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THESE

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THEME

QUELQUES NOUVELLES INEGALITES INTEGRALES APPLIQUEