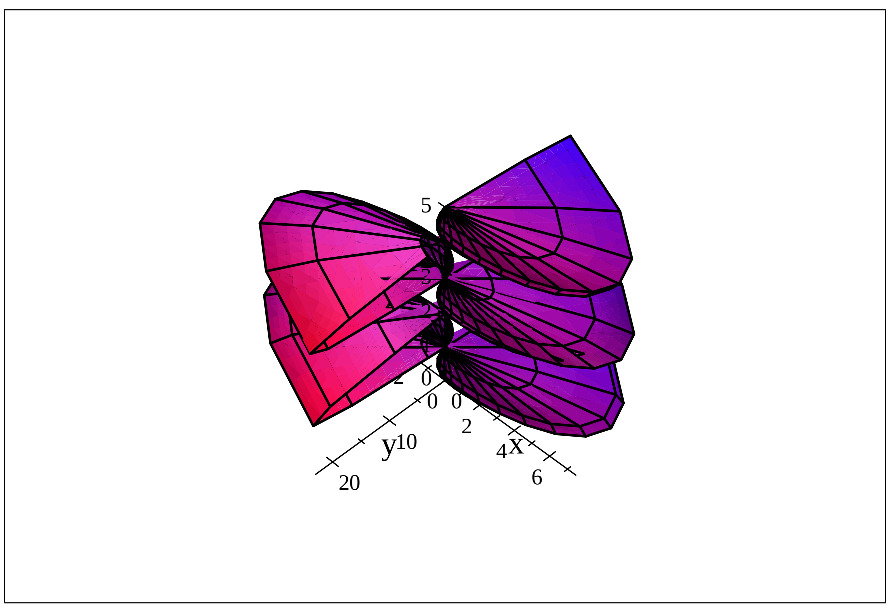
Variation of $u_{ex} = x^6 + t^6$ for different values of x and t

$u_{Ad} = x^6 + t^6 - 120t^3$



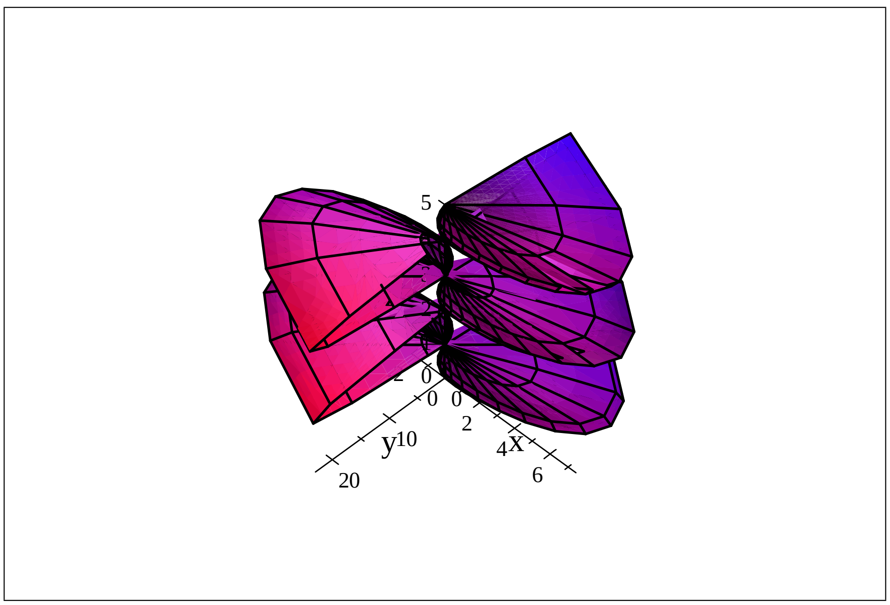
Variation of $u_{Ad} = x^6 + t^6 - 120t^3$ for different values of x and t

$u_{ex} = \exp(t) \times \sin(\pi x)$



variation of $u_{ex} = \sin(\pi x)\exp(t)$ for different values of x and t

$u = \pi^8 \times \sin(\pi x) \times (-4.2514 \times 10^{-6}) + \exp(t) \times \sin(\pi x)$



Variation of $u_{Ad} = (-4.2514 \times 10^{-6})\pi^8 \sin(\pi x) + \sin(\pi x)\exp(t)$ for different values of x and t

Table 2 *Comparing Absolute Error* $h_x = \frac{1}{10},\ h_t = \frac{1}{100}$
For 4 terms of Adomian solution

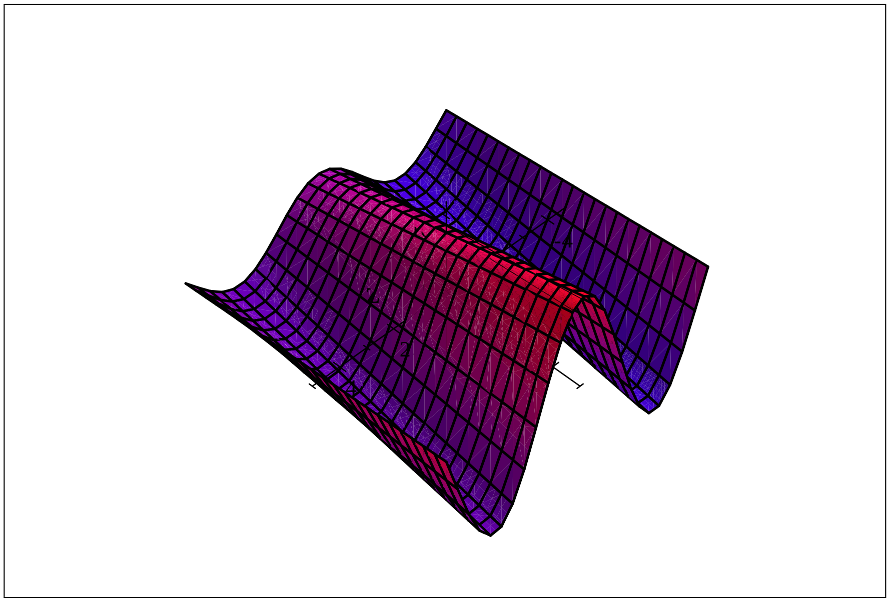
x	u_{Ad}	u_{ex}	$\|u_{ex} - u_{Ad}\|$
0.0	0.0	0.0	0.0
0.1	0.4	0.31214	0.08786
0.2	0.6	0.59373	0.00627
0.3	0.8	0.81719	0.101719
0.4	0.9	0.96067	0.06067
0.5	1.0	1.0101	0.0101
0.6	0.9	0.96067	0.06067
0.7	0.8	0.81719	0.01719
0.8	0.6	0.59373	0.00627
0.9	0.4	0.31214	0.08786
1.0	0.0	0.0	0.0

Example 5

$$h_x = \frac{1}{10}, \qquad h_t = 0.25$$

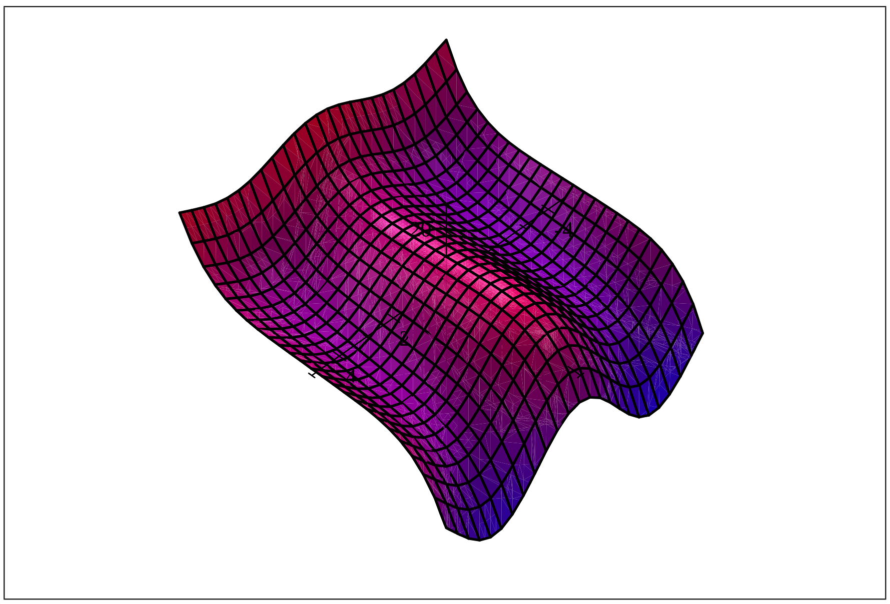
x_i	u_{ex}	$4 - Iterates$	u_{Ad}	$\lvert u_{ex} - u_{Ad} \rvert$
0.0	9.9096		9.9096	0.0
0.1	9.8601		9.8601	0.0
0.2	9.7121		9.7121	0.0
0.3	9.467		9.467	0.0
0.4	9.1274		9.1274	0.0
0.5	8.6965		8.6965	0.0
0.6	8.1787		8.1787	0.0
0.7	7.5793		7.5793	0.0
0.8	6.9041		6.9041	0.0
0.9	6.1599		6.1699	0.0
1.0	5.3542		5.3542	0.0

$u_{ex} = (\pi^2 + t) \times \cos(x)$



Variation of $u_{ex} = (\pi^2 + t)\cos(x)$ for different values of x and t

$u_{Ad} = (\pi^2 + t) \times \cos(x) - \frac{t^5}{120}$



Variation of $u_{Ad} = (\pi^2 + t)\cos(x) - \frac{t^5}{120}$ for different values of x and t

2.8 One-dimensional nonhomogeneous diffusion equation with derivative boundary conditions

Statement of the Problem

In this chapter, we consider the one dimensional nonhomogeneous heat equation with derivative boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t), (x,t)\epsilon]0,1[\times]0,T[\tag{129}$$

$$u(x,0) = g(x), 0 \leq x \leq 1 \tag{130}$$

$$u_x(0,t) = f_1(t), 0 < t \leq T \tag{131}$$

$$u_x(1,t) = f_2(t), 0 < t \leq T \tag{132}$$

Where $g(x), f_1(t), f_2(t)$ and $q(x,t)$ are known. Many authors have proposed numerical methods for solving nonlocal problems [6], [7-10], [18-22], [24-26], [30-31]. Later Akram[11], has proposed an $0(h^3 + t^3)$ $L_0 - stable$ parallel algorithm for solving the problem. In this work we propose a new technique based on the Adomian decomposition series solution [36,40,47]. The numerical examples show that results obtained coincide with the exact ones [1]. The organization of this chapter is the following.

In this section, we give a brief definition of this method, in section 3 the accuracy and the effeciency of the Adomian decomposition method are investigated with numerical illustration, the section 4 consists of a brief conclusion.

2.9 Adomian decomposition method

Reweriting the problem (4,1) in the following operator form

$$L_t(u(x,t)) = L_{xx}(u(x,t)) + q(x,t) \tag{133}$$

where

$$L_t(.) = \frac{\partial}{\partial t}(.), L_{xx}(.) = \frac{\partial^2}{\partial x^2}(.)$$

$L^{-1}$is regarded as the inverse operator of L and is defined by a defined integration from 0 to t, i.e.

$$L_t^{-1}(.) = \int_0^t (.)dt \tag{134}$$

Operating on both sides of equation (4,5) with L_t^{-1} using the initial condition yields

$$L_t^{-1}L_t(u(x,t)) = L_t^{-1}(L_{xx}(u(x,t))) + L_t^{-1}(q(x,t))$$

Or

$$u(x,t) = u(x,0) + L_t^{-1}(L_{xx}(u(x,t))) + L_t^{-1}(q(x,t)) \qquad (135)$$

Now we decompose the unkown function $u(x,t)$ by a sum of components defined by the following series with u_0 identified as $u(x,0)$

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) \qquad (136)$$

The components u_k are obtained by the recursive formula

$$\sum_{k=0}^{\infty} u_k(x,t) = g(x) + L_t^{-1}(L_{xx}(\sum_{k=0}^{\infty} u_k(x,t))) + L_t^{-1}(q(x,t))$$

Or

$$u_0 = g(x) + L_t^{-1}(q(x,t)) \qquad (137)$$

$$u_{k+1} = L_t^{-1}(L_{xx}(u_k(x,t))), k \geq 0 \qquad (138)$$

From the equations (4,9) and (4,10), we get

$$u_0 = g(x) + L_t^{-1}(q(x,t))$$
$$u_1 = L_t^{-1}(L_{xx}(u_0(x,t)))$$
$$u_2 = L_t^{-1}(L_{xx}(u_1(x,t)))$$
$$u_3 = L_t^{-1}(L_{xx}(u_2(x,t)))$$

....

and so on. The componenets u_0,$u_1, u_2, u_3, ...,$ are identified and the series solution thus entirely determined. However in many cases the exact solution in a closed form may be obtained. For numerical purposes, we can use the approximation

$$u(x,t) = lim_{m\longrightarrow\infty}\psi_m \qquad (139)$$

where

$$\psi_m = \sum_{k=0}^{m-1} u_k(x,t) \qquad (140)$$

Evaluating more componenets of $u(x,t)$, we obtain a more accurate solution. Noting that the convergence of this method has been proved by Adomian [47], Adomian and Rach[40] and Wazwaz [32] have investigated the phenomenon of the self-canceling "Noise" terms where sum of components vanishes in the limit, we observe that "Noise" terms appear for nonhomogeneouse cases only.

2.10 Numerical examples

Example 1

We consider the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t), 0 < x < 1, t > 0 \tag{141}$$

$$q(x,t) = -2e^{x-t}$$
$$u_x(0,t) = f_1(t) = e^{-t}, 0 < t \leq T$$
$$u_x(1,t) = f_2(t) = e^{1-t}, 0 < t \leq T$$
$$u(x,0) = g(x) = e^x, 0 \leq x \leq 1 \tag{142}$$

Rewriting equation (4,13) in operator form

$$L_t(u(x,t)) = L_{xx}(u(x,t))) + q(x,t) \tag{143}$$

where

$$L_t(.) = \frac{\partial}{\partial t}(.), L_{xx}(.) = \frac{\partial^2}{\partial^2}(.), L_t^{-1}(.) = \int_0^t (.)dt$$

Applying L_t^{-1} on both sides of (4,15), we have

$$u(x,t) = u(x;0) + L_t^{-1}(L_{xx}(u(x,t))) + L_t^{-1}(q(x,t)) \tag{144}$$

Now we get the recursive formula as follows

$$u_0(x,t) = g(x) + L_t^{-1}(q(x,t))$$

Or

$$u_0(x,t) = e^x + L_t^{-1}(-2e^{x-t}) = e^x(-1 + 2e^{-t}) \tag{145}$$
$$u_{k+1}(x,t) = L_t^{-1}(L_{xx}(u_k(x,t))), k \geq 0$$
$$u_1(x,t) = L_t^{-1}(L_{xx}(u_0(x,t))) = e^x(2 - t - 2e^{-t}) \tag{146}$$
$$u_2(x,t) = L_t^{-1}(L_{xx}(u_1(x,t))) = e^x(-2 + 2t - \frac{t^2}{2} + 2e^{-t}) \tag{147}$$
$$u_3(x,t) = L_t^{-1}(L_{xx}(u_2(x,t))) = e^x(2 - 2t + t^2 - \frac{t^3}{3!} - 2e^{-t})$$

Then the solution, in the series form, is given by

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

Or

$$u(x,t) = e^x(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + ...) = e^{x-t}$$

This solution is the exact one.

Example 2

Consider the problem (1) with the follownig conditions

$$q(x,t) = xt^2$$

$$u(x,0) = sin(x) \tag{148}$$

$$u_x(0,t) = 1$$

$$u_x(1,t) = sin(t)$$

Rewriting the given problem in an operator form as

$$L_t(u(x,t)) = L_{xx}(u(x,t)) + q(x,t) \tag{149}$$

where

$$L_t(.) = \frac{\partial}{\partial t}(.), L_{xx}(.) = \frac{\partial^2}{\partial x^2}(.) \ and \ L_t^{-1}(.) = \int_0^t (.)dt$$

Applying the inverse operator L_t^{-1}on both sides of equation (4,21), we get

$$u(x,t) = u(x,0) + L_t^{-1}(L_{xx}(u(x,t))) + L_t^{-1}(q(x,t)) \tag{150}$$

Now the recursive formula is

$$u_0(x,t) = g(x) + L_t^{-1}(q(x,t))$$

Or

$$u_0(x,t) = sin(x) + \int_0^t xt^2 dt = sin(x) + x(\frac{t^3}{3}) \tag{151}$$

$$u_{k+1}(x,t) = L_t^{-1}(L_{xx}(u(x,t))), k \geq 0$$

Then

$$u_1(x,t) = L_t^{-1}(L_{xx}(u(x,t))) = \int_0^t -sin(x)dt = -t * sin(x) \tag{152}$$

$$u_2(x,t) = L_t^{-1}(L_{xx}(u_1(x,t))) = \int_0^t tsin(x)dt = \frac{t^2}{2!}sin(x) \tag{153}$$

$$u_3(x,t) = L_t^{-1}(L_{xx}(u_2(x,t))) = \int_0^t -\frac{t^2}{2} sin(x)dt = -\frac{t^3}{3!} sin(x) \quad (154)$$

....

And so on, the solution in the series formula is given by

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

Or

$$u(x,t) = \frac{t^3}{3}x + (1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + ...)sin(x) = \frac{t^3}{3}x + e^{-t}sin(x) \quad (155)$$

this is the exact solution.

Example 3

Consider the problem (1) with the following boundary and initial conditions

$$\begin{aligned} q(x,t) &= 0 \\ u(x,0) &= sin(\pi x) \\ u_x(0,t) &= \pi e^{-\pi^2 t} \\ u_x(1,t) &= -\pi e^{\pi^2 t} \end{aligned} \quad (156)$$

We rewrite the equation (1) in an operator form as following

$$L_t(u(x,t)) = L_{xx}(u(x,t)) + q(x,t) \quad (157)$$

where

$$L_t(.) = \frac{\partial}{\partial t}(.), L_{xx}(.) = \frac{\partial^2}{\partial x^2} \; and \; L_t^{-1}(.) = \int_0^t (.)dt$$

Operating on both sides of equation (4,28) with the inverse operator L_t^{-1}, we have

$$u(x,t) = u(x,0) + L_t^{-1}(L_{xx}(u(x,t))) + L_t^{-1}(q(x,t)) \quad (158)$$

Proceeding as before, we find the recursive formula as follows

$$u_0(x,t) = g(x) + L_t^{-1}(q(x,t))$$

Or

$$u_0(x,t) = sin(\pi x)$$
$$u_{k+1}(x,t) = L_t^{-1}(L_{xx}(u_k(x,t))), k \geq 0$$

So, the components of the series solution are computed using the recursive formula as follows

$$u_0(x,t) = sin(\pi x) \tag{159}$$

$$u_1(x,t) = L_t^{-1}(L_{xx}(u_0(x,t))) = -\pi^2 t sin(\pi x)$$

$$u_2(x,t) = L_t^{-1}(L_{xx}(u_1(x,t))) = \frac{t^2}{2!}\pi^4 sin(\pi x)$$

$$u_3(x,t) = L_t^{-1}(L_{xx}(u_2(x,t))) = -\frac{t^3}{3!}\pi^6 sin(\pi x)$$

....

And so on, the solution in the series formula is given by

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

Or

$$u(x,t) = sin(\pi x)(1 - \frac{\pi^2 t}{1!} + \frac{\pi^4 t^2}{2!} - \frac{\pi^6 t^3}{3!} + ...) = sin(\pi x)e^{-\pi^2 t}$$

This solution coincides with the analytic one.

.EXAMPLE 4

Consider the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, 0 < x < 1 \tag{160}$$

Subjec to the intial condition

$$u(x;0) = sinx, 0 < x < 1 \tag{161}$$

and the boundary conditions

$$u_x(1,t) = -\pi e^{-\pi^2 t}, 0 < t < T$$

$$\int_0^b u(x,t)dx = \frac{1}{\pi}(\frac{1}{\sqrt[2]{2}} + 1)e^{-\pi^2 t} \tag{162}$$

we rewrite the equation (4,31) in an operator form

$$L_t(u(x,t)) - L_{xx}(u(x,t)) = 0, 0 < x < 1 \tag{163}$$

Following Adomian, appliyng the inverse operator L_t^{-1} to both sides of equation (4,34) one obtains

$$u(x,t) = u_0(x,0) + L_t^{-1}(L_{xx}(u(x,t)) \tag{164}$$

According to Adomian's method, one assumes that the unknown function $u(x,t)$ can be expressed by an infinite sum of components of the form,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{165}$$

Substituting equation (4,36) into equation (4,35) one obtains

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0(x,0) + L_t^{-1}(L_{xx}(\sum_{n=0}^{\infty} u_n(x,t))) \tag{166}$$

To determine the components of $u_n(x,t), n = 0,1,2,...,$ Adomian's technique can employ the recursive relation defined by

$$u_0 = u_0(x) = sinx \tag{167}$$

$$u_1 = L_t^{-1}(L_{xx}(u_0)) = -\int_0^t sinxdt = -tsinx \tag{168}$$

$$u_2 = L_t^{-1}(L_{xx}(u_1)) = sinx \int_0^t tdt = \frac{t^2}{2!} sinx \tag{169}$$

$$u_3 = L_t^{-1}(L_{xx}(u_2)) = -sinx \int_0^t \frac{t^2}{2!} dt = -\frac{t^3}{3!} sinx \tag{170}$$

$$...$$

And so on, the solution obtained is given by

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + ... + . = sinx(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + .. + ..) = e^{-t} sinx \tag{171}$$

2.11 Conclusion

The results obtained in this chapter compared to those obtained by Akram[11] show that the Adomian decomposition method is more accurate. In adddition the computation of the components of the solution are easy and take less time in comparison with other classical methods.

$$Example\ \ 1$$
$$h_x = \frac{1}{10}, \qquad h_t = \frac{1}{25}$$

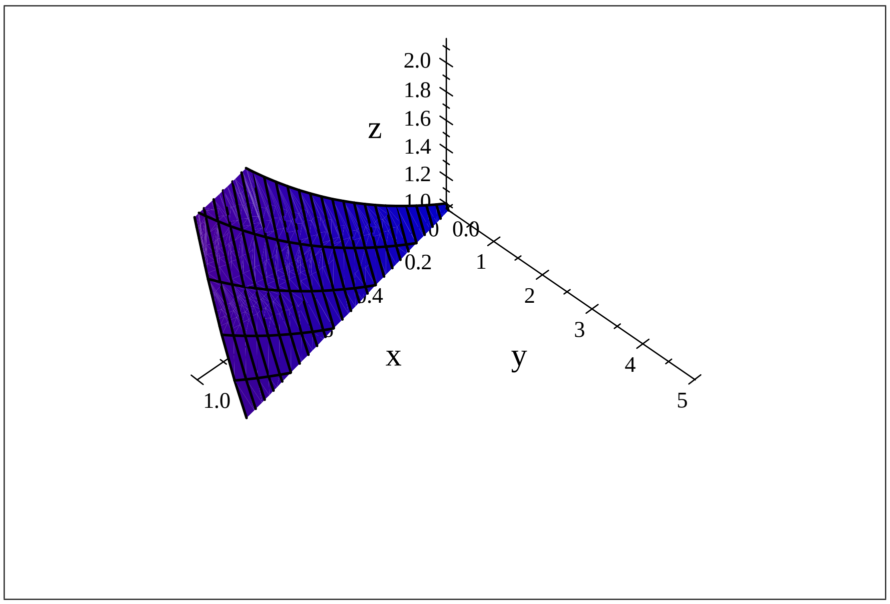
x_i	u_{ex}	$4 - Iterates\ \ u_{Ad}$	$\lvert u_{ex} - u_{Ad} \rvert$
0.0	0.96079	0.96079	0.0
0.1	1.0618	1.0618	0.0
0.2	1.1735	1.1735	0.0
0.3	1.2969	1.2969	0.0
0.4	1.4333	1.4333	0.0
0.5	1.5841	1.5841	0.0
0.6	1.7507	1.7507	0.0
0.7	1.9348	1.9348	0.0
0.8	2.1383	2.1383	0.0
0.9	2.3632	2.3632	0.0
1.0	2.1617	2.1617	0.0

Example 2

$h_x = \frac{1}{10},\ h_t = \frac{1}{25}$

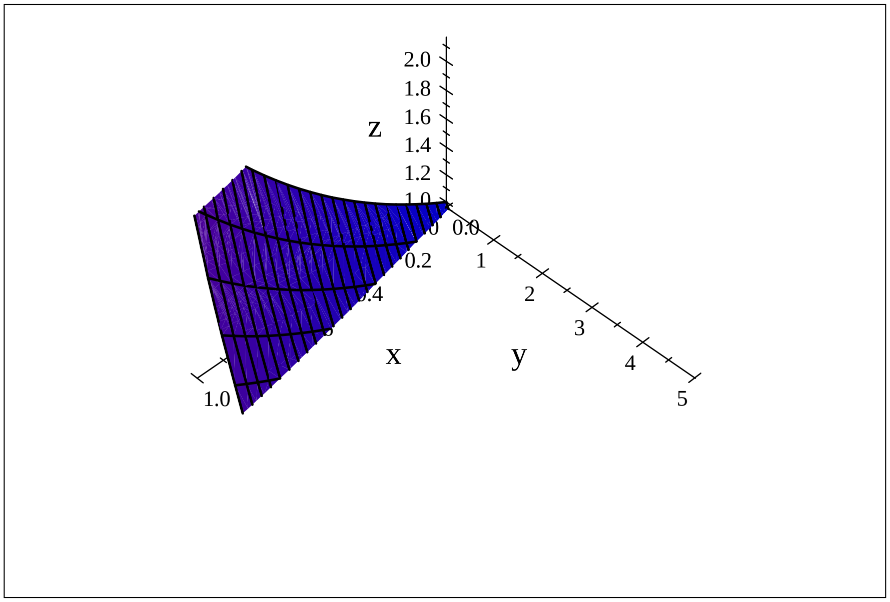
x_i	u_{ex}	u_{Ad} $4 - Iterates$	$\lvert u_{ex} - u_{Ad} \rvert$
0.0	0.0	0.0	0.0
0.1	0.10391	9.5921×10^{-2}	7.989×10^{-3}
0.2	0.20678	0.19088	0.0159
0.3	0.30759	0.28394	0.02365
0.4	0.40532	0.37416	0.03116
0.5	0.499	0.46064	0.03836
0.6	0.58770	0.54252	0.04518
0.7	0.67052	0.61897	0.05155
0.8	0.74665	0.68925	0.0574
0.9	0.81531	0.75263	0.06268
1.0	0.87583	0.80850	0.06733

$u_{ex} = \exp(x - t)$



Variation of $u_{ex} = \exp(x - t)$ for different values of x and t

$u_{Ad} = \exp(x) \times (1 - t + \frac{t^2}{2!} - \frac{t^3}{3!})$



Variation of $u_{Ad} = \exp(x) - (1 - \frac{t}{1} + \frac{t^2}{2!} - \frac{t^3}{3!})$ for different values of x and t

Example 1

$$h_x = \frac{1}{10},\ h_t = \frac{1}{100}$$

x_i	u_{ex}	$u_{Ad}\_4\ iterate$	$\|u_{ex} - u_{Ad}\|$
0.0	0.99005	0.99005	0
0.1	1.0942	1.0942	0
0.2	1.2092	1.2092	0
0.3	1.3364	1.3365	0.0001
0.4	1.4770	1.4770	0.0
0.5	1.6323	1.6323	0.0
0.6	1.8040	1.8040	0.0
0.7	1.9937	1.9938	0.0001
0.8	2.2034	2.2034	0.0
0.9	2.4351	2.4351	0.0
1.0	2.6912	2.6913	0.0001

Example 2

$$h_x = \frac{1}{10},\ h_t = \frac{1}{100}$$

x_i	u_{ex}	u_{Ad} $4-iterate$	$\lvert u_{ex} - u_{Ad}\rvert$
0.0	0.0	0.0	0.0
0.1	0.9884	0.9884	0.0
0.2	0.19669	0.19669	0.0
0.3	0.29258	0.29258	0.0
0.4	0.38555	0.38555	0.0
0.5	0.47516	0.47516	0.0
0.6	0.55962	0.55962	0.0
0.7	0.63781	0.63781	0.0
0.8	0.71022	0.71022	0.0
0.9	0.77554	0.77554	0.0
1.0	0.8331	0.8331	0.0

Example 3

$$h_x = \frac{1}{10},\ h_t = \frac{1}{250}$$

x_i	u_{ex}	u_{Ad}	$5-iterate$	$\lvert u_{ex}-u_{Ad}\rvert$
0.0	0.0	0.0		0.0
0.1	0.29706	0.29705		0.00001
0.2	0.56503	0.56503		0.0
0.3	0.7777	0.77770		0.0
0.4	0.91424	0.91424		0.0
0.5	0.96129	0.96129		0.0
0.6	0.91424	0.91424		0.0
0.7	0.7777	0.77770		0.0
0.8	0.56503	0.56503		0.0
0.9	0.29706	0.29706		0.00001
1.0	0.0	0.0		0.0

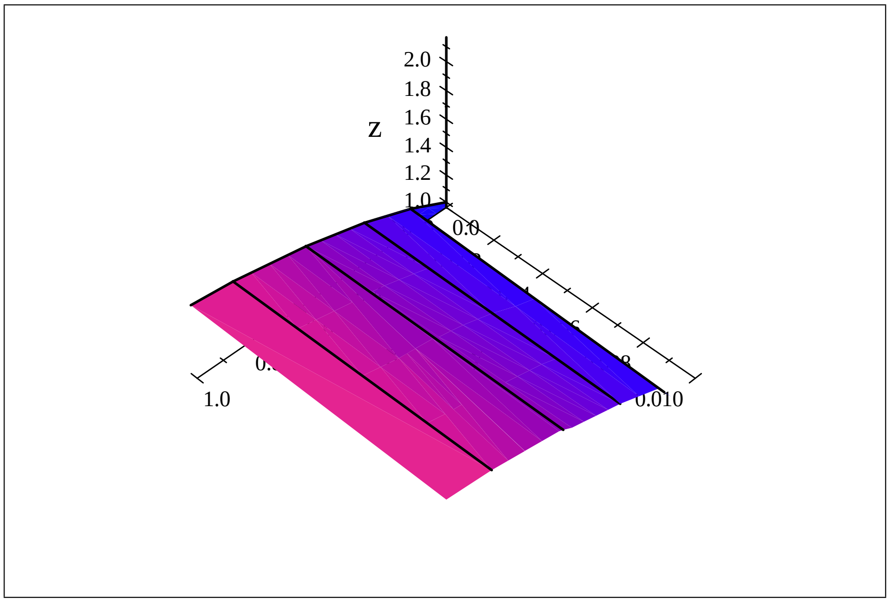
Example 4

$h_x = \frac{1}{10},\ h_t = \frac{1}{250}$

x_i	u_{ex}	u_{Ad} $5-iterate$	$\lvert u_{ex}-u_{Ad}\rvert$
0.0	0.0	0.0	0.0
0.1	9.9435×10^{-2}	9.9435×10^{-2}	0.0
0.2	0.19788	0.19788	0.0
0.3	0.29434	0.29434	0.0
0.4	0.38786	0.38786	0.0
0.5	0.47751	0.47751	0.0
0.6	0.56239	0.56239	0.0
0.7	0.64165	0.64165	0.0
0.8	0.71449	0.71449	0.0
0.9	0.78020	0.78020	0.0
1.0	0.83811	0.83811	0.0

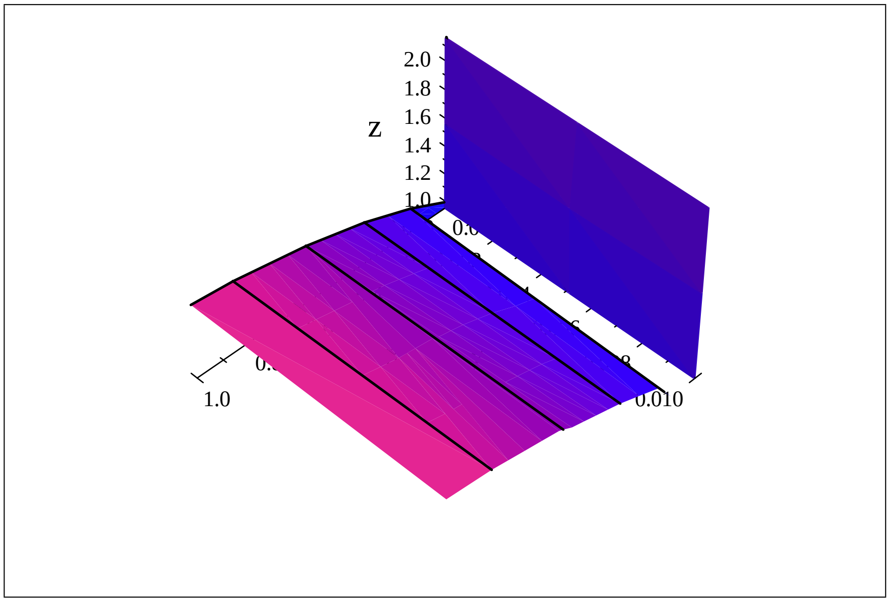
Example 1

$u_{ex} = \exp(x - t)$



Variation of $u_{ex} = \exp(x - t)$ for different values of x and t

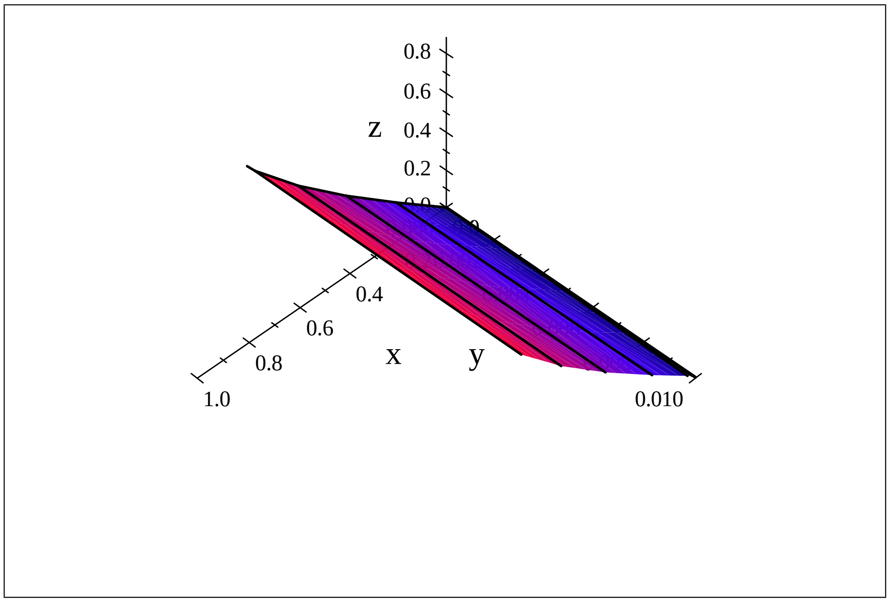
$u_{Ad} = \exp(x)(1 - t + \frac{t^2}{2} - \frac{t^3}{3!})$



Variation of $u_{Ad} = \exp(x)(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!})$ for different values of x and t

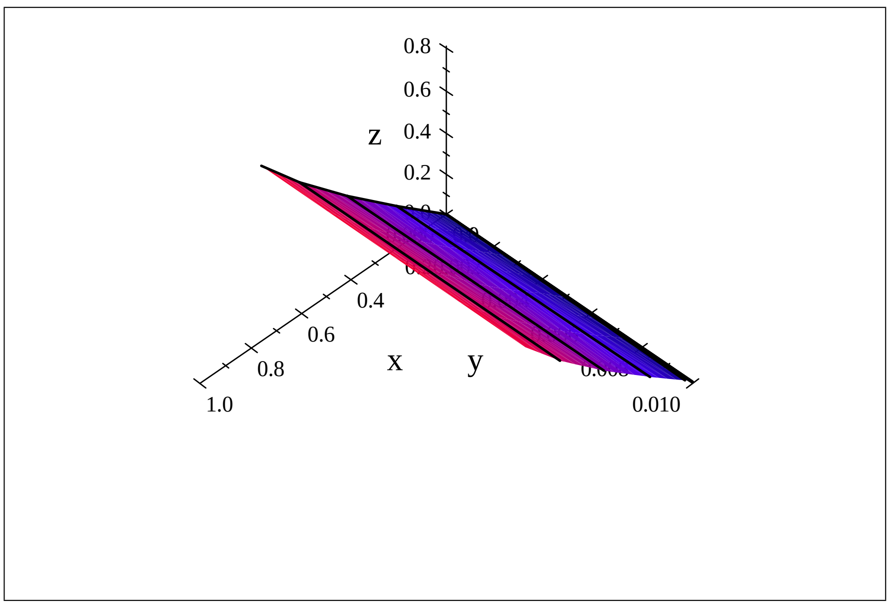
Example 2

$u_{ex} = \frac{xt^3}{3} + \exp(-t)\sin(x)$



Variation of $u_{ex} = \frac{xt^3}{3} + \sin(x)\exp(-t)$ for different values of x and t

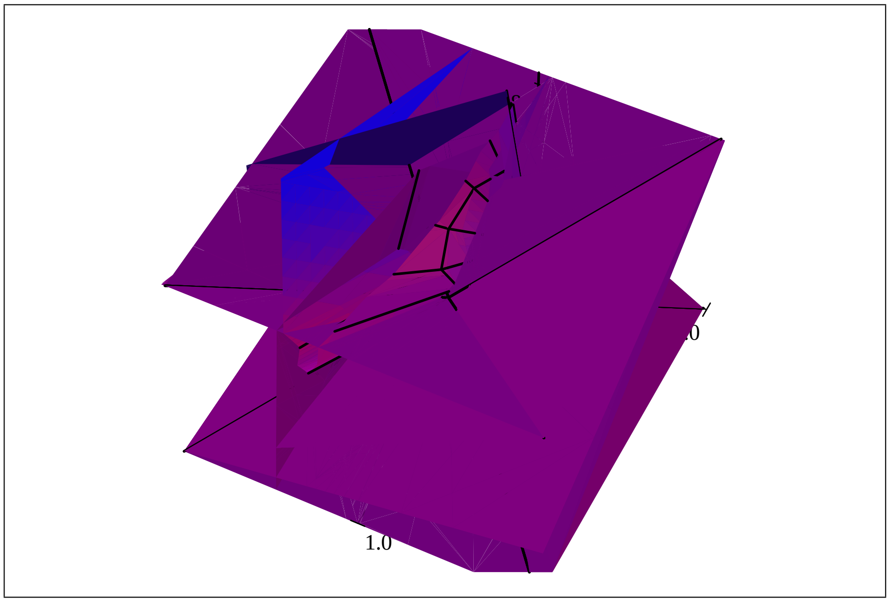
$u_{Ad} = \frac{xt^3}{3} + \sin(x)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!})$



Variation of $u_{Ad} = \frac{xt^3}{3} + \sin(x)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!})$ for different values of x and t

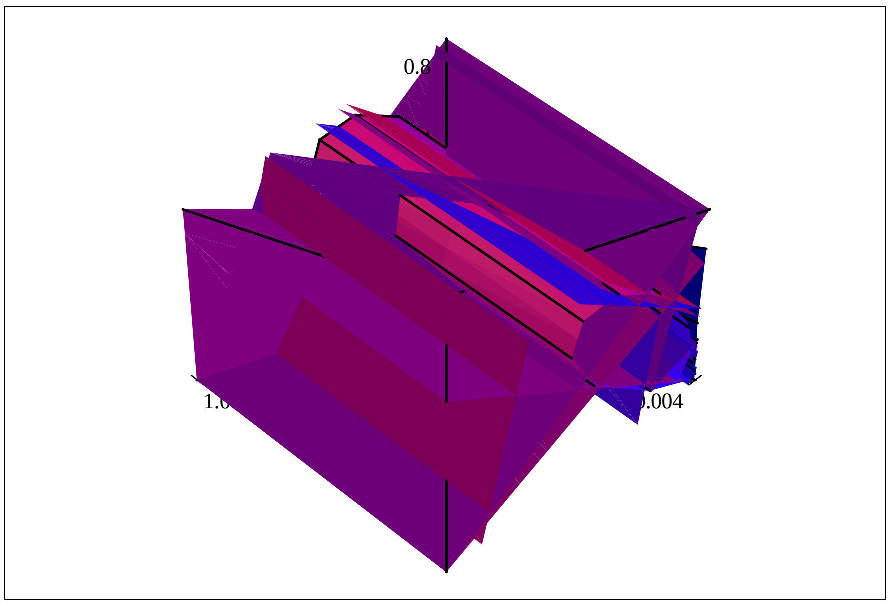
Example 3

$u_{ex} = \sin(\pi x) \times \exp(-\pi^2 t)$



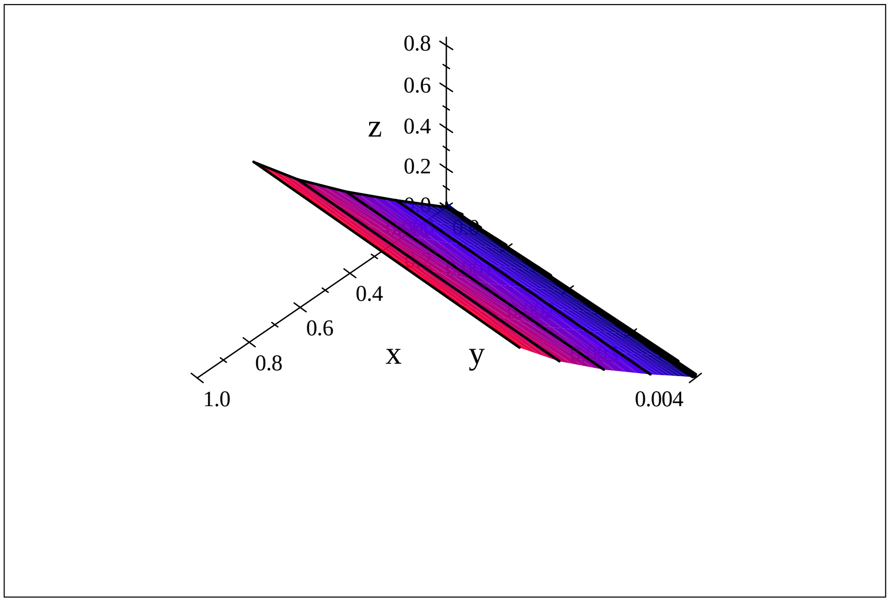
Variation of $u_{ex} = \sin(\pi x) \exp(-\pi^2 t)$ for different values of x and t

$u_{Ad} = \sin(\pi x) \times (1 - \frac{\pi^2 t}{1!} + \frac{\pi^4 t^2}{2!} - \frac{\pi^6 t^3}{3!})$



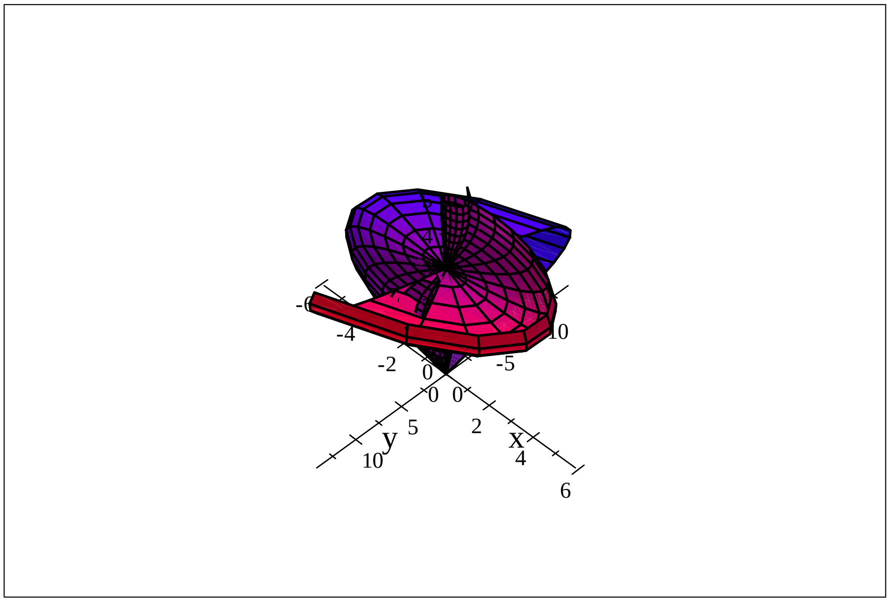
Variation of $u_{Ad} = \sin(\pi x)(1 - \frac{\pi^2 t}{1!} + \frac{\pi^4 t^2}{2!} - \frac{\pi^6 t^3}{3!})$ for different values of x and t

$u_{ex} = \sin(x)\exp(-t)$



Variation of $u_{ex} = \sin(x)\exp(-t)$ for different values of x and t

$u_{ex} = \sin(x)(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!})$



Variation of $u_{Ad} = \sin(x)(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!})$ for different values of x and t

2.12 A two-dimensional diffusion equation

Statement of the Problem

Partial differetial equations with non local boundary conditions and partial integro-differential arise in many fields of science and engineering, as chemical diffusion, heat conduction processes, population dynamics, thermoelasticity, medical science, electrochemistry and control theory , [10,24,31,37, 39], [41-45], [50], [52-55] and [57]. A detailed description of the occurence of such equations is given in [54]. The present paper deals with a two-dimensional diffusion equation with non local boundary conditions. We apply the decomposition method for solving this problem [36, , 40, 47]. This type of problem, was solved by many searchers using traditional numerical methods, B.A. Wade et al [19] have proposed a fourth-order Padè-scheme. The purpose of this work is to study and use Adomian decomposition method, we obtain an analytic solution. These results show that the decomposition method is more accurate, efficient and reliable in comparison with the tradional methods, like finite difference method, etc.

We consider the two-dimensional diffusion equation, that is given by

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}, 0<x,y<1, t>0 \tag{172}$$

Initial conditions are assumed to be of the form

$$u(x,y,0)=f(x,y),(x,y)\in\Omega\cup\partial\Omega$$

And the Dirichelet time-dependent boundary conditions are

$$\begin{aligned} u(0,y,t)&=\psi_0(y,t),\ 0\le t\le T,\ 0\le y\le 1\\ u(1,y,t)&=\psi_1(y,t),\ 0\le t\le T,\ 0\le y\le 1\\ u(x,0,t)&=\varphi_0(x)\gamma(t),\ 0\le t\le T,\ 0\le x\le 1\\ u(x,1,t)&=\varphi_1(x,t),\ 0\le t\le T,\ \ 0\le x\le 1 \end{aligned} \tag{173}$$

and non local boundary condition.

$$\int_0^1\int_0^1 u(x,y,t)dxdy=m(t),\ (x,y)\in\Omega\cup\partial\Omega \tag{174}$$

where $f,\psi_0,\psi_1,\varphi_0,\ \varphi_1$ and m are known functions. $\gamma(t)$ is to be determined.

2.13 Adomian decomposition method

First, we rewrite the problem (1) in an operator form

$$L_t(u(x,y,t) = L_{xx}(u(x,y,t)) + L_{yy}(u(x,y,t)) \tag{175}$$

where

$$L_t = \frac{\partial}{\partial t}(.),\ L_{xx} = \frac{\partial^2}{\partial x^2}(.),\ L_{yy} = \frac{\partial^2}{\partial y^2}(.)$$

L_t^{-1} is the inverse operator of L_t and is defined by

$$L_t^{-1} = \int_0^t (.)dt \tag{176}$$

Operating on both sides of equation (4) with the inverse operator L_t^{-1} using the initial condition we have

$$u(x,y,t) = L_t^{-1}((L_{xx}(u(x,y,t)) + L_{yy}(u(x,y,t)))$$

Or

$$u(x,y,t) = u(x,y,0) + L_t^{-1}(((L_{xx}(u(x,y,t)) + L_{yy}(u(x,y,t))) \tag{177}$$

Decomposing the uknown function $u(x,y,t)$ by a sum of components defined by the following series

$$u(x,y,t) = \sum_{k=0}^{\infty} u_k(x,y,t) \tag{178}$$

Where u_0 is identified as $u(x,y,0)$ the components u_k are obtained by the recursive relation

$$\sum_{k=0}^{\infty} u_k(x,y,t) = f(x,y) + L_t^{-1}(L_{xx}(\sum_{k=0}^{\infty} u_k(x,y,t)) + L_{yy}(\sum_{k=0}^{\infty} u_k(x,y,t)))$$

Or

$$u_0 = f(x,y) \tag{179}$$

$$u_{k+1} = L_t^{-1}(L_{xx}(u_k(x,y,t)) + L_{yy}(u_k(x,y,t))),\ k \geq 0 \tag{180}$$

From the equations (8) and (9), we get

$$u_0 = f(x,y)$$

$$u_1 = L_t^{-1}(L_{xx}(u_0(x,y,t)) + L_{yy}(u_0(x,y,t)))$$

$$u_2 = L_t^{-1}(L_{xx}(u_1(x,y,t)) + L_{yy}(u_1(x,y,t)))$$
$$u_3 = L_t^{-1}(L_{xx}(u_2(x,y,t)) + L_{yy}(u_2(x,y,t)))$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

And so on as result the componenets $u_0, u_1, u_2, u_3, ...$ are identified and the series solution is determined. However, in many cases the exact solution may be obtained as we can see in the numrical examples.

2.14 Numerical examples

Example 1

We consider the two-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2}, \quad 0 < x, y < 1 \tag{181}$$

In which $u = u(x,y,t)$, with dirichlet time-dependent boundary conditions on the boundary $\partial\Omega$ of the square Ω defined by the lines $x = 0,\ y = 0,\ x = 1,\ y = 1$, given by

$$u(0,y,t) = e^{(y+2t),\ 0\leq t\leq T,\quad 0\leq y\leq 1}$$

$$\begin{aligned} u(1,y,t) &= e^{(1+y+2t)}, \quad 0 \leq t \leq T, \quad 0 \leq y \leq 1 \\ u(x,0,t) &= e^{(x+2t)}, \quad 0 \leq t \leq T, \quad 0 \leq x \leq 1 \\ u(x,1,t) &= e^{(1+x+2t)}, \quad 0 \leq t \leq T, \quad 0 \leq x \leq 1 \end{aligned} \tag{182}$$

and non local boundary condition

$$\int_0^1 \int_0^1 u(x,y,t)dxdy = (e-1)^2 e^{2t} \tag{183}$$

with the initial conditions

$$u(x,y,0) = e^{(x+y)} \tag{184}$$

Theoretical solution is given by

$$u(x,y,t) = e^{(x+y+2t)} \tag{185}$$

We write the given problem in an operator form as the following

$$L_t(u(x,y,t)) = L_{xx}(u(x,y,t)) + L_{yy}(u(x,y,t)) \tag{186}$$

where

$$L_t(.) = \frac{\partial}{\partial t}(.), \quad L_{xx} = \frac{\partial^2}{\partial x^2}(.), \quad L_{yy} = \frac{\partial^2}{\partial y^2}, \quad L_t^{-1} = \int_0^t (.)dt$$

Applying the inverse operator to both sides of (15) one obtains

$$u(x,y,t) = u(x,y,0) + L_t^{-1}(L_{xx}(u(x,y,t))) + L_t^{-1}(L_{yy}(u(x,y,t))) \quad (187)$$

According to Adomian's method, one assumes that the unkown function $u(x,y,t)$ can be expressed by an infinte sum of components of the form

$$u(x,y,t) = \sum_{k=0}^{\infty} u_k(x,y,t) \quad (188)$$

Substituting equation (17) into equation (16), one obtains

$$\sum_{k=0}^{\infty} u_k(x,y,t) = u_0 x,y) + L_t^{-1}(L_{xx}(\sum_{k=0}^{\infty} u_k(x,y,t)) + L_{yy}(\sum_{k=0}^{\infty} u_k(x,y,t))) \quad (189)$$

To determine the componenents of $u_k(x,y,t)$,$k = 0,1,2,...$ Adomian's technique can employ the recursive relation defined by

$$u_0(x,y,t) = f(x,y)$$

or

$$u_0(x,y,t) = e^{(x+y)} \quad (190)$$

and

$$u_{k+1}(x,y,t) = L^{-1}(L_{xx}(u_k(x,y,t)) + L_{yy}(u_k(x,y,t))), \quad k \geq 0$$

which gives

$$u_1 = L_t^{-1}(L_{xx}(u_0) + L_{yy}(u_0)) = 2\int_0^t e^{x+y}dt = 2te^{x+y} \quad (191)$$

$$u_2 = L_t^{-1}(L_{xx}(u_1) + L_{yy}(u_1)) = 2\int_0^t te^{x+y}dt = 2t^2e^{x+y} \quad (192)$$

$$u_3 = L_t^{-1}(L_{xx}(u_2) + L_{yy}(u_2)) = 4\int_0^t t^2e^{x+y}dt = \frac{4}{3}t^3e^{x+y} \quad (193)$$

$$u_4 = L_t^{-1}(L_{xx}(u_3) + L_{yy}(u_3)) = \frac{4}{3}\int_0^t t^3e^{(x+y)}dt = \frac{2}{3}t^4e^{(x+y)} \quad (194)$$

$$u_5 = L_t^{-1}(L_{xx}(u_4)+L_{yy}(u_4)) = \frac{2}{3}\int_0^t t^4 e^{(x+y)}dt = \frac{4}{3\times 5}t^5 e^{(x+y)}$$

$$u_6 = L_t^{-1}(L_{xx}(u_5) + L_{yy}(u_5)) = \frac{4}{3\times 5}\int_0^t t^5 e^{(x+y)}dt = \frac{8}{3\times 5\times 6}t^6 e^{\prime x+y)}$$

$$u_7 = L_t^{-1}(L_{xx}(u_6) + L_{yy}(u_6)) = \frac{16}{3\times 5\times 6\times 7}t^7 e^{(x+y)} \quad (195)$$

Substituting (19)-(24) into equation (17). The solution $u(x, y, t)$ of (10) in a series form

$$u(x, y, t) = e^{(x+y)}(1 + \frac{2}{1!}t + \frac{4}{2!}t^2 + \frac{4\times 2}{3!}t^3 + \frac{2\times 2\times 4}{4!}t^4 + \quad (196)$$
$$+\frac{4\times 4\times 2}{5!}t^5 \quad +\frac{4\times 2\times 8}{6!}t^6 + \frac{\times 24\times 16}{7!}t^7 + ...)$$

Follows immediately. After some tedious algebra factoring equation (25) can be rewritten as

$$u(x, y, t) = e^{(x+y)}(1 + \frac{2t}{1!} + \frac{(2)^2}{2!}t^2 + \frac{(2)^3}{3!}t^3 + \frac{(2)^4}{4!}t^4 + ..) \quad (197)$$

It can be easily observed that (26) is equivalent to the exact solution

$$u(x, y, t) = e^{(x+y)}e^{2t} = e^{(x+y+2t)} \quad (198)$$

Example 2

Consider the two-dimensional nonhomogeneous diffusion problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - e^{-t}(x^2 + y^2 + 4), \quad t > 0, \quad 0 < x, y < 1 \quad (199)$$

with the initial condition

$$u(x, y, 0) = 1 + x^2 + y^2 \quad (200)$$

And the boundary conditions

$$u(0, y, t) = 1 + y^2 e^{-t}, \quad 0 \le t \le 1, \quad 0 \le y \le 1,$$

$$u(1, y, t) = 1 + (1 + y^2)e^{-t}, \quad 0 \le t \le 1, \ 0 \le y \le 1 \quad (201)$$

$$u(x, 0, t) = 1 + x^2 e^{-t}, \quad 0 \le t \le 1, \ 0 \le x \le 1$$

$$u(x,1,t) = 1 + (1+x^2)e^{-t}, \quad 0 \le t \le 1,\ 0 \le x \le 1$$

and nonlocal boundary condition

$$\int_0^1 \int_0^t u(x,y,t)dt = 1 + \frac{2}{3}e^{-t}, \quad 0 \le x \le 1,\ 0 \le y \le 1 \tag{202}$$

The exact solution is

$$u(x,y,t) = 1 + e^{-t}(x^2+y^2) \tag{203}$$

Writing the problem in operator form

$$L_t(u(x,y,t)) = L_{xx}(u(x,y,t)) + L_{yy}(u(x,y,t)) - e^{-t}(x^2+y^2+4) \tag{204}$$

Where $L_t,\ L_{xx}$, and L_{yy} are the linear differential operators defined by the following form

$$L_t = \frac{\partial}{\partial t}(.), \quad L_{xx} = \frac{\partial^2}{\partial x^2}(.), \quad L_{yy} = \frac{\partial^2}{\partial y^2}(.) \tag{205}$$

And the inverse operator given by the defined integral with respect to t from 0 to t.

$$L_t^{-1}(.) = \int_0^t (.)dt \tag{206}$$

Applying the inverse operator on both sides of the equation (33), we get

$$L_t^{-1}(L_t(u(x,y,t,))) = L_t^{-1}(L_{xx}u(x,y,t))) + L_t^{-1}(L_{yy}u(x,y,t))) + L_t^{-1}(-e^{-t}(x^2+y^2+4)$$

$$L_t^{-1}(L_t(u(x,y,0))) = u(x,y,0) \tag{207}$$

From equations (36) and (37) we have

$$\begin{aligned} u(x,y,t) = & u(x,y,0) + L_t^{-1}((L_{xx}+L_{yy})(u(x,y,t)) + .. \\ & + L_t^{-1}(-e^{-t}(x^2+y^2+4)) \end{aligned} \tag{208}$$

And the zeroth-component given by

$$u_0(x,y,t) = u(x,y,0) + L_t^{-1}(-e^{-t}(x^2+y^2+4)) \tag{209}$$

Decomposing the unkown function $u(x,y,t)$ into a sum components defined by the infinite series

$$u(x,y,t)=\sum_{k=0}^{\infty}u_k(x,y,t) \tag{210}$$

One the zeroth-component determined, the remaining components u_1,$u_2,u_3,.,u_k...$, entirely determined. Now substituting (40) into (38) leads to the recursive relations

$$u_0(x,y,t)=u(x,y,0)+L_t^{-1}(-e^{-t}(x^2+y^2+4))$$

.

$$u_{k+1}(x,y,t)=L_t^{-1}((L_{xx}1+L_{yy})(u(x,y,t)))$$

Or

$$u_0(x,y,t)=1+x^2+y^2-(x^2+y^2+4)\int_0^t e^{-t}dt= \tag{211}$$

$$(1+x^2+y^2)+(4+x^2+y^2)e^{-t}-(4+x^2+y^2)$$

$$=-3+(4+x^2+y^2)e^{-t} \tag{212}$$

$$u_1(x,y,t)=L_t^{-1}((L_{xx}+L_{yy})(u_0(x,y,t)))=\int_0^t 4e^{-t}dt \tag{213}$$

$$=-4e^{-t}+4$$

$$u_2(x,y,t)=L_t^{-1}((L_{xx}+L_{yy})(u_1))=\int_0^t(0)dt=0 \tag{214}$$

$$u_k(x,y,t)=0,\quad k\geq 2 \tag{215}$$

One the components $u_0,u_1,u_2,u_3,...,u_k,...$, are determined then, the series solution completely determined as following

$$u(x,y,t)=u_0(x,y,t)+u_1(x,y,t)+\sum_{k=2}^{\infty}u_k(x,y,t)$$

Or

$$u(x,y,t)=1+(x^2+y^2)e^{-t} \tag{216}$$

it should be noted that this solution coincides with the exact one.

Example 3

Consider the two-dimensonal diffusion problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad 0 < x, y < 1, \quad t > 0 \tag{217}$$

Subject to the initial condition

$$u(x, y, 0 = (1 - y)e^x, \quad 0 \le x, y \le 1 \tag{218}$$

and the boundary conditions

$$u(0, y, t) = (1 - y)e^t, \quad 0 \le t \le 1, \quad 0 \le y \le 1 \tag{219}$$

$$u(1, y, t) = (1 - y)e^{1+t}, \quad 0 \le t \le 1, \quad 0 \le y \le 1 \tag{220}$$

$$u(x, 0, t) = e^{1+t}, \quad 0 \le t \le 1, \quad 0 \le x \le 1 \tag{221}$$

$$u(x, 1, t) = 0, \quad 0 \le t \le 1, \quad 0 \le x \le 1 \tag{222}$$

Rewriting the equation (48) in an operator form as the following

$$L_t(u(x, y, t)) = L_{xx}(u(x, y, t)) + L_{yy}(u(x, y, t)) \tag{223}$$

where

$$L_t(.) = \frac{\partial}{\partial t}(.), L_{xx}(.) = \frac{\partial^2}{\partial x^2}(.), \ L_{yy}(.) = \frac{\partial^2}{\partial y^2}(.)$$

L_t^{-1} is regarderd as the inverse operator of L_t and is defined by a defined integration from 0 to t, i.e

$$L_t^{-1}(.) = \int_0^t (.)dt \tag{224}$$

Operating on both sides of equation (54) with the inverse operator L_t^{-1} using the initial condition yields

$$L_t^{-1}(L_t(u(x, y, t)) = L_t^{-1}((L_{xx} + L_{yy})(u(x, y, t)))$$

Or

$$u(x, y, t) = u(x, y, 0) + L_t^{-1}((L_{xx} + L_{yy})(u(x, y, t))) \tag{225}$$

Now decomposing the unkown function $u(x, y, t)$ by a sum of components defined by the following series with u_0 identified as $u(x, y, 0)$

$$u(x, y, t) = \sum_{k=0}^{\infty} u_k(x, y, t) \tag{226}$$

from the recursive relations we obtain the components u_k as following

$$\sum_{k=0}^{\infty} u_k(x,y,t) = u(x,y,0) + L_t^{-1}((L_{xx}+L_{yy})(\sum_{k=0}^{\infty} u_k(x,y,t)))$$

Or

$$u_0(x,y,t) = u(x,y,0) \tag{227}$$

$$u_{k+1}(x,y,t) = L_t^{-1}((L_{xx}+L_{yy})(u_k(x,y,t))), \quad k \geq 0 \tag{228}$$

From (58) and (59), we have

$$u_0(x,y,t) = (1-y)e^x \tag{229}$$

$$\begin{aligned} u_1(x,y,t) &= L_t^{-1}((L_{xx}+L_{yy})(u_0(x,y,t))) = \int_0^t (1-y)e^x dt = \\ &= (1-y)te^x \end{aligned} \tag{230}$$

$$\begin{aligned} u_2(x,y,t) &= L_t^{-1}((L_{xx}+L_{yy})(u_1(x,y,t))) = \\ &= \int_0^t (1-y)e^x dt = (1-y)e^x(\frac{t^2}{2!}) \end{aligned}$$

$$\begin{aligned} u_3(x,y,t) &= L_t^{-1}((L_{xx}+L_{yy})(u_2(x,y,t))) = \\ &= \int_0^t (1-y)e^x(\frac{t^2}{2!})dt = (1-y)e^x(\frac{t^3}{3!}) \end{aligned}$$

x

....

$$\begin{aligned} u_k(x,y,t) &= L_t^{-1}((L_{xx}+L_{yy})(u_{k-1}(x,y,t))) = \\ &= \int_0^t (1-y)e^x(\frac{t^{k-1}}{(k-1)!}dt = (1-y)e^x\frac{t^k}{k!} \end{aligned}$$

Then the solution in the series form is given by

$$u(x,y,t) = \sum_{k=0}^{\infty} u_k(x,y,t) = (1-y)e^x(\sum_{k=0}^{\infty}\frac{t^k}{k!}) = (1-y)e^{x+t} \tag{231}$$

As we can verify by substitution, this solution is equivalent to the theoritical one

Example 1

$$h_x = h_y = \frac{1}{10},\ h_t = \frac{1}{250}$$

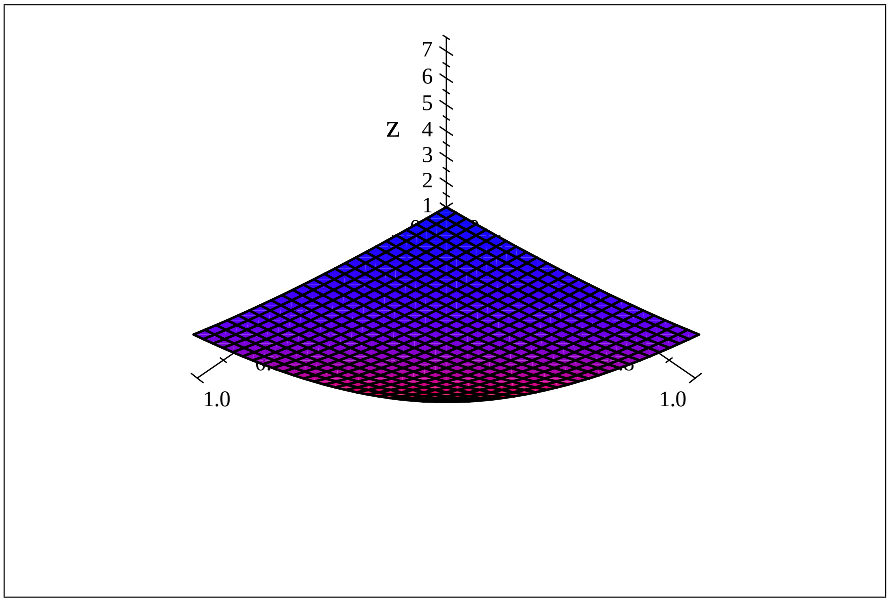
x_i,	y_j	u_{ex}	u_{Ad}	$5-iterate$	$\lvert u_{ex}-u_{Ad}\rvert$
0.0	0.0	1.008	1.008		0.0
0.1	0.1	1.2312	1.2312		0.0
0.2	0.2	1.5038	1.5038		0.0
0.3	0.3	1.8368	1.8368		0.0
0.4	0.4	2.2434	2.2434		0.0
0.5	0.5	2.7401	2.7401		0.0
0.6	0.6	3.3468	3.3468		0.0
0.7	0.7	4.0878	4.0878		0.0
0.8	0.8	4.9928	4.9928		0.0
0.9	0.9	6.0982	6.0982		0.0
1.0	1.0	7.4484	7.4484		0.0

Example 3

$h_x = h_y = \dfrac{1}{10},\ h_t = \dfrac{1}{250}$

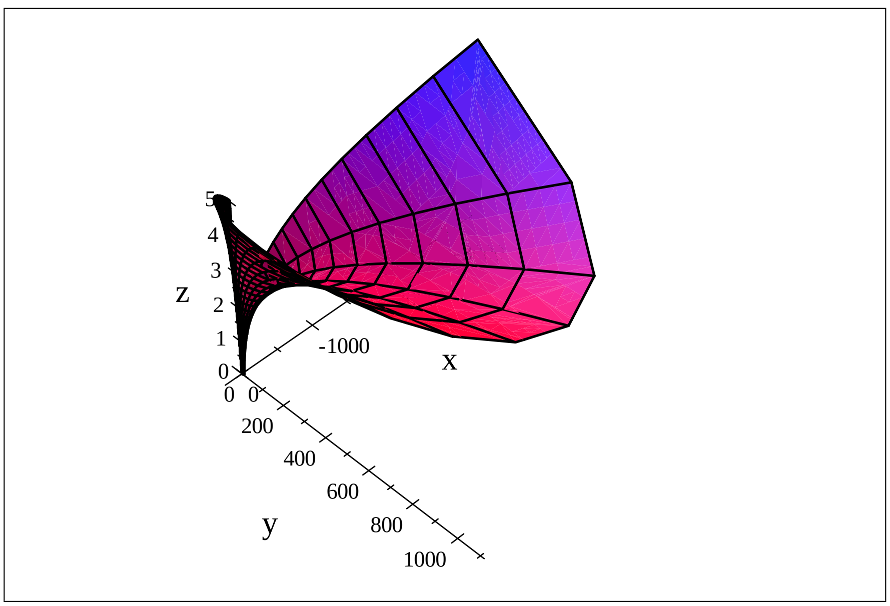
x_i	y_j	u_{ex}	u_{Ad}	$5-iterate$	$\|u_{ex} - u_{Ad}\|$
0.0	0.0	1.004	1.004		0.0
0.1	0.1	0.99864	0.99864		0.0
0.2	0.2	0.98104	0.98104		0.0
0.3	0.3	0.94869	0.94869		0.0
0.4	0.4	0.89868	0.89868		0.0
0.5	0.5	0.82766	0.82766		0.0
0.6	0.6	0.73177	0.73177		0.0
0.7	0.7	0.60655	0.60655		0.0
0.8	0.8	0.44689	0.44689		0.0
0.9	0.9	0.24695	0.24695		0.0
1.0	1.0	0.0	0.0		0.0

$u_{ex} = \exp(\frac{1}{250}) \times \exp(x + y)$



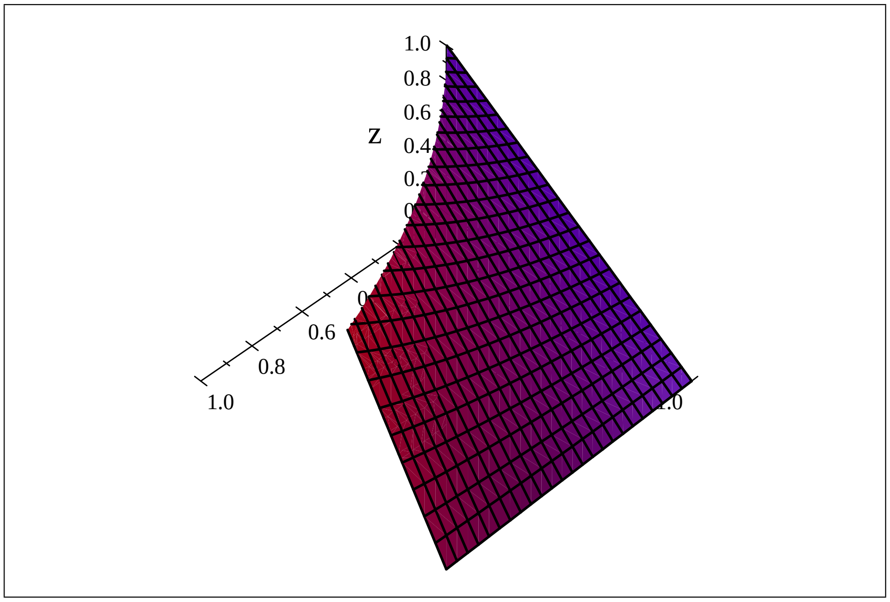
Variation of $u_{ex} = \exp(x + y + \frac{1}{250})$ for different values of x and y

$u_{Ad} = \exp(x+y) \times (1 + \frac{1}{1!} \times \frac{2}{250} + \frac{1}{2!} \times (\frac{2}{250})^2 + \frac{1}{3!} \times (\frac{2}{250})^3 + \frac{1}{4!} \times (\frac{2}{250})^4)$



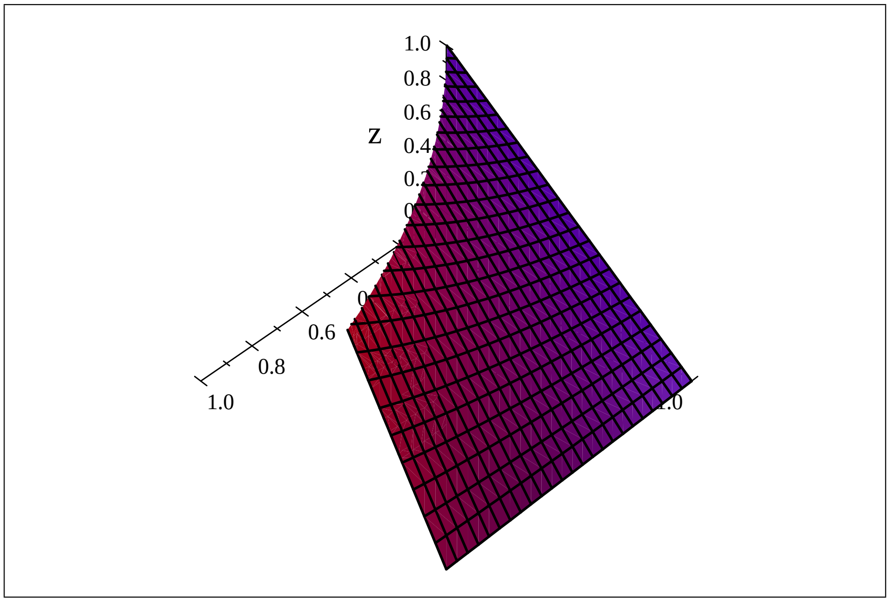
Variation of u_{Ad} for diferent values of x and y

$u_{ex} = (1 - y) \times \exp(x + \frac{1}{250})$



Variation of $u_{ex} = (1 - y) \exp(x + \frac{1}{250})$ for values of x and y

$$u_{Ad} = (1-y)\times\exp(x)\times(1+\frac{1}{250}+\frac{1}{2!}\times(\frac{1}{250})^2+\frac{1}{3!}\times(\frac{1}{250})^3+\frac{1}{4!}\times(\frac{1}{250})^4)$$



Variation of u_{Ad} for different values of x and y

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2.15 A three-dimensional Diffusion equation

Statement of the Problem

In this chapter we are dealing with A three-dimensional Diffusion equation. For solving this problem we use the Adomian decomposition method presented in the previous Chapters. We consider the three-dimensional diffusion equation given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, 0 < x, y, z < 1, t > 0 \tag{232}$$

initial condition is given by

$$u(x, y, z, 0) = f(x, y, z), (x, y, z) \in \Omega \cup \partial\Omega$$

And the Dirichelet time-dependent boundary conditions are

$$u(0, y, z, t) = \psi_0(y, z, t), 0 \leq t \leq T, 0 \leq y, z \leq 1 \tag{233}$$

$$u(1, y, z, t) = \psi_1(y, z, t), 0 \leq t \leq T, 0 \leq y, z \leq 1$$

$$u(x, 0, z, t) = \varphi_0(x, z) \times \gamma(t), 0 \leq t \leq T, 0 \leq x, z \leq 1$$

$$u(x, 1, z, t) = \varphi_1(x, z, t), 0 \leq t \leq T, 0 \leq x, z \leq 1$$

$$u(x, y, 0, t) = \phi_0(x, y, t), 0 \leq t \leq T, 0 \leq x, y \leq 1$$

$$u(x, y, 1, t) = \phi_1(x, y, t), 0 \leq t \leq T, 0 \leq x, y \leq 1$$

And nonlocal boundary condition

$$\int_0^1 \int_0^1 \int_0^1 u(x, y, z, t)dxdydz = m(t), (x, y, z) \in \Omega \cup \partial\Omega \tag{234}$$

Where $f, \psi_0, \psi_1, \varphi_0, \varphi, \phi_0, \phi_1$ and m are known functions and $\gamma(t)$ is to be determined.

2.16 ADOMIAN DECOMPOSITION METHOD

A Operator form

In this section, we outline the steps to obtain a solution to the above problem using the Adomian decomposition method, which was initiated by G. Adomian [36,40,47]. For this purpose we reformulate the problem in an operator form

$$L_t(u(x,y,z,t)) = L_{xx}(u(x,y,z,t)) + L_{yy}(u(x,y,z,t)) + L_{zz}(u(x,y,z,t)) \tag{235}$$

Where the differential operators $L_{t(.)=\frac{\partial}{\partial t}}$, $L_{xx}(.) = \frac{\partial^2}{\partial x^2}$, $L_{yy}(.) = \frac{\partial^2}{\partial y^2}$, $L_{zz}(.) = \frac{\partial^2}{\partial z^2}$. Assuming that the inverse operator L_t^{-1} exists and is defined as:

$$L_t^{-1} = \int_0^t (.)dt \tag{236}$$

B. Application to the problem

Applying the inverse operator on both the sides of (6,4) and using the initial condition, yields:

$$u(x,y,z,t) = L_t^{-1}(L_{xx}(u(x,y,z,t)) + L_{yy}(u(x,y,z,t)) + L_{zz}(u(x,y,z,t))$$

or

$$u(x,y,z,t) = u(x,y,z,0) + L_t^{-1}(L_{xx}(u(x,y,z,t) + L_{yy}(u(x,y,z,t)) + L_{zz}(u(x,y,z,t))) \tag{237}$$

Now , we decompose the unknown function $u(x,y,z,t)$ as a sum of components defined by the series

$$u(x,y,z,t) = \sum_{k=0}^{\infty} u_k(x,y,z,t) \tag{238}$$

Where $u_0(x,y,z,t)$ is identifined as $u(x,y,z,0)$. Substituting equation (7) into equation (6) one obtains

$$\sum_{k=0}^{\infty} u(x,y,z,t) = L_t^{-1}(L_{xx}(\sum_{k=0}^{\infty} u_k(x,y,z,t) + L_{yy}(\sum_{k=0}^{\infty} u_k(x,y,z,t) + L_{zz}(\sum_{k=0}^{\infty} u_k(x,y,z,t)) \tag{239}$$

The components are obtained by the recursive formula

$$u_0(x,y,z) = f(x,y,z) \tag{240}$$

$$u_{k+1}(x,y,z,t) = L_t^{-1}(L_{xx}(u_k(x,y,z,t)+L_{yy}(u_k(x,y,z,t)+L_{zz}(u_k(x,y,z,t)), k \geq 0 \quad (241)$$

From equations (9) and (10) we obtain the first few terms as:

$$u_0 = f(x,y,z)$$

$$u_1 = L_t^{-1}(L_{xx}(u_0) + L_{yy}(u_0) + L_{zz}(u_0))$$
$$u_2 = L_t^{-1}(L_{xx}(u_1) + L_{yy}(u_1) + L_{zz}(u_1))$$
$$u_3 = L_t^{-1}(L_{xx}(u_2) + L_{yy}(u_2) + L_{zz}(u_2))$$

and so on. As a result the components $u_0,\ u_1,\ u_2,\ u_3, ...$are identified and the series solution is thus entirely determined. However , in many cases the exact solution in a closed form may be obtained as we can see in our examples.

2.17 NUMERICAL EXAMPLES

Example 1

We consider the three-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

In which $u = u(x,y,z,t)$. The Dirichelet time-dependent boundary conditions on the boundary $\partial\Omega$ of the cube Ω defined by the lines $x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 1,\ z = 1$ are given by

$$u(0,y,z,t) = exp(y+z+3t), 0 \leq t \leq T, 0 \leq y,z \leq 1 \quad (242)$$

$$u(1,y,z,t) = exp(1+y+z+3t), 0 \leq t \leq T, 0 \leq y,z \leq 1$$
$$u(x,0,z,t) = exp(x+z+3t), 0 \leq t \leq T,\ 0 \leq x,z \leq 1$$
$$u(x,1,z,t) = exp(1+x+z+3t), 0 \leq t \leq T, 0 \leq x,z \leq 1$$
$$u(x,y,0,t) = exp(x+y+3t), 0 \leq t \leq T, 0 \leq x,y \leq 1$$
$$u(x,y,1,t) = exp(x+y+1+3t)$$

And nonlocal boundary condition

$$\int_0^1 \int_0^1 \int_0^1 u(x,y,z,t)dxdydz = (e-1)^3 exp(3t) \quad (243)$$

With the initial condition

$$u(x,y,z,0) = exp(x+y+z) \quad (244)$$

Analytic solution is given by

$$u(x, y, z, t) = e^{(x+y+z+3t)} \tag{245}$$

Using the decomposition method, described above, equation (9) gives the first component

$$u_0 = f(x, y, z) = e^{(x+y+z)} \tag{246}$$

And equation (10) gives the following components of the series

$$L_{xx}(u_0) + L_{yy}(u_0) + L_{zz}(u_0) = 3e^{(x+y+z)}$$

$$u_1 = L_t^{-1}(3e^{(x+y+z)}) = \int_0^t 3e^{(x+y+z)} dt = 3te^{(x+y+z)} \tag{247}$$

$$u_2 = L_t^{-1}(L_{xx}(u_1) + L_{yy}(u_1) + L_{zz}(u_1))$$

$$u_2 = L_t^{-1}(9te^{(x+y+z)}) = \int_0^t 9te^{(x+y+z)} dt = \frac{3^2}{2!} t^2 e^{(x+y+z)} \tag{248}$$

$$u_3 = L_t^{-1}(L_{xx}(u_2) + L_{yy}(u_2) + L_{zz}(u_2)$$

$$u_3 = L_t^{-1}(\frac{27}{2} t^2 e^{(x+y+z)}) = \int_0^t \frac{27}{2} t^2 e^{(x+y+z)} dt = \frac{3^3}{3!} t^3 e^{(x+y+z)} \tag{249}$$

$$u_4 = L_t^{-1}(L_{xx}(u_3) + L_{yy}(u_3) + L_{zz}(u_3)$$

$$u_4 = L_t^{-1}(\frac{3^2}{2} t^3 e^{(x+y+z)}) = \int_0^t \frac{3^2}{2} t^3 e^{(x+y+z)} dt = \frac{3^4}{4!} t^4 e^{(x+y+z)} \tag{250}$$

$$u_5 = L_t^{-1}(L_{xx}(u_4) + L_{yy}(u_4) + L_{zz}(u_4))$$

$$u_5 = L_t^{-1}(\frac{3^4}{4!} t^4 e^{(x+y+z)}) = \int_0^t \frac{3^4}{4!} t^4 e^{(x+y+z)} dt = \frac{3^5}{5!} t^5 e^{(x+y+z)} \tag{251}$$

$$u_6 = L_t^{-1}(L_{xx}(u_5) + L_{yy}(u_5) + L_{zz}(u_5))$$

$$u_6 = L_t^{-1}(\frac{3^5}{5!} t^5 e^{(x+y+z)}) = \int_0^t \frac{3^5}{5!} t^5 e^{(x+y+z)} dt = \frac{3^6}{6!} t^6 e^{(x+y+z)} \tag{252}$$

$$u_7 = L_t^{-1}(L_{xx}(u_6) + L_{yy}(u_6) + L_{zz}(u_6))$$

$$u_7 = L_t^{-1}(\frac{3^6}{6!} t^6 e^{(x+y+z)}) = \int_0^t \frac{3^6}{6!} t^6 e^{(x+y+z)} dt = \frac{3^7}{7!} t^7 e^{(x+y+z)} \tag{253}$$

Substituting (6,15)-(6,22) into equation (6,7) we obtain the solution $u(x, y, z, t)$ of (6,1)-(6,13) in series form as:

$$u(x, y, z, t) = e^{(x+y+z)}(1 + \frac{3t}{1!} + \frac{3^2}{2!} t^2 + \frac{3^3}{3!} t^3 + \frac{3^4}{4!} t^4 + \frac{3^5}{5!} t^5 + ..) \tag{254}$$

Which can be rewritten as:

$$u(x,y,z,t) = e^{(x+y+z)}e^{3t} = e^{(x+y+z+3t)} \tag{255}$$

It can be easily observed that (6,24) is equivalent to the exact solution.

Eample 2

Consider the three-dimensional non homogeneous diffusion problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - e^{-t}(x^2+y^2+z^2+4), t > 0, 0 < x, y, z < 1 \tag{256}$$

With the initial condition

$$u(x,y,z,0) = 1 + x^2 + y^2 + z^2 \tag{257}$$

And the boundary conditions

$$u(0,y,z,t) = 3 + (y^2 + z^2 - 2)e^{-t}, 0 \leq t \leq T, 0 \leq y, z \leq 1$$

$$u(1,y,z,t) = 3 + (-1 + y^2 + z^2)e^{-t}, 0 \leq t \leq T, 0 \leq y, z \leq 1$$

$$u(x,0,z,t) = 3 + (x^2 + z^2 - 2)e^{-t}, 0 \leq t \leq T, 0 \leq x, z \leq 1$$

$$u(x,1,z,t) = 3 + (-1 + x^2 + z^2)e^{-t}, 0 \leq t \leq T, 0 \leq x, z \leq 1$$

$$u(x,y,0,t) = 3 + (x^2 + y^2 - 2)e^{-t}, 0 \leq t \leq T, 0 \leq x, y \leq 1$$

$$u(x,y,1,t) = 3 + (-1 + x^2 + y^2)e^{-t}, 0 \leq t \leq T, 0 \leq x, y \leq 1 \tag{258}$$

And the non local boundary condition

$$\int_0^1 \int_0^1 \int_0^1 u(x,y,z,t)dxdydz = 3 - e^{-t}, 0 \leq t \leq T \tag{259}$$

Theoritical solution is given by

$$u(x,y,z,t) = 3 + (x^2 + y^2 + z^2 - 2)e^{-t} \tag{260}$$

Writing the problem in operator form and applying the inverse operator one obtains;

$$L_t^{-1}(L_t(u(x,y,z,t))) = L_t^{-1}(L_{xx}u(x,y,z,t) + L_{yy}u(x,y,z,t) + L_{zz}u(x,y,z,t)) \tag{261}$$

$$+L_t^{-1}(-e^{-t}(x^2 + y^2 + z^2 + 4)) \tag{262}$$

$$L_t^{-1}(L_t(u(x,y,z,0))) = u(x,y,z,0) \tag{263}$$

From which we obtain

$$u(x,y,z,t) = u(x,y,z,0) + L^{-1}(L_{xx}u(x,y,z,t) + L_{yy}u(x,y,z,t) + L_{zz}u(x,y,z,t)) \tag{264}$$
$$+L_t^{-1}(-e^{-t}(x^2+y^2+z^2+4)) \tag{265}$$

Using Adomian decomposition, the zeroth component is given by:

$$u_0(x,y,z,t) = u(x,y,z,0) + L_t^{-1}(-e^{-t}(x^2+y^2+z^2+4)) \tag{266}$$

And

$$u_{k+1}(x,y,z,t) = L_t^{-1}(L_{xx}u_k(x,y,z,t)+L_{yy}u_k(x,y,z,t)+L_{zz}u_k(x,y,z,t)), k \geq 0 \tag{267}$$

Applying these formula, we obtain the components of the series solution as:

$$u_0(x,y,z,t) = 1+x^2+y^2+z^2+\int_0^t -e^{-t}(x^2+y^2+z^2+4)dt = -3+(x^2+y^2+z^2+4)e^{-t} \tag{268}$$

$$u_1(x,y,z,t) = L_t^{-1}(L_{xx}u_0(x,y,z,t)+L_{yy}u_0(x,y,z,t)+L_{zz}u_0(x,y,z,t)) = \int_0^t 6e^{-t}dt = 6-6e^{-t} \tag{269}$$

$$u_2(x,y,z,t) = L_t^{-1}(L_{xx}u_1(x,y,z,t) + L_{yy}u_1(x,y,z,t) + L_{zz}u_1(x,y,z,t))$$
$$u_2(x,y,z,t) = \int_0^t 0dt = 0 \tag{270}$$

Then

$$u_k(x,y,z,t) = 0, k \geq 2 \tag{271}$$

Finally, we obtain the aproximate solution:

$$u(x,y,z,t) = u_0(x,y,z,t) + u_1(x,y,z,t)$$
$$u(x,y,z,t) = -3 + (x^2+y^2+z^2++4) + 6 - 6e^{-t}$$
$$u(x,y,z,t) = 3 + (x^2+y^2+z^2-2)e^{-t} \tag{272}$$

And we can observe that the obtained result is exact.

Example 1

$$h_x = h_y = h_z = \frac{1}{10},\ h_t = \frac{1}{250}$$

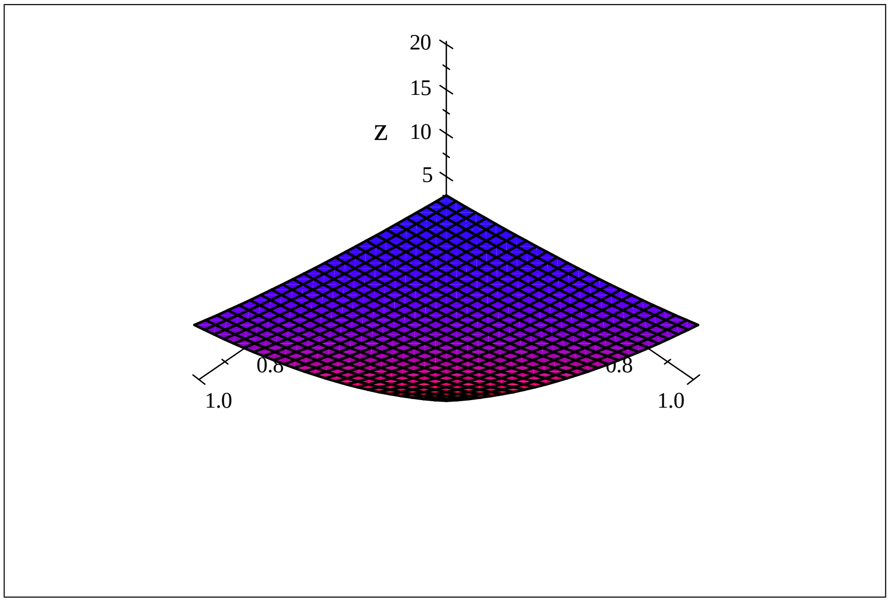
x_i,	y_j,	z_k	u_{ex}	u_{Ad}	$5 - iterate$	$\lvert u_{ex} - u_{Ad} \rvert$
0.0	0.0	0.0	1.021	1.021		0.0
0.1	0.1	0.1	1.3662	1.3662		0.0
0.2	0.2	0.2	1.8441	1.8441		0.0
0.3	0.3	0.3	2.4893	2.4893		0.0
0.4	0.4	0.4	3.3602	3.3602		0.0
0.5	0.5	0.5	4.5358	4.5358		0.0
0.6	0.6	0.6	6.1227	6.1227		0.0
0.7	0.7	0.7	8.2648	8.2648		0.0
0.8	0.8	0.8	11.156	11.156		0.0
0.9	0.9	0.9	15.059	15.059		0.0
1.0	1.0	1.0	20.328	20.328		0.0

Example 2

$$h_x = h_y = h_z = \frac{1}{10}, \; h_t = \frac{1}{250}$$

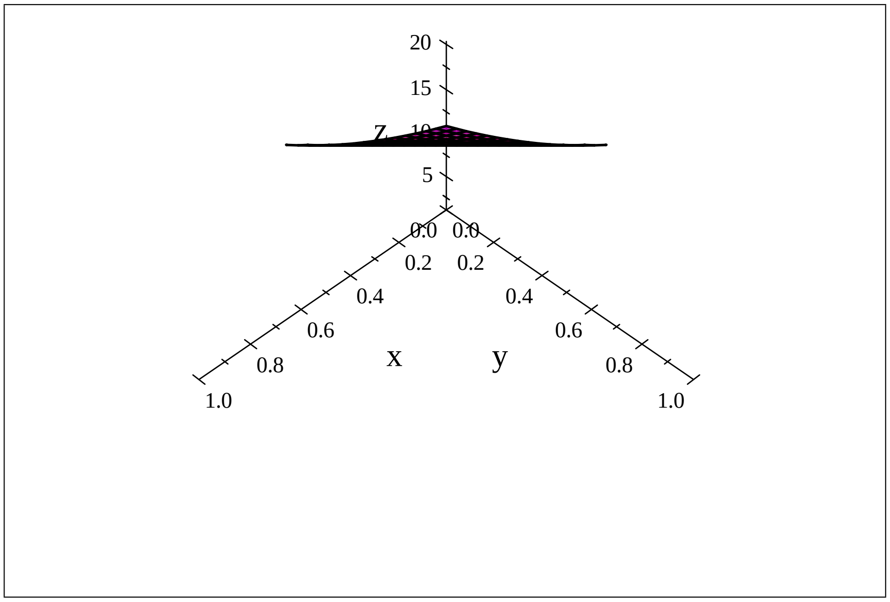
x_i	y_j	z_k	u_{ex}	u_{Ad} $1 - iterate$	$\lvert u_{ex} - u_{Ad} \rvert$
0.0	0.0	0.0	1.0080	0.98403	0.02397
0.1	0.1	0.1	1.0976	1.0737	0.0239
0.2	0.2	0.2	1.3665	1.3426	0.0239
0.3	0.3	0.3	1.8148	1.7908	0.024
0.4	0.4	0.4	2.4422	2.4183	0.0239
0.5	0.5	0.5	3.249	3.225	0.024
0.6	0.6	0.6	4.235	4.2111	0.0239
0.7	0.7	0.7	5.4004	5.3764	0.024
0.8	0.8	0.8	6.7450	6.721	0.024
0.9	0.9	0.9	8.2689	8.2449	0.024
1.0	1.0	1.0	9.9721	9.9481	0.024

$u_{ex} = \exp(x + y + \frac{253}{250})$



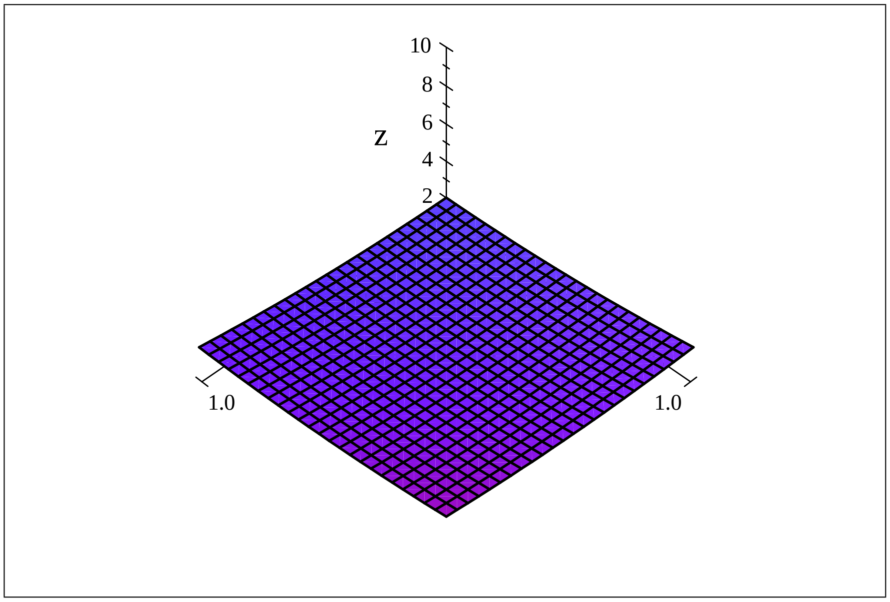
Variation of $u_{ex} = \exp(x + y + \frac{253}{250})$ for different values of x and y

$u_{Ad} = \exp(x+y+1)(1+\frac{3}{1!}+\frac{1}{2!}\times(\frac{3}{250})^2+\frac{1}{3!}\times(\frac{3}{250})^3+\frac{1}{4!}\times(\frac{3}{250})^4)$



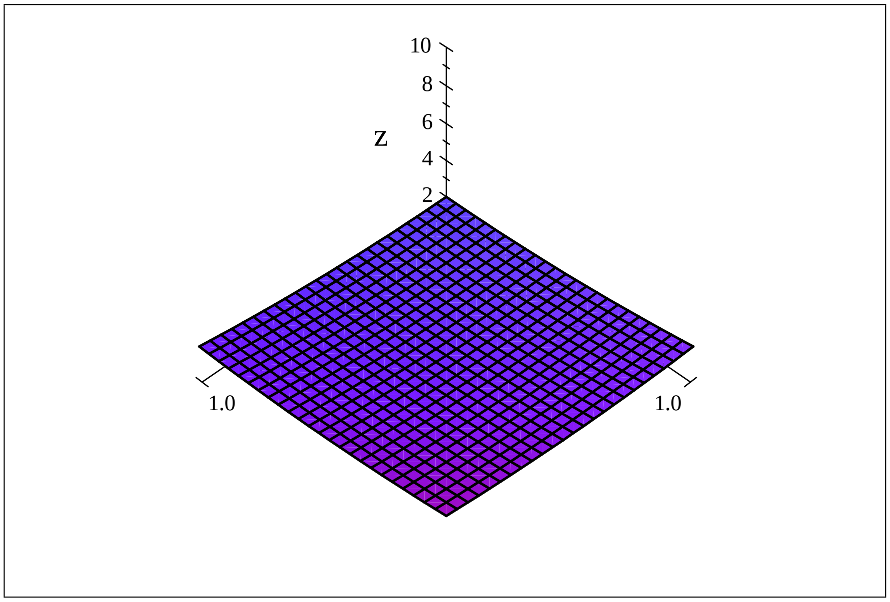
Variation of u_{Ad} for different values of x and y

$u_{ex} = 3 + (x^2 + y^2 - 1)\exp(\frac{-1}{250})$



Variation of $u_{ex} = (3 + x^2 + y^2 - 1)\exp(\frac{-1}{250})$ for different values of x and y

$u_{Ad} = -3 + (x^2 + y^2 + 5)\exp(\frac{-1}{250})$



Variation of $u_{Ad} = -3 + (x^2 + y^2 + 5)\exp(\frac{-1}{250})$ for different values of x and y

Chapter 3

3 Numerical method for solving hyperbolic equation with an integral boundary condition

In this chapter we will deal with a new kind of non classical boundary value problems that is, the solution of hyperbolic partial differential equations with nonlocal boundary specifications. These nonlocal conditions arise mainly when the data on the boundary cannot be measured directly. Many physical phenomena are modeled by nonclassical hyperbolic boundary value problems with nonlocal boundary conditions. Numerical solution of hyperbolic partial differential equations with an integral condition on the boundary still to be a major research area with widespread application in engineering, physic and technology.

We consider the following one-dimentional wave-equation with non classical boundary specification

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = q(x,t), 0 < x < 1, 0 < t \leq T \tag{273}$$

With the initial conditions

$$u(x,0) = r(x), 0 \leq x \leq 1 \tag{274}$$

$$u_t(x,0) = s(x), 0 \leq x \leq 1 \tag{275}$$

And the boundary conditions

$$u(0,t) = p(t), 0 < t \leq T \tag{276}$$

$\int_0^1 u(x,t)dx = q(t), 0 < t \leq T$

Where r, s, p and q are known functions, we suppose that f is suffeciently smooth to produce a smooth classical solution.

3.1 Adomian decomposition analysis

The Adomian decomposition method has been shown [36, 40, 46] to solve effectively, easily, and accrately a large classe of linear and non linear, ordinary, partial differantial equations with approximate solutions which converge rapidly to accurate solutions. In recent years, many papers were devoted to the problem of approximate solution of one-dimensional wave equation with non local boundary conditions [39]. The basic motivation of this work is to apply the Adomian decomposition method for solving the one-dimentional wave equation with a non local boundary condition. It is well known now in the literature that this algorithm provides the solution in rapidly convergent series. The implementation of the Adomian method in [17] and [40-43] has shown reliable results in that few terms only are needed to obtain accurate solutions.

References

[1] Consider equation (7,1)-(7,5) written in the form

$$L_{tt}(u(x,t)) = L_{xx}(u(x,t)) + q(x,t), 0 < x < 1, 0 < t \leq T \qquad (277)$$

Where the differential operators L_{xx} and L_{tt} are given as

$$L_{xx} = \frac{\partial^2}{\partial x^2}, \ L_{tt} = \frac{\partial^2}{\partial t^2}$$

The inverse operator L_{tt}^{-1} is therefore considered a two-fold integral operator defined by

$$L_{tt}^{-1}(.) = \int_0^t \int_0^t (.)dt \qquad (278)$$

Operating with L_{tt}^{-1} on equation (7,6), it then follows

$$L_{tt}^{-1}(L_{tt}(u(x,t)) = L_{tt}^{-1}(L_{xx}(u(x,t)) + L_{tt}^{-1}(q(x,t)) \qquad (279)$$

The result can be simplified as:

$$u(x,t) = r(x) + ts(x) + L_{tt}^{-1}(L_{xx}(u(x,t) + q(x,t))) \qquad (280)$$

The standard ADM defines the solution in the form of

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) \qquad (281)$$

Where the components u_k, $(k = 0, 1, 2, 3, ..)$. Are determined recursively by using the relation

$$u_0 = r(x) + ts(x) + L_{tt}^{-1}(q(x,t)) \qquad (282)$$

And

$$u_{k+1} = L_{tt}^{-1}[L_{xx}(u_k)], k \geq 0 \qquad (283)$$

If the series converges in a suitable way, then the general solution is obtained as

$$u(x,t) = lim_{n\to\infty} \sum_{k=0}^{n} u_k(x,t) \qquad (284)$$

3.2 Numerical Examples

References

[1] Example 1

We consider the following wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, 0 < x < 1, 0 < t < 0.5 \quad (285)$$

With the initial boundary conditions

$$u(x,0) = 0, 0 \leq x \leq 1 \quad (286)$$

$$u_t(x,0) = \pi cos(\pi x), 0 \leq x \leq 1 \quad (287)$$

And the boundary conditions

$$u(0,t) = p(t) = sin(\pi t), 0 < t < 0.5$$

$$\int_0^1 u(x,t)dt = q(t) = 0 \quad (288)$$

Substituting in the equations (7,11) and (7,12) we obtain the following equations

$$u_0 = t(\pi cos\pi x) \quad (289)$$

$$u_{k+1} = L_{tt}^{-1}[L_{xx}(u_k)], k \geq 0 \quad (290)$$

We can then proceed to compute the first few terms of the series (7,10)

$$u_0 = t(\pi cos\pi x) \quad (291)$$

$$u_1 = L_{tt}^{-1}[L_{xx}(u_0)] = \pi cos\pi x \int_0^t \int_0^t tdt = cos(\pi x)(-\pi^3 \frac{t^3}{3!}) \quad (292)$$

$$u_2 = L_{tt}^{-1}[L_{xx}(u_1)] = cos\pi x \int_0^t \int_0^t \pi^5 \frac{t^3}{3!} dt = cos\pi x(\pi^5 \frac{t^5}{5!}) \quad (293)$$

$$u_3 = L_{tt}^{-1}[L_{xx}(u_2)] = cos\pi x \int_0^t \int_0^t (-\pi^7 \frac{t^5}{5!}) dt = cos\pi x(-\pi^7 \frac{t^7}{7!}) \quad (294)$$

$$u_4 = L_{tt}^{-1}[L_{xx}(u_3)] = cos\pi x \int_0^t \int_0^t \pi^9 \frac{t^7}{7!} dt = cos\pi x(-\frac{t^9}{9!} \quad (295)$$

....

....

And so on

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + u_4 + ... \tag{296}$$

Hence

$$u(x,t) = cos\pi x[\pi t - \frac{(\pi t)^3}{3!} + \frac{(\pi t)^5}{5!} - \frac{(\pi t)^7}{7!} + \frac{(\pi t)^9}{9!} - ...] \tag{297}$$

Or

$$u(x,t) = cos(\pi x)sin(\pi t) \tag{298}$$

This result shows that, this method provides excellent approximation to the solution of this problem.
Example 2
Consider the following wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{4}\frac{\partial^2 u}{\partial x^2} = 0, 0 < x < 1, t > 0 \tag{299}$$

With the initial conditions

$$u(x,0) = x, 0 < x < 1 \tag{300}$$

$$u_t(x,0) = e^x, 0 < x < 1 \tag{301}$$

And the boundary conditions

$$u_x(0,t) = 2sinh(\frac{t}{2}), t > 0 \tag{302}$$

$$u_x(1,t) = 2e(\ sinh(\frac{t}{2}) + 1), t > 0 \tag{303}$$

It can be verified that the exact solution is

$$u(x,t) = 2e^x\ sinh(\frac{t}{2}) + x$$

Writing the problem (7,28) in an operator form yields

$$L_{tt}(u(x,t)) - \frac{1}{4}L_{xx}(u(x,t) = 0 \tag{304}$$

Operating with the inverse operator L_{tt}^{-1} on both sides of equation (7,33) and impose the initial conditions (7,29)-(7,30) we obtain

$$L_{tt}^{-1}[L_{tt}(u(x,s))] = u(x,t) - u(x,0) - tu_t(x,0)$$

$$u(x,t) = u(x,0) + tu_t(x,0) + L_{tt}^{-1}[L_{xx}(u(x,t))] \tag{305}$$

Starting with

$$u_0 = u(x,0) + tu_t(x,0) = x + te^x \tag{306}$$

And using

$$u_{k+1} = L_{tt}^{-1}[L_{xx}(u_k)], k \geq 0$$

We can obtain

$$u_1 = L_{tt}^{-1}[L_{xx}(u_0)] = 2e^x \int_0^t \int_0^t (\frac{t}{2})dt = 2e^x \frac{(\frac{t}{2})^3}{3!} \tag{307}$$

$$u_2 = L_{tt}^{-1}[L_{xx}(u_1)] = 2e^x \int_0^t \int_0^t \frac{(\frac{t}{2})^3}{3!} dt = 2e^x \frac{(\frac{t}{2})^5}{5!} \tag{308}$$

$$u_3 = L_{tt}^{-1}[L_{xx}(u_2)] = 2e^x \int_0^t \int_0^t \frac{(\frac{t}{2})^5}{5!} dt = 2e^x \frac{(\frac{t}{2})^7}{7!} \tag{309}$$

$$\cdots$$

By continuing the iteration, we find that

$$u_k = 2e^x \frac{(\frac{t}{2})^{2k+1}}{(2k+1)!} \tag{310}$$

Which implies that

$$u(x,t) = x + 2e^x \sum_{k=0}^{\infty} u_k(x,t) = x + 2e^x sinh(\frac{t}{2}) \tag{311}$$

Which converges to the exact solution.
Eample 3
Consider the following wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, 0 < x < 1, 0 < t \leq 0.5 \tag{312}$$

with the initial conditions

$$u(x,0) = cos\pi x, 0 < x < 1 \tag{313}$$

$$u_t(x,0) = 0, 0 < x < 1 \tag{314}$$

And the boundary conditions

$$u_x(0,t) = 0 \tag{315}$$

$$\int_0^1 u(x,t)dx = 0, 0 < t \leq 05 \tag{316}$$

Which easily seen to have the exact solution

$$u(x,t) = cos(\pi x)\ cos(\pi t)$$

Rewriting the equation (7,41) in an operator form

$$L_{tt}(u(x,t)) = L_{xx}(u(x,t)) \tag{317}$$

We operate with the inverse operator L_{tt}^{-1} on both sides of equation we get the following equations

$$L_{tt}^{-1}[L_{tt}(u(x,t)] = L_{tt}^{-1}[L_{xx}(u(x,t)] \tag{318}$$

$$L_{tt}^{-1}[L_{tt}(u(x,t)] = \int_0^t dx \int_0^t \frac{\partial^2 u}{\partial t^2} dx = u(x,t) - u(x,0) - tu_t(x,0) \tag{319}$$

Then

$$u(x,t) - u(x,0) - tu_t(x,0) = L_{tt}^{-1}[L_{xx}(u(x,t)] \tag{320}$$

Or

$$u(x,t) = u(x,0) + tu_t(x,0) + L_{tt}^{-1}[L_{xx}(u(x,t))] \tag{321}$$

From equation (7,10), (7,11) and (7,12) we can get the different terms of the approximate series solution

$$u_0 = u(x;0) + tu_t(x,0) = cos\pi x \tag{322}$$

$$u_1 = L_{tt}^{-1}[L_{xx}(u_0)] = -\pi^2 cos(\pi x) \int_0^t \int_0^t dt = cos(\pi x)(-\frac{(\pi t)^2}{2!}) \tag{323}$$

$$u_2 = L_{tt}^{-1}[L_{xx}(u_1))] = cos(\pi x) \int_0^t \int_0^t \frac{(\pi t)^2}{2!} dt = cos(\pi x)(\frac{(\pi t)^4}{4!}) \tag{324}$$

$$u_3 = L_{tt}^{-1}[L_{xx}(u_2)] = cos(\pi x) \int_0^t \int_0^t (-\frac{(\pi t)^4}{4!}) dt = cos(\pi x)(-\frac{(\pi t)^6}{6!}) \tag{325}$$

...

$$u_k = cos(\pi x)(-1)^k \frac{(\pi t)^{2k}}{(2k)!} \tag{326}$$

Hence, the approximate series solution is given by

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + ... + u_k + .. \tag{327}$$

Or

$$u(x,t) = cos(\pi x)(1 - \frac{(\pi t)^2}{2!} + \frac{(\pi t)^4}{4!} - \frac{(\pi t)^6}{6!} + .. + ((-1)^k \frac{(\pi t)^{2k}}{(2k)!} + ..) \tag{328}$$

Which convrges to the exact solution

$$u(x,t) = cos(\pi x)cos(\pi t) \tag{329}$$

3.3 *Conclusion*

In this chapter, we employed new thechnique for the solution of some hyperbolic equations with nonlocal boundary conditions(wave equation). this technique is reliable , efficient, accurate and gives us the solution in a closed form.

$Example\ 1$

$$h_x = \frac{1}{10},\ h_t = \frac{1}{250}$$

x_i	u_{ex}	u_{Ad} $5-iterate$	$\lvert u_{ex}-u_{Ad}\rvert$
0.0	1.2566×10^{-2}	1.2566×10^{-2}	0.0
0.1	1.1951×10^{-2}	1.1951×10^{-2}	0.0
0.2	1.0166×10^{-2}	1.0166×10^{-2}	0.0
0.3	7.3851×10^{-3}	7.3851×10^{-3}	0.0
0.4	3.8831×10^{-3}	3.8831×10^{-3}	0.0
0.5	0.0	0.0	0.0
0.6	-3.8831×10^{-3}	-3.8831×10^{-3}	0.0
0.7	-7.3861×10^{-3}	-7.3861×10^{-3}	0.0
0.8	-1.0166×10^{-2}	-1.0166×10^{-2}	0.0
0.9	-1.1951×10^{-2}	-1.1951×10^{-2}	0.0
1.0	-1.2566×10^{-2}	-1.2566×10^{-2}	0.0

Example 2

$$h_x = \frac{1}{10},\ h_t = \frac{1}{250}$$

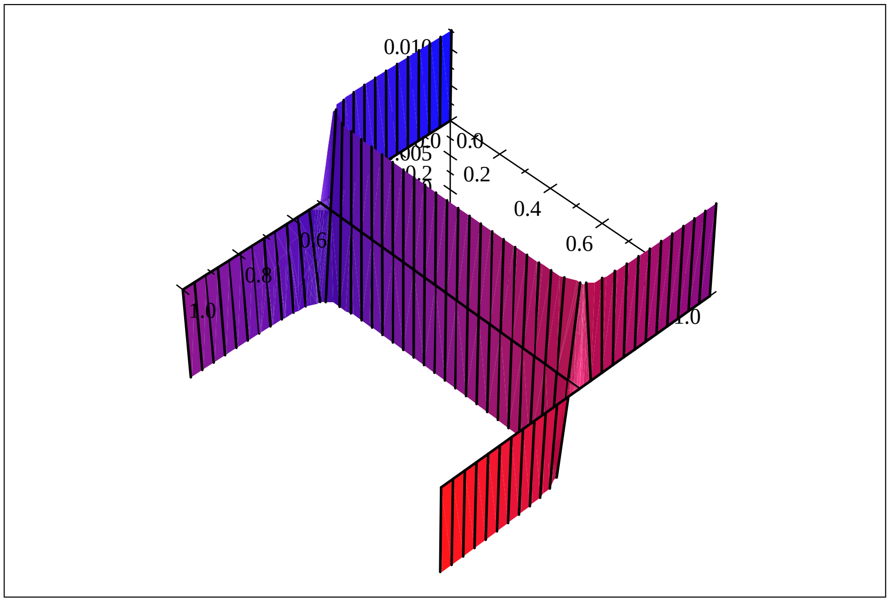
x_i	u_{ex}	u_{Ad} $5-iterate$	$\|u_{ex-u_{Ad}}\|$
0.0	0.004	0.004	0.0
0.1	0.10442	0.10442	0.0
0.2	0.20489	0.20489	0.0
0.3	0.30540	0.30540	0.0
0.4	0.40597	0.40597	0.0
0.5	0.50659	0.50659	0.0
0.6	0.60729	0.60729	0.0
0.7	0.70806	0.70806	0.0
0.8	0.8089	0.8089	0.0
0.9	0.90984	0.90984	0.0
1.0	1.0109	1.0109	0.0

Example 3

$$h_x = \frac{1}{10},\ h_t = \frac{1}{250}$$

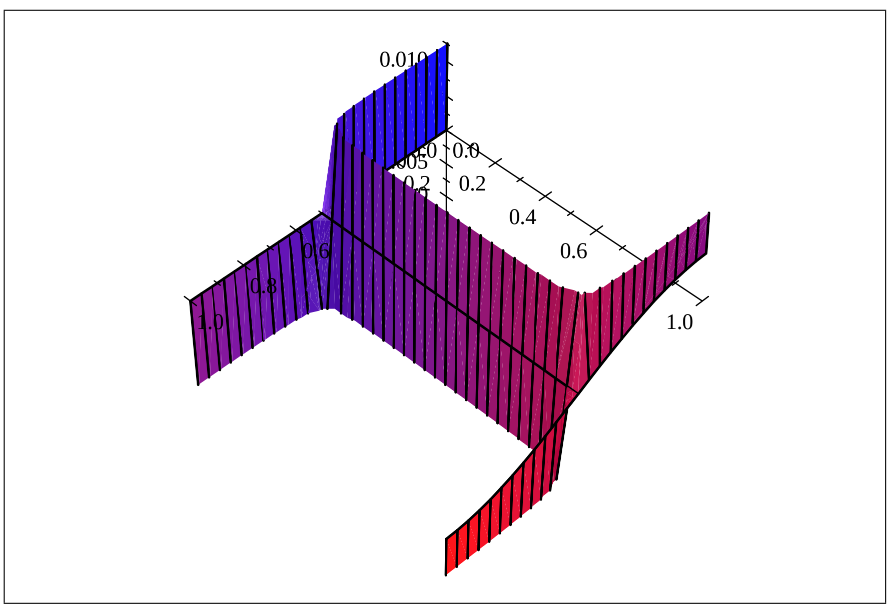
x_i	u_{ex}	u_{Ad}	$5-iterate$	$\lvert u_{ex}-u_{Ad}\rvert$
0.0	0.99992	0.99992		0.0
0.1	0.95098	0.95098		0.0
0.2	0.80895	0.80895		0.0
0.3	0.58774	0.58774		0.0
0.4	0.30899	0.30899		0.0
0.5	0.0	0.0		0.0
0.6	− 0.30899	− 0.30899		0.0
0.7	− 0.58774	− 0.58774		0.0
0.8	− 0.80895	− 0.80895		0.0
0.9	− 0.95098	− 0.95098		0.0
1.0	− 0.99992	− 0.99992		0.0

$u_{ex} = \cos(\pi x) \times \sin(\pi t)$



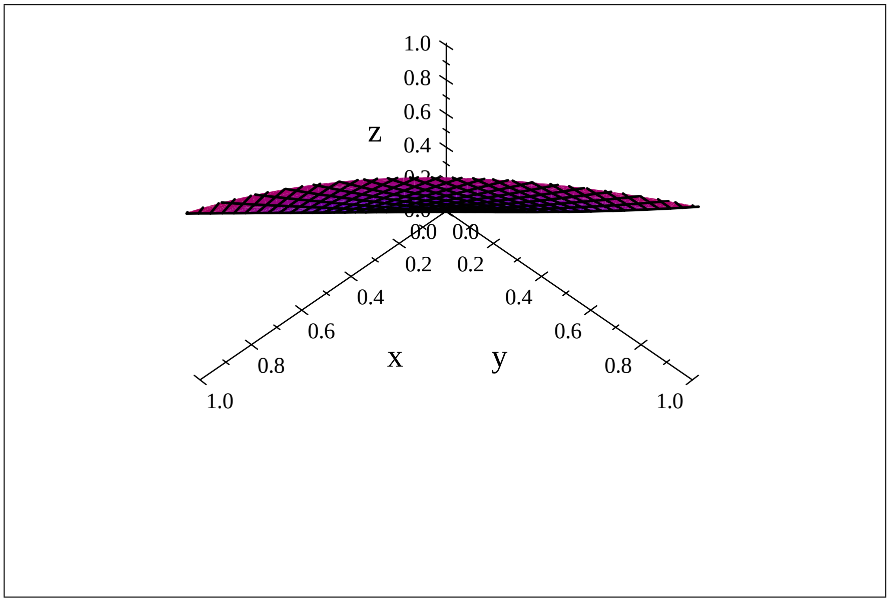
Variation of $u_{ex} = \cos(\pi x)\cos(\pi t)$ for different values of x and t

$u_{Ad} = \cos(\pi x) \times (\pi t - \frac{(\pi t)^3}{3!} + \frac{(\pi t)^5}{5!} - \frac{(\pi t)^7}{7!} + \frac{(\pi t)^9}{9!})$



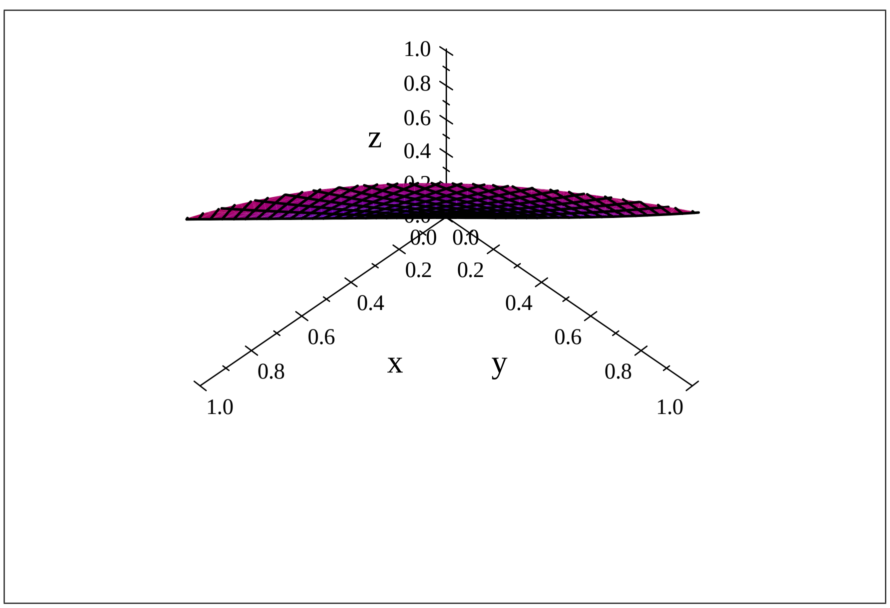
Variation of u_{Ad} for different values of x and t

$u_{ex} = x + 2\exp(x) \times \sinh(\frac{t}{2})$



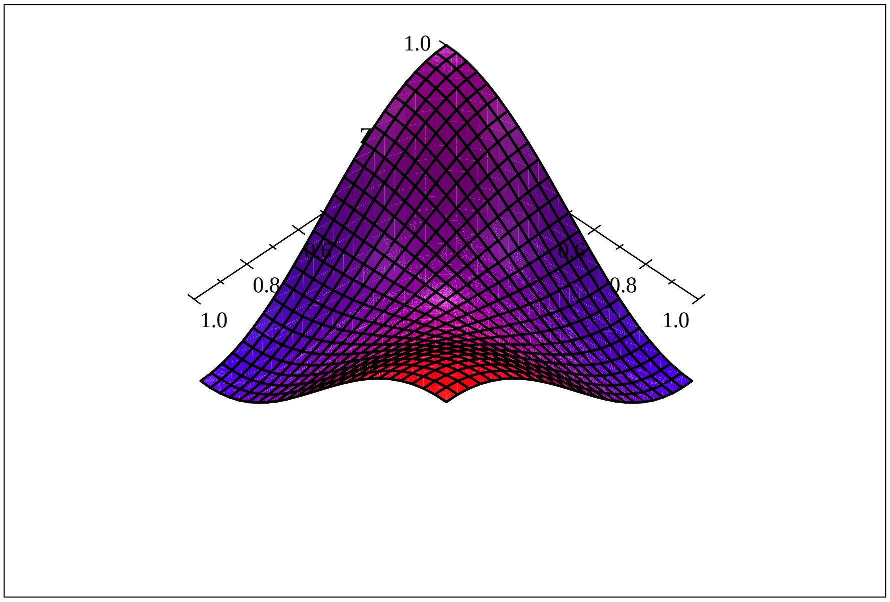
Variation of $u_{ex} = x + 2\exp(x)\sinh(\frac{t}{2})$ for different values of x and t

$u_{Ad} = x + 2\exp(x)(\frac{t}{2} + \frac{1}{3!}(\frac{t}{2})^3 + \frac{1}{5!}(\frac{t}{2})^5 + \frac{1}{7!}(\frac{t}{2})^7 + \frac{1}{9!}(\frac{t}{2})^9)$



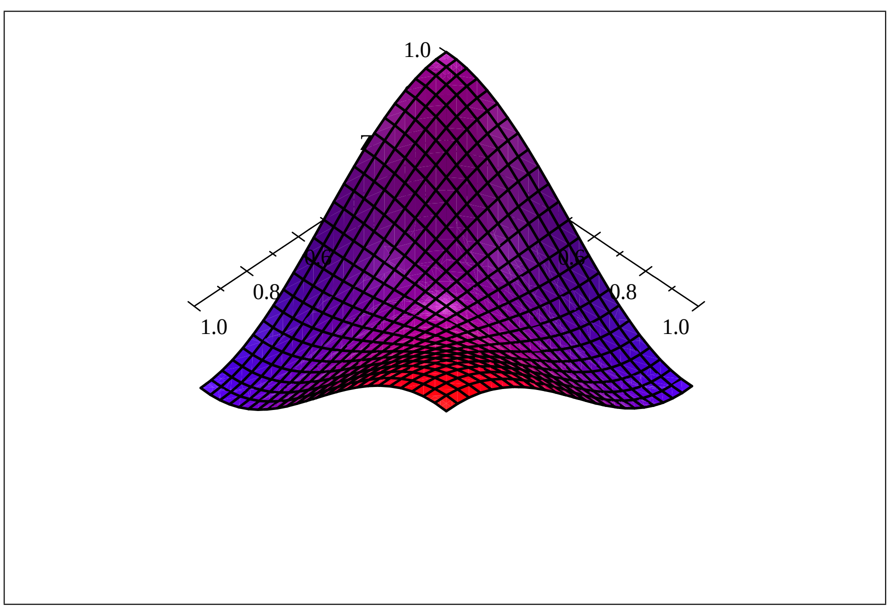
Variation of $u_{Ad} = x + 2\exp(x)(\frac{t}{2} + \frac{1}{3!}(\frac{t}{2})^3 + \frac{1}{5!}(\frac{t}{2})^5 + \frac{1}{7!}(\frac{t}{2})^7 + \frac{1}{9!}(\frac{t}{2})^9$ for different values of x and t

$u_{ex} = \cos(\pi x)\cos(\pi t)$



Variation of $u_{ex} = \cos(\pi x)\cos(\pi t)$ for different values of x and t

$u_{Ad} = \cos(\pi x)(1 - \frac{(\pi t)^2}{2!} + \frac{(\pi t)^4}{4!} - \frac{(\pi t)^6}{6!} + \frac{(\pi t)^8}{8!})$



Variation of $u_{Ad} = \cos(\pi x)(1 - \frac{(\pi t)^2}{2!} + \frac{(\pi t)^4}{4!} - \frac{(\pi t)^6}{6!} + \frac{(\pi t)^8}{8!})$ for different values of x and t

3.4 The homotopy perturbation method for solving nonlocal problems

In the last two decades wih the rapid development of differential equations. There has appeared ever-increasing interest of scientists and engineers in the analytical techniques for linear and nonlinear problems with nonlocal boundary conditions. The widely applied techniques are perturbation methods . J.He [29] has proposed a new perturbation technique coupled with the homotopy technique, which is called the homotopy perturbation method (HPM). In contrast to the traditional perturbation methods. HPM does not depend upon a small parameter in the equation. By the homotopy technique in topology. A homotopy is constructed with an embeding parameter $p \in [0,1]$, which is considered as a small parameter. HPM has been used in numerous works. He [23], has obtained a solution to Blasius equation by the HPM. He [15], applied HPM to solve boundary value problems which is governed by the nonlinear ordinary(Partial) differential equation. Furthermore, HPM is applied to solve the Helmhotz equation. And the results show that this method is effecient and simple. Thus, the main goal of this work is to apply the homotopy perturbation method (HPM) for solving linear or nonlinear Initial boundary value problems with nonlocal boundary conditions. The general form of equation is given as follows

$$\frac{\partial u}{\partial t} - G(x,t,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}) = 0, a < x < b, 0 < t < T \tag{330}$$

Subject to the initial condition

$$u(x;0) = f(x), 0 < t < T \tag{331}$$

And nonlocal boundary conditions

$$u(a,t) = \int_a^b \varphi(x,t)u(x,t)dx + g_0(t), 0 < t < T \tag{332}$$

$$u(b,t) = \int_a^b \psi(x,t)u(x,t)dx + g_1(t), 0 < t < T \tag{333}$$

Where f, φ, ψ, g_0 and g_1 are known functions and G is continuous function.

3.5 Analysis of homotopy perturbation method

To ullistrate the basic ideas, let X, and Y be the topological spaces. If f and g are continuous maps of the spaces X into Y, it is said that f is homotopic to g,if there is continuous map $F: X \times [0,1] \rightarrow Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$, for each $x \in X$, then the map is called homotopy between f and g. We consider the following nonlinear partial differential equation

$$A(u) - f(r) = 0, r \in \Omega \tag{334}$$

Subject to the boundary conditions

$$B(u, \frac{\partial u}{\partial \eta}) = 0, r \in \Gamma \tag{335}$$

Where A is a general differential operator. f is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial \eta}$ denotes directional derivative in outward normal direction to Ω. The operator A, generally divided into two parts, L and N, where L is linear , while N is nonlinear. using $A = L + N$, eq(8,5) can be rewritten as follws

$$L(u) + N(u) - f(r) = 0 \tag{336}$$

By the homotpy technique, we construct a homotopy defined as

$$H(u,p) : \Omega \times [0,1] \rightarrow R \tag{337}$$

Which satisfies

$$H(u,p) = (1-p)[L(u) - L(u_0)] + p[A(u) - f(r)], p \in [0,1], r \in \Omega \tag{338}$$

Or

$$H(u,p) = L(u) - L(u_0) + pL(u_0) + p[N(u) - f(r)] = 0, p \in [0,1], r \in \Omega \tag{339}$$

Where $p \in [0,1]$ is an embedding parameter , u_0 is an initial approximation of equation (8,5), which satisfies the boundary conditions. It follows from the equation (8,10) that

$$H(u,0) = L(u) - L(u_0) = 0 \tag{340}$$

$$H(u,1) = A(u) - f(r) = 0 \tag{341}$$

The changing process of p from 0 to 1 monotonically as a trivial problem. $H(u,0) = L(u) - L(u_0) = 0$ is continuously transformed to the original problem

$$H(u,1) = A(u) - f(r) = 0 \tag{342}$$

In topology, this process is known as continuous deformation . $L(u)-L(u_0)$ and $A(u)-f(r)$ are called homotopic. We use the embedding parameter p as a small parameter , and assume that the solution of equation (8,10) can be written as a power series of p :

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + ... + p^nu_n + ... \tag{343}$$

Setting $p = 1$ we obtain the approximate solution of equation (8,5) as

$$u = lim_{p\to 1}\ u = u_0 + u_1 + u_2 + u_3 + ... + u_n + ... \tag{344}$$

The series of equation (8,15) is convergent for most of the cases. However, the rate of the convergence depends on the nonlinear operator $N(u)$. The following suggestions have already been made by He (1999):

- The second derivative of $N(u)$ with respect to u should be small because the parameter may be relatively large i.e $p \to 1$ and the norm of $L^{-1}(\frac{\partial N}{\partial u})$ must be smaller than one in order that the series converges.

3.6 Numerical examples

Example 1

Consider the equation (8,1) with the following data

$$G(x,t,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \tag{345}$$

$$u(x,0) = e^x - x \tag{346}$$

$$u(0,1) = \int_0^1 u(x,t)dx = e + t, a = 0, b = 1, \varphi(x,t) = 1, g_0 = \frac{3}{2} \tag{347}$$

$$u(1,t) = \int_0^1 \frac{1}{2}u(x,t)dt = \frac{1}{2}(e+t), a = 0, b = 1, \psi(x,t) = \frac{1}{2}, g_1 = \frac{3}{4} \tag{348}$$

We solve the problem using the homotopy perturbation method (HPM), we construct the homotopy as the following

$$H(u,p) = (1-p)(\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t}) + p(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}) = 0 \tag{349}$$

Suppose that the solution of (8,1) with (8,16)-(8,19) is given in the following form

$$u(x,t) = u_0(x,t) + pu(x,t) + p^2u(x,t) + p^3u(x,t) + ... \tag{350}$$

Substituting the equation (8,1) into equation (8,20) and equating the coefficients of like powers of p, we get

$$\frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, u_0 = u(x,0) = e^x - x \tag{351}$$

$$\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial u_0}{\partial x} = 0, u_1(x,0) = 0, u_1 = t \tag{352}$$

$$\frac{\partial u_2}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial u_1}{\partial x} = 0, u_2(x,0) = 0 \tag{353}$$

From equation (8,17) and (8,20) we can get

$$u_0 = e^x - x, u_1 = t, u_2 = 0, u_3 = 0, ..., u_n = 0, ... \tag{354}$$

Supposing that $p \to 1$ we obtain the solution of the problem (8,1),(8,16)-(8,19) as follows

$$u = e^x - x + t$$

Which coincides with the exact solution.

Example 2

We consider the problem

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + 2(t-x), 0 < x < 1, 0 < t < T \tag{355}$$

With the initial condition

$$u(x,0) = x^2, \frac{\partial u}{\partial t}(x,0) = 0, 0 < x < 1, 0 < t < T \tag{356}$$

And the boundary conditions

$$u(0,t) = \int_0^1 \varphi(x,t)u(x,t)dt + g_0(t) = 1 + \frac{1}{4}t^2, where\ , \varphi(x,t) = \frac{1}{4}, g_0 = \frac{11}{12}, \tag{357}$$

$$u(1,t) = \int_0^1 \psi(x,t)u(x,t)dt + g_1(t) = 1 + \frac{1}{6}t^2,\ where, \psi(x,t) = \frac{1}{6},\ g_1 = \frac{17}{18} \tag{358}$$

From (8,20) and (8,21) we can get

$$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = 0, u_0 = u(x,0) = x^2 \tag{359}$$

$$\frac{\partial u_1}{\partial t} + \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial u_0}{\partial x} - \frac{\partial^2 u_0}{\partial x^2} - 2(t-x) = 0, u_1(x,0) = 0, \tag{360}$$

$$\frac{\partial u_0}{\partial x} = 2x,\ \frac{\partial^2 u_0}{\partial x^2} = 2,\ \frac{\partial^2 u_0}{\partial t^2} = 0$$

$$\frac{\partial u_1}{\partial t} = 2 + 2t$$

Hence, we obtain

$$u_1 = 2t + t^2 \tag{361}$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial u_1}{\partial x} = 0, u_2(x,0) = 0 \tag{362}$$

$$\frac{\partial^2 u_1}{\partial t^2} = 2,\ \frac{\partial u_1}{\partial x} = \frac{\partial^2 u_1}{\partial x^2} = 0$$

$$\frac{\partial u_2}{\partial t} = -2$$

Then, we have

$$u_2 = -2t \tag{363}$$

$$\frac{\partial u_3}{\partial t} + \frac{\partial^2 u_2}{\partial t^2} - \frac{\partial u_2}{\partial x} - \frac{\partial^2 u_2}{\partial x^2} = 0,\ u_3(x,0) = 0 \tag{364}$$

$$\frac{\partial^2 u_2}{\partial t^2} = 0,\ \frac{\partial u_2}{\partial x} = \frac{\partial^2 u_2}{\partial x^2} = 0,\ \frac{\partial u_3}{\partial t} = 0$$

And so on, we obtain the approximate solution as follows

$$u(x,t) = u_0 + u_1 + u_2 + ..$$

Or

$$u(x,t) = x^2 + 2t + t^2 - 2t = x^2 + t^2 \tag{365}$$

Which is the exact solution of equation (8,26)-(8,29).

Example 3

Consider the following nonlinear reaction-diffusion equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u^2 - (\frac{\partial u}{\partial x})^2,\ 0 < x < 1,\ 0 < t \leq T \tag{366}$$

Subject to the initial condition

$$u(x,0) = e^x,\ 0 < x < 1 \tag{367}$$

And the nonlocal boundary conditions

$$u(0,t) = \int_0^1 \varphi(x,t)u(x,t)dx + g_0(t) = e^{1+t},\ \varphi = 1,\ g_0 = e^t \tag{368}$$

$$u(1,t)=\int_0^1 \psi(x,t)u(x,t)dx+g_1(t)=\frac{1}{2}e^{1+t},\ \psi=\frac{1}{2},\ g_1=\frac{1}{2}e^t \quad (369)$$

Solving the equation (8,37) with the initial condition (8,38), yields

$$u_0=e^x$$

$$u_1=te^x$$

$$u_2=\frac{t^2}{2!}e^x$$

$$u_3=\frac{t^3}{3!}e^x$$

$$.$$

$$u_n=\frac{t^n}{n!}e^x \quad (370)$$

Substituting equation (8,41) into equation (8,15)nyields

$$u(x,t)=u_0+u_1+u_2+u_3+...+u_n+..$$

Or

$$u(x,t)=e^x(1+\frac{t}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+...+\frac{t^n}{n!}+..)$$

Finally we obtain the approximate solution

$$u(x,t)=e^{x+t} \quad (371)$$

Which coincides with the exact solution .

Example 4
We consider the problem

$$\frac{\partial u}{\partial t}=\frac{1}{2}(x^2\frac{\partial^2 u}{\partial x^2}),0<x<1,0<t\leq T \quad (372)$$

Subject to the initial condition

$$u(x,0)=x^2 \quad (373)$$

And the boundary conditions

$$u(0,t)=\int_0^1 u(x,t)dx+g_1=\frac{1}{3}e^t,\ g_1=0 \quad (374)$$

$$u(1,t)=\int_0^1 u(x,t)dx+g_2=\frac{1}{3}(e^t+1),\ g_2=\frac{1}{3} \quad (375)$$

According to equations (8,10)-(8,15) and equation (8,44) the following terms are calculated successively

$$\frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, u_0 = x^2 \tag{376}$$

$$\frac{\partial u_1}{\partial t} - \frac{1}{2}x^2\frac{\partial^2 u_0}{\partial x^2} = 0 \tag{377}$$

$$\frac{\partial u_1}{\partial t} = x^2 \tag{378}$$

$$u_1 = x^2 t \tag{379}$$

$$\frac{\partial u_2}{\partial t} - \frac{1}{2}x^2\frac{\partial^2 u_1}{\partial x^2} = 0, \frac{\partial u_2}{\partial t} = x^2 t \tag{380}$$

$$u_2 = x^2\frac{t^2}{2} \tag{381}$$

$$\frac{\partial u_3}{\partial t} - \frac{1}{2}x^2\frac{\partial^2 u_2}{\partial x^2} = 0, \frac{\partial u_3}{\partial t} = x^2\frac{t^2}{2!}$$

$$u_3 = x^2\frac{t^3}{3!} \tag{382}$$

$$.$$

$$u_n = x^2\frac{t^n}{n!}$$

And so on. To obtain the solution in the series form

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + ... + u_n + ...$$

Or

$$u(x,t) = x^2(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + ... + \frac{t^n}{n!} + ...)$$

And in a closed form given by

$$u(x,t) = x^2 e^t \tag{383}$$

Which is the exact solution.

Example 5

Consider the problem

$$\frac{\partial u}{\partial t} = \frac{1}{4}(x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2}), 0 < x, y < 1, 0 < t \le T \tag{384}$$

With the following initial condition

$$u(x,y,0) = x^2y^2 \tag{385}$$

And the boundary conditions

$$u(0,y,t) = \int_0^1\int_0^1 u(x,y,t)dxdy + g_1 = \frac{1}{9}e^t + \frac{1}{6},\ g_1 = \frac{1}{6}$$

$$u(1,y,t) = \int_0^1\int_0^1 u(x,y,t)dxdy + g_2 = \frac{1}{9}e^t + \frac{1}{3},\ g_2 = \frac{1}{3}$$

$$u(x,0,t) = \int_0^1\int_0^1 u(x,y,t)dxdy + g_3 = \frac{1}{9}e^t + \frac{1}{4},\ g_3 = \frac{1}{4}$$

$$u(x,1,t) = \int_0^1\int_0^1 u(x,y,t)dxdy = \frac{1}{9}e^t \tag{386}$$

Using (8,14),(8,15) for (8,5) we have

$$\frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, u_0 = x^2y^2 \tag{387}$$

$$\frac{\partial u_1}{\partial t} - \frac{1}{4}(x^2\frac{\partial^2 u_0}{\partial x^2} + y^2\frac{\partial^2 u_0}{\partial y^2}) = 0$$

$$\frac{\partial u_1}{\partial t} = x^2y^2$$

$$u_1 = x^2y^2t$$

$$\frac{\partial u_2}{\partial t} - \frac{1}{4}(x^2\frac{\partial^2 u_1}{\partial x^2} + y^2\frac{\partial^2 u_1}{\partial y^2}) = 0$$

$$\frac{\partial u_2}{\partial t} = (x^2y^2)t$$

$$u_2 = (x^2y^2)\frac{t^2}{2!}$$

$$\frac{\partial u_3}{\partial t} - \frac{1}{4}(x^2\frac{\partial^2 u_2}{\partial x^2} + y^2\frac{\partial^2 u_2}{\partial y^2}) = 0$$

$$\frac{\partial u_3}{\partial t} = x^2y^2\frac{t^2}{2!}$$

$$u_3 = x^2y^2\frac{t^3}{3!}$$

.

$$\frac{\partial u_n}{\partial t} - \frac{1}{4}(x^2\frac{\partial^2 u_{n-1}}{\partial x^2} + y^2\frac{\partial^2 u_{n-1}}{\partial y^2}) = 0$$

$$\frac{\partial u_n}{\partial t} = x^2y^2\frac{t^{n-1}}{(n-1)!}$$

$$u_n = x^2y^2\frac{t^n}{n!}$$

The solution in the series form is given by

$$u(x,y,t) = x^2y^2(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + ... + \frac{t^n}{n!} + ...) \tag{388}$$

And in a closed form by

$$u(x,y,t) = x^2y^2e^t \tag{389}$$

This solution coincides with the exact one.

Example 6
Consider the problem

$$\frac{\partial u}{\partial t} = \frac{1}{6}(x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}), 0 < x,y,z < 1, 0 < t \leq T \tag{390}$$

Subject to the initial condition

$$u(x,y,z,0) = x^2y^2z^2 \tag{391}$$

And the boundary conditions

$$u(0,y,z,t) = \int_0^1\int_0^1\int_0^1 u(x,y,z,t)dxdydz + g_1 = \frac{1}{27}e^t,\ g_1 = 0$$

$$u(1,y,z,t) = \int_0^1\int_0^1\int_0^1 u(x,y,z,t)dxdydz + g_2 = \frac{1}{27}e^t + \frac{1}{2}t,\ g_2 = \frac{1}{2}t$$

$$u(x,0,z,t) = \int_0^1\int_0^1\int_0^1 u(x,y,z,t)dxdydz + g_3 = \frac{1}{27}(e^t + 1),\ g_3 = \frac{1}{27}$$

$$u(x,1,z,t) = \int_0^1\int_0^1\int_0^1 u(x,y,z,t)dxdydz + g_4 = \frac{1}{27}(e^t + 3),\ g_4 = \frac{1}{9}$$

$$u(x,y,0,t) = \int_0^1\int_0^1\int_0^1 u(x,y,z,t)dxdydz + g_5 = \frac{1}{27}e^t + \frac{1}{6},\ g_5 = \frac{1}{6}$$

$$u(x,y,1,t)=\int_0^1\int_0^1\int_0^1 u(x,y,z,t)dxdydz+g_6=\frac{1}{27}e^t+\frac{1}{5}t,\ \ g_6=\frac{1}{5}t \tag{392}$$

From equations (8,10)-(8,15) we get the following equations

$$\frac{\partial u_0}{\partial t}-\frac{\partial u_0}{\partial t}=0, u_0=x^2y^2z^2 \tag{393}$$

$$\frac{\partial u_1}{\partial t}-\frac{1}{6}(x^2\frac{\partial^2 u_0}{\partial x^2}+y^2\frac{\partial^2 u_0}{\partial y^2}+z^2\frac{\partial^2 u_0}{\partial z^2})=0$$

$$\frac{\partial u_1}{\partial t}=\frac{1}{6}(2x^2y^2z^2+2x^2y^2z^2+2x^2y^2z^2)=x^2y^2z^2$$

$$u_1=x^2y^2z^2\frac{t}{1!}$$

$$\frac{\partial u_2}{\partial t}-\frac{1}{6}(x^2\frac{\partial^2 u_1}{\partial x^2}+y^2\frac{\partial u_1}{\partial y^2}+z^2\frac{\partial u_1}{\partial z^2})=0$$

$$\frac{\partial u_2}{\partial t}=\frac{1}{6}(2x^2y^2z^2+2x^2y^2z^2+2x^2y^2z^2)t=x^2y^2z^2t$$

$$u_2=x^2y^2z^2\frac{t^2}{2!}$$

$$\frac{\partial u_3}{\partial t}-\frac{1}{6}(x^2\frac{\partial^2 u_2}{\partial x^2}+y^2\frac{\partial^2 u_2}{\partial y^2}+z^2\frac{\partial^2 u_2}{\partial z^2})=0$$

$$\frac{\partial u_3}{\partial t}=\frac{1}{6}(2x^2y^2z^2+2x^2y^2z^2+2x^2y^2z^2)\frac{t^2}{2!}=x^2y^2z^2\frac{t^2}{2!}$$

$$u_3=x^2y^2z^2\frac{t^3}{3!}$$

.

$$\frac{\partial u_n}{\partial t}-\frac{1}{6}(x^2\frac{\partial^2 u_{n-1}}{\partial x^2}+y^2\frac{\partial^2 u_{n-1}}{\partial y^2}+z^2\frac{\partial^2 u_{n-1}}{\partial z^2})=0$$

$$\frac{\partial u_n}{\partial t}=\frac{1}{6}(x^2\frac{\partial^2 u_{n-1}}{\partial x^2}+y^2\frac{\partial^2 u_{n-1}}{\partial y^2}+z^2\frac{\partial^2 u_{n-1}}{\partial z^2})\frac{t^{n-1}}{(n-1)!}=x^2y^2z^2\frac{t^{n-1}}{(n-1)!}$$

$$u_n=x^2y^2z^2\frac{t^n}{n!} \tag{394}$$

Hece, the approximate solution is given by

$$u(x,y,z,t)=u_0+u_1+u_2+u_3+..+u_n+..$$

Or

$$u(x,y,z,t)=x^2y^2z^2(1+\frac{t}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+..+\frac{t^n}{n!}+..)$$

Therefore we obtain

$$u(x,y,z,t) = x^2y^2z^2e^t \tag{395}$$

So, the exact solution of the problem is obtained.

Example 7

consider the problem

$$u_{tt} = (u^{-1}u_x)_x, 0 < x < 1, 0 < t \leq T \tag{396}$$

With the initial condition

$$u(x,0) = \frac{1}{(1+x)^2}, u_t(x,0) = 0 \tag{397}$$

According to the HPM, we have

$$H(u,p) = (1-p)(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2}) + p(\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}(u^{-1}\frac{\partial u}{\partial x})) \tag{398}$$

$$= 0 \tag{399}$$

By equating the terms with the identical powers of p , yields

$$p^0: \ \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = 0, \frac{\partial^2 u_0}{\partial t^2} = 0, u_0 = \frac{1}{(1+x)^2} \tag{400}$$

$$p^1: \ \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial}{\partial x}(u_0^{-1}\frac{\partial u_0}{\partial x}) = 0$$

$$\frac{\partial^2 u_1}{\partial t^2} = \frac{2}{(1+x)^2}$$

$$u_1 = \frac{2}{(1+x)^2}\frac{t^2}{2!}$$

$$p^2: \ \frac{\partial^2 u_2}{\partial t^2} - \frac{\partial}{\partial x}(u_1^{-1}\frac{\partial u_1}{\partial x}) = 0$$

$$\frac{\partial^2 u_2}{\partial t^2} = \frac{2}{(1+x)^2}$$

$$u_2 = u_1$$

Then,

$$u = u_0 + u_1$$

Or

$$u(x,t) = \frac{1+t^2}{(1+x)^2} \tag{401}$$

Which is the exact solution.

Example 8

Consider the following problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}(u^{-2}\frac{\partial u}{\partial x}) \tag{402}$$

Subject to the initial condition

$$u(x,0) = \frac{1}{2+x}, \frac{\partial u}{\partial t}(x,0) = \frac{-t}{2+x} \tag{403}$$

We suggest $u_0(x,t) = \frac{1-t}{2+x}$. doing as above we can get the following equations

$$\frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = 0, u_0 = \frac{1-t}{2+x} \tag{404}$$

$$\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial}{\partial x}(u_0^{-2}\frac{\partial u_0}{\partial x}) = 0$$

$$\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial}{\partial x}(u^{-2}\frac{\partial u_0}{\partial x}) = 0$$

$$u_1 = 0$$

Then, we obtain the solution

$$u(x,t) = \frac{1-t}{2+x} \tag{405}$$

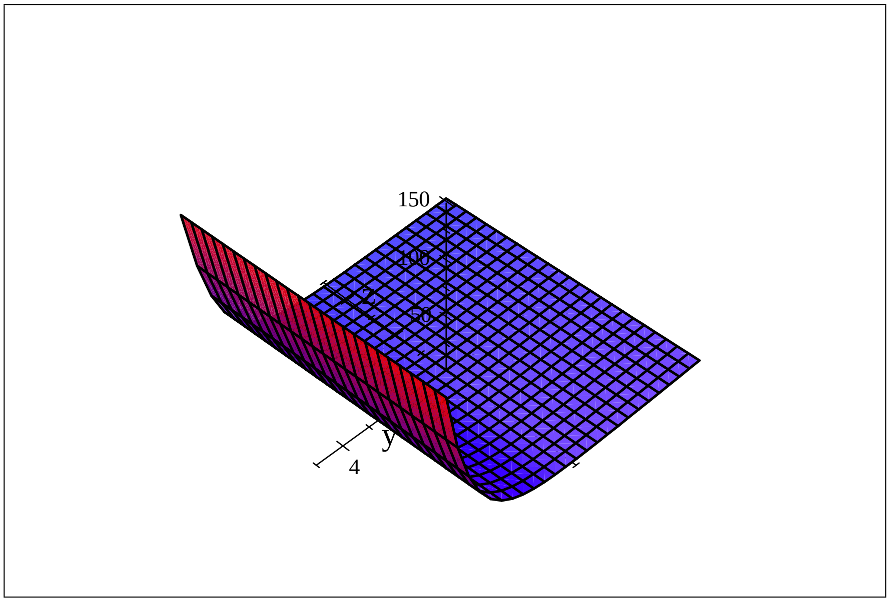
Which is the exact solution.

Example 1

$$h_x = \frac{1}{10},\ h_t = \frac{1}{250}$$

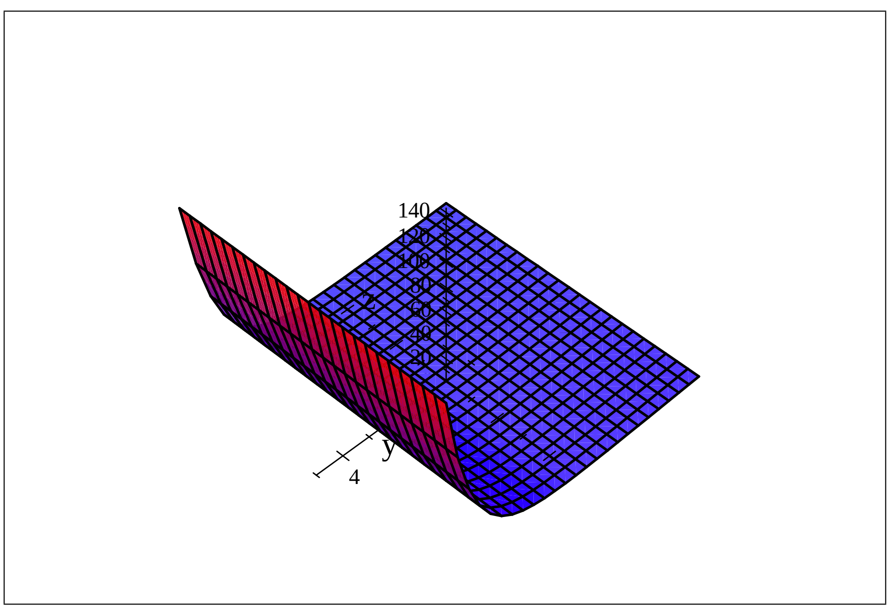
x_i	u_{ex}	u_{hpm}	$1 - iterate$	$\lvert u_{ex} - u_{hpm} \rvert$
0.0	1.004	1.0		0.004
0.1	1.0092	1.0052		0.004
0.2	1.0254	1.0214		0.004
0.3	1.0539	1.0499		0.004
0.4	1.0958	1.0918		0.004
0.5	1.1527	1.1487		0.004
0.6	1.2261	1.2221		0.004
0.7	1.3178	1.3138		0.004
0.8	1.4295	1.4255		0.004
0.9	1.5636	1.5596		0.004
1.0	1.7223	1.7183		0.004

$u_{ex} = \exp(x) - x + t$



Variation of $u_{ex} = \exp(x) - x + t$ for different values of x and t

$u_{hpm} = \exp(x) - x$



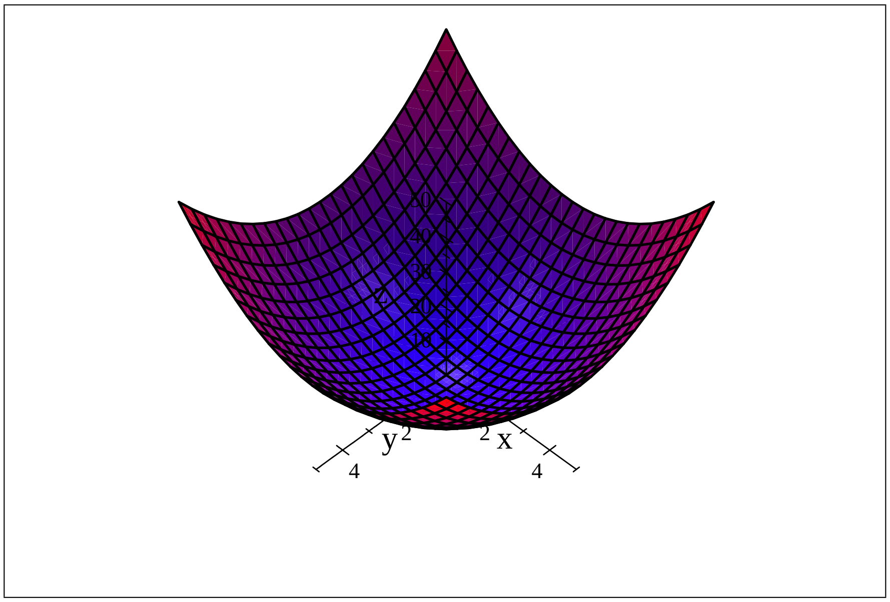
Variation of $u_{hpm} = \exp(x) - x$ for different values of x

Example 2

$$h_x = \frac{1}{10},\ h_t = \frac{1}{250}$$

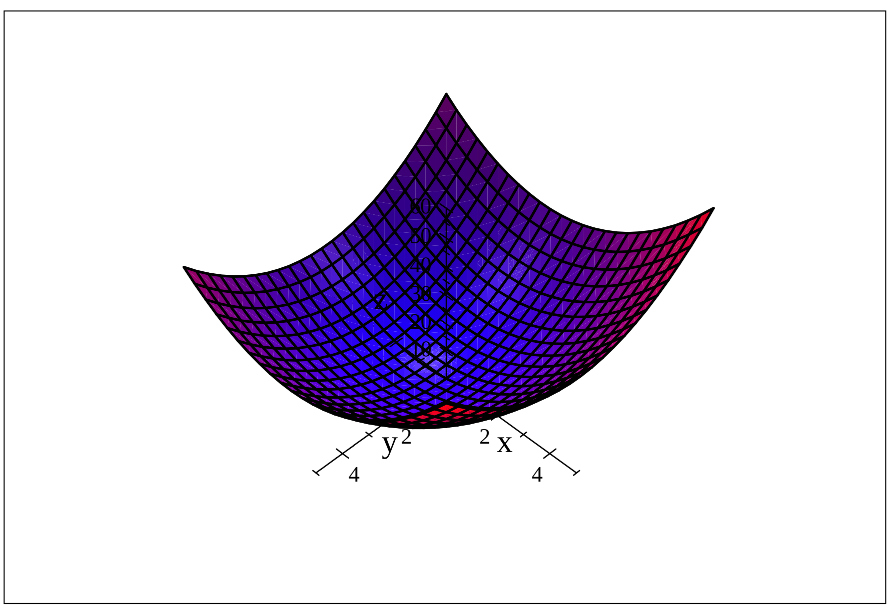
x_i	u_{ex}	u_{hpm} $2-iterates$	$\lvert u_{ex} - u_{hpm}\rvert$
0.0	1.6×10^{-5}	8.016×10^{-3}	0.008
0.1	1.0016×10^{-2}	1.8016×10^{-2}	0.008
0.2	4.0016×10^{-2}	4.8016×10^{-2}	0.008
0.3	9.0016×10^{-2}	9.8016×10^{-2}	0.008
0.4	0.16002	0.16002	0.0
0.5	0.25002	0.25802	0.008
0.6	0.36002	0.36802	0.008
0.7	0.49002	0.49802	0.008
0.8	0.64002	0.64802	0.008
0.9	0.81002	0.81802	0.008
1.0	1.0	1.008	0.008

$u_{ex} = x^2 + t^2$



Variation of $u_{ex} = x^2 + t^2$ for different values of x and t

$u_{hpm} = x^2 + t^2 + 2t$



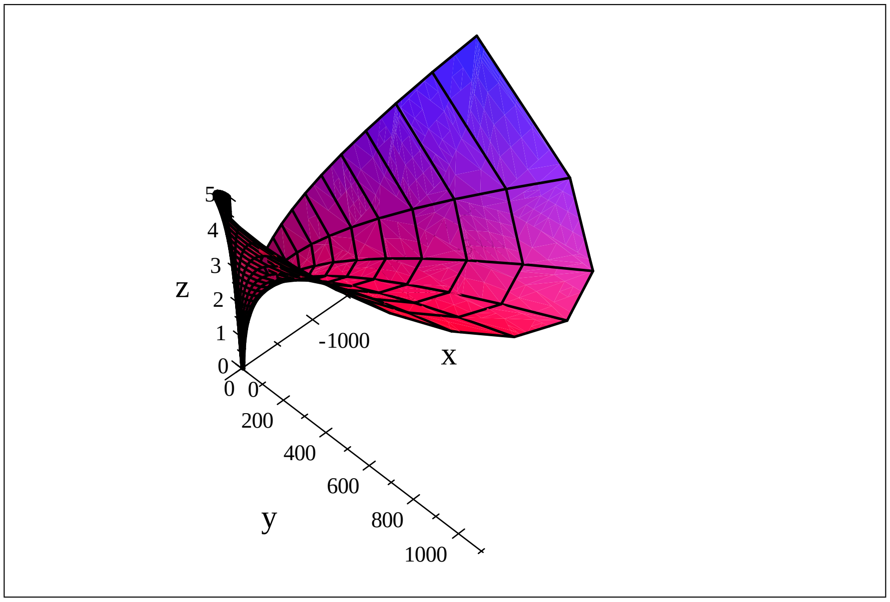
Variation of $u_{hpm} = x^2 + t^2 + 2t$ for different values of x and t

Example 3

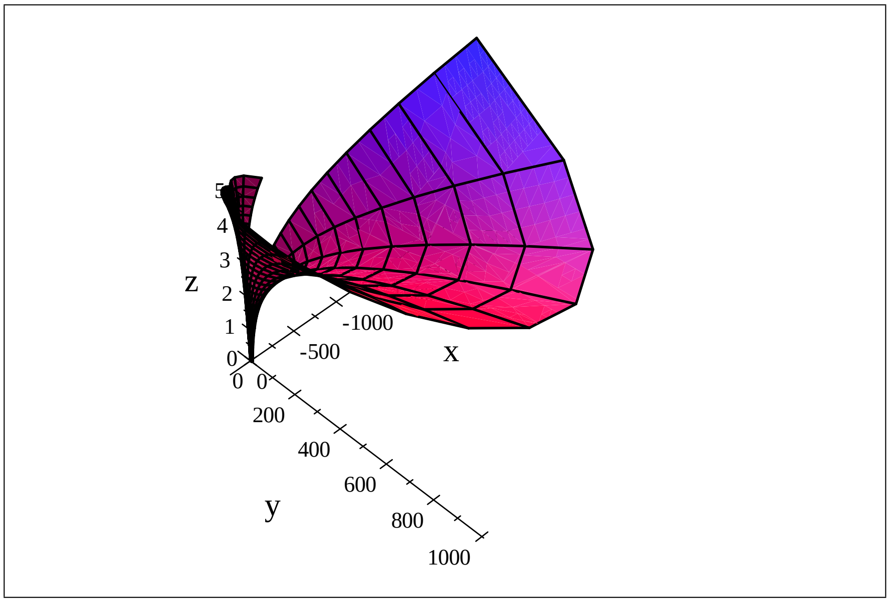
$$h_x = \frac{1}{10},\ h_t = \frac{1}{250}$$

x_i	u_{ex}	u_{hpm}	$5 - iterates$	$\lvert u_{ex} - u_{hpm} \rvert$
0.0	1.004	1.004		0.0
0.1	1.1096	1.1096		0.0
0.2	1.2263	1.2263		0.0
0.3	1.3553	1.3553		0.0
0.4	1.4978	1.4879		0.0
0.5	1.6553	1.6553		0.0
0.6	1.8294	1.8294		0.0
0.7	2.0218	2.0218		0.0
0.8	2.2345	2.2345		0.0
0.9	2.4695	2.4695		0.0
1.0	2.7292	2.7292		0.0

$u_{ex} = \exp(x+t)$



Variation of $u_{ex} = \exp(x+t)$ for different values of x and t
$u_{hpm} = \exp(x)(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!})$



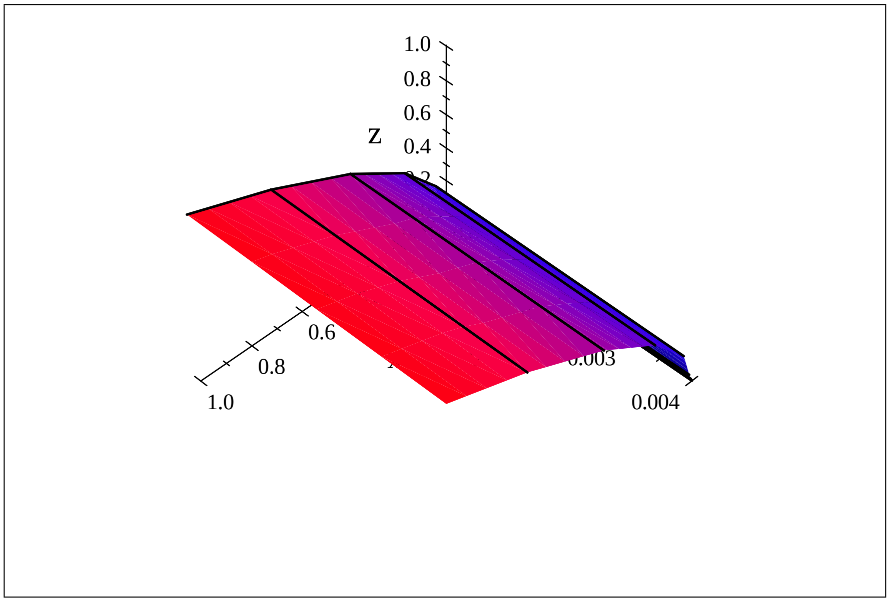
Variation of $u_{hpm} = \exp(x)(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!})$

Example 4

$$h_x = \frac{1}{10},\ h_t = \frac{1}{250}$$

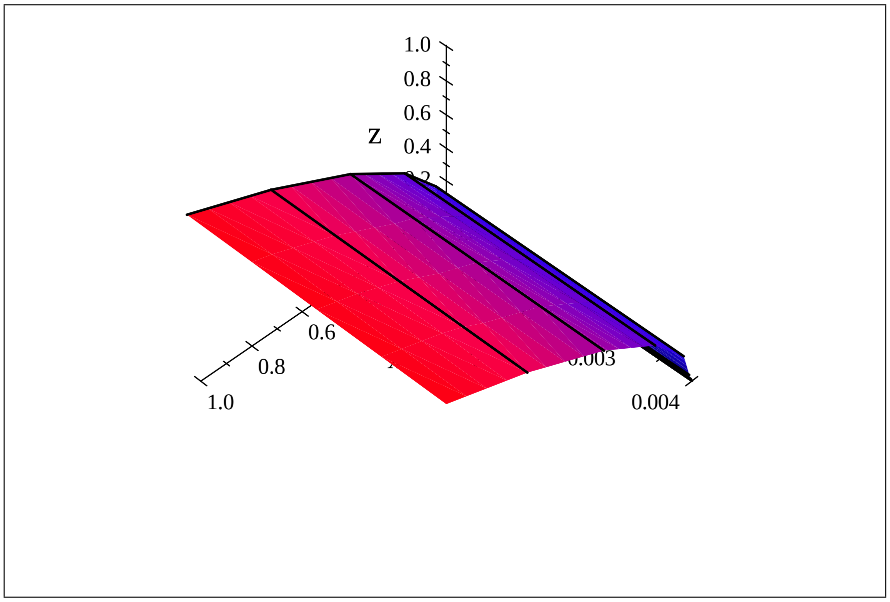
x_i	u_{ex}	u_{hpm}	$5 - iterate$	$\|u_{ex} - u_{hpm}\|$
0.0	0.0	0.0		0.0
0.1	0.01004	0.01004		0.0
0.2	0.04016	0.04016		0.0
0.3	9.036×10^{-2}	9.036×10^{-2}		0.0
0.4	0.16064	0.16064		0.0
0.5	0.251	0.231		0.0
0.6	0.36144	0.36144		0.0
0.7	0.49196	0.49196		0.0
0.8	0.64257	0.64257		0.0
0.9	0.81325	0.81325		0.0
1.0	1.004	1.004		0.0

$u_{ex} = x^2 \exp(t)$



Variation of $u_{ex} = x^2 \exp(t)$ for different values of x and t

$u_{hpm} = x^2(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!})$



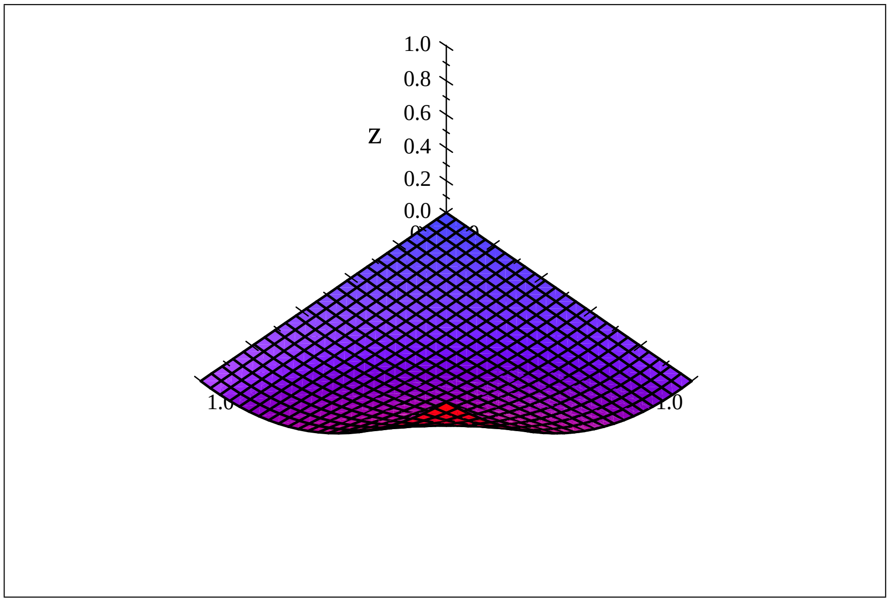
Variation of $u_{hpm} = x^2(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!})$ for different values of x and t

Example 5

$$h_x = \frac{1}{10},\ h_t = \frac{1}{250}$$

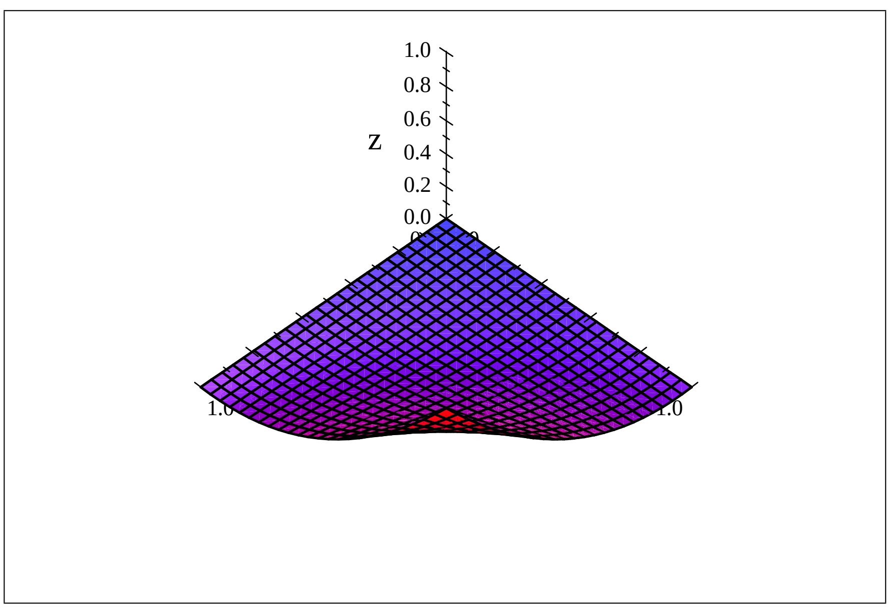
x_i	u_{ex}	u_{hpm}	$5-iterate$	$\lvert u_{ex}-u_{hpm}\rvert$
0.0	0.0	0.0		0.0
0.1	1.004×10^{-4}	1.004×10^{-4}		0.0
0.2	1.6064×10^{-3}	1.6064×10^{-3}		0.0
0.3	8.1325×10^{-3}	8.1325×10^{-3}		0.0
0.4	2.5703×10^{-2}	8.5703×10^{-2}		0.0
0.5	6.2751×10^{-2}	8.2751×10^{-2}		0.0
0.6	0.13012	0.13012		0.0
0.7	0.24106	0.24106		0.0
0.8	0.41124	0.41124		0.0
0.9	0.65873	0.65873		0.0
1.0	1.004	1.004		0.0

$u_{ex} = x^2y^2\exp(\frac{1}{250})$



Variation of $u_{ex} = x^2y^2\exp(\frac{1}{250})$ for different values of x and y

$u_{hpm} = x^2y^2(1 + \frac{1}{250} + \frac{1}{2!}(\frac{1}{250})^2 + \frac{1}{3!}(\frac{1}{250})^3 + \frac{1}{4!}(\frac{1}{250})^4)$



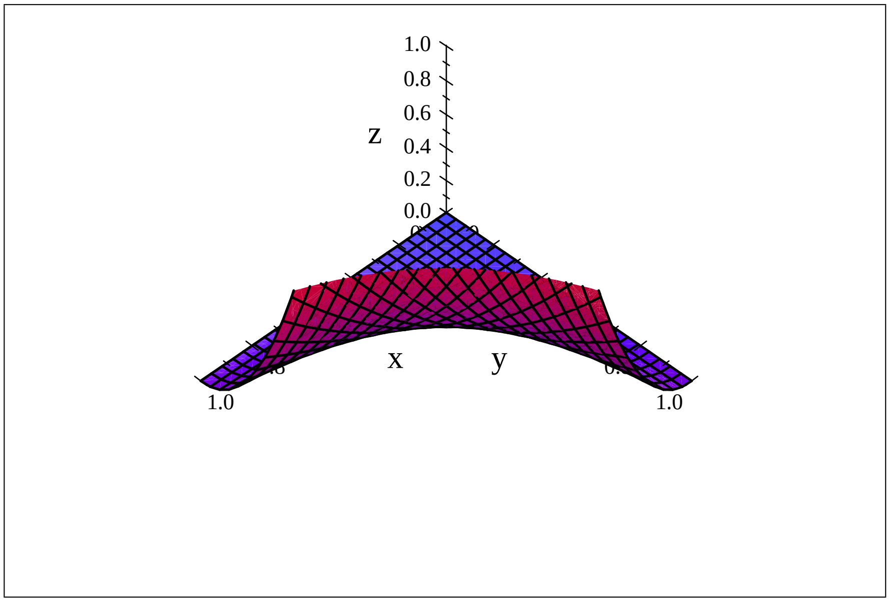
Variation of $u_{hpm} = x^2y^2(1 + \frac{1}{250} + \frac{1}{2!}(\frac{1}{250})^2 + \frac{1}{3!}(\frac{1}{250})^3 + \frac{1}{4!}(\frac{1}{250})^4)$ for different values of x and y

Example 6

$$h_x = h_y = h_z = \frac{1}{10},\ h_t = \frac{1}{250}$$

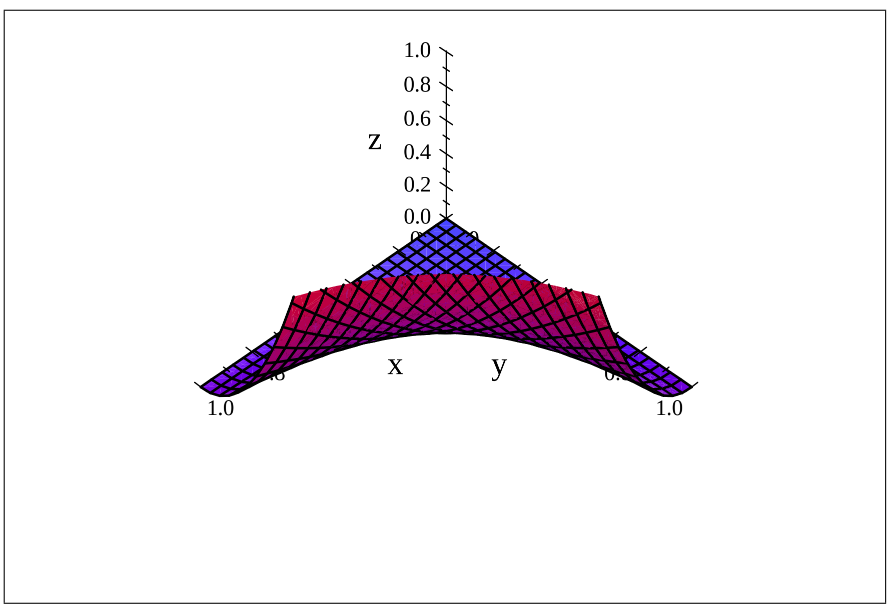
x_i	y_j	z_k	u_{ex}	u_{hpm} $5-iterate$	$\lvert u_{ex} - u_{hpm} \rvert$
0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.1	0.1	1.004×10^{-6}	1.004×10^{-6}	0.0
0.2	0.2	0.2	6.4257×10^{5}	6.4257×10^{-5}	0.0
0.3	0.3	0.3	7.3192×10^{-4}	7.3192×10^{-4}	0.0
0.4	0.4	0.4	4.1124×10^{-3}	4.1124×10^{-3}	0.0
0.5	0.5	0.5	1.5688×10^{-2}	1.5688×10^{-2}	0.0
0.6	0.6	0.6	4.6843×10^{-2}	4.6843×10^{-2}	0.0
0.7	0.7	0.7	0.11812	0.11812	0.0
0.8	0.8	0.8	0.26319	0.26319	0.0
0.9	0.9	0.9	0.53357	0.53357	0.0
1.0	1.0	1.0	1.004	1.004	0.0

$u_{ex} = (5.5)x^2y^2\exp(\frac{1}{250})$



Variation of $u_{ex} = (5.5)x^2y^2\exp(\frac{1}{250})$ for different values of x and y

$u_{hpm} = (5.5)x^2y^2(1 + \frac{1}{250} + \frac{1}{2!}(\frac{1}{250})^2 + \frac{1}{3!}(\frac{1}{250})^3 + \frac{1}{4!}(\frac{1}{250})^4$



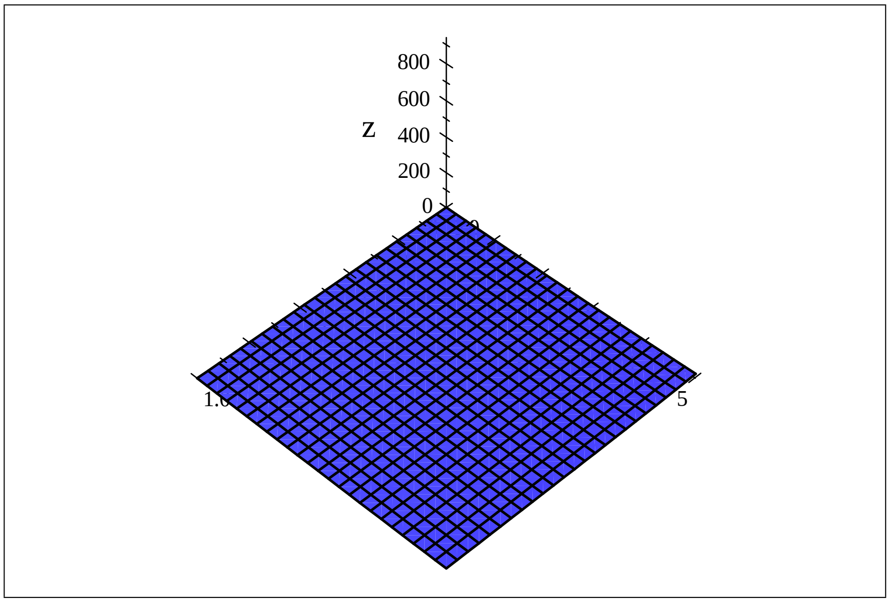
Variation of u_{hpm} for different values of x and y

Example 7

$$h_x = \frac{1}{10},\ h_t = \frac{1}{250}$$

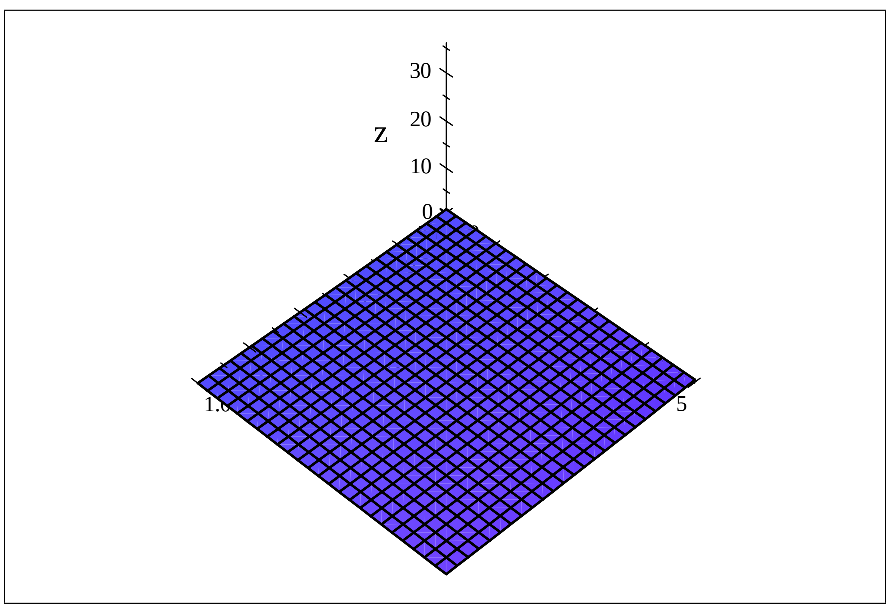
x_i	u_{ex}	u_{hpm}	$1-iterate$	$\lvert u_{ex}-u_{hpm}\rvert$
0.0	1.0	1.0		0.0
0.1	0.82646	0.82645		0.00001
0.2	0.69446	0.69444		0.00002
0.3	0.59173	0.59172		0.00001
0.4	0.51021	0.51020		0.00001
0.5	0.44445	0.44444		0.00001
0.6	0.39063	0.39063		0.0
0.7	0.34603	0.34602		0.00001
0.8	0.30865	0.30864		0.00001
0.9	0.27701	0.27701		0.0
1.0	0.25	0.25		0.0

$u_{ex} = \frac{1+t^2}{(1+x)^2}$



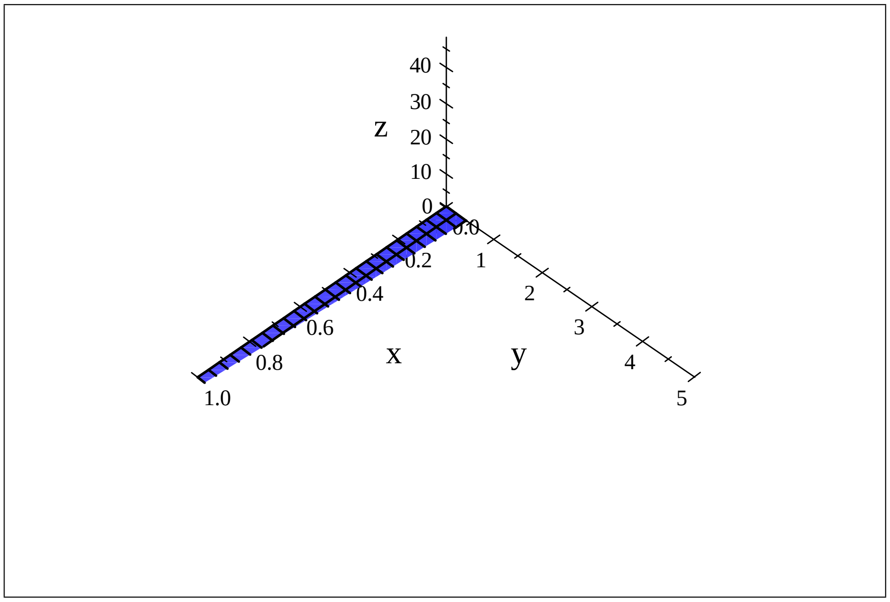
Variation of $u_{ex} = \frac{1+t^2}{(1+x)^2}$ for different values of x and t

$u_{0hpm} = \frac{1}{(1+x)^2}$



Variation of $u_{0hpm} = \frac{1}{(1+x)^2}$ for different values of x and t

$u_{ex} = \frac{1-t}{2+x}$



variation of $u_{ex} = \frac{1-t}{1+x}$ for different values of x and t

3.7 CONCLUSION

In this thesis we made a detailed study of some relatively new techniques along with some of their applications. In partucular, we focused on high-order finite difference schemes HOFS, Adomian decomposition method ADM and homotopy pertubation method HPM and discussed in lenght their respective applications in solving various diversified initial and boundary value problems. first, we proposed and employed HOFS unconditionally stable, the results obtained are of order $O(h_x^6 + h_t^4)$ [6]. The next methods are employed without using linearization, discretization, transformation, or restrictive assumptions, absorb the positve features, the coupled techniques and hence are very much compatible with the diversified and versatile nature of the physical problems, the results obtained are all in good agreement with the exact solutions of the problems under study [1-5]. moreover these methods are easier to implement and are more user friendly as compared to the tradional techniques. It may be concluded that the relatively new techniques can be treated as alternatives for solving a wide class of non linear problems. We would like to mention that the techniques and ideas presented in this thesis can be extended for finding the analytic solution of the obstacle, unilateral, free, moving, and contact problems which arise in various branches of mathematical, physical, medical, structure analysis, and engineering sciences. These problems can be studied in the general, natural, and unified framework of variational inequalities. In a variational inequality framework of such problems, the location of contact area (free or moving boundary) becoms an integral part of the solution and no special techniques are needed to obtain it. It is well-known that if the obstacle is known then the variational inequalities can be characterized by system of variational equations. This area of research is not yet developed and offers a wealth of new opportunities for further research.

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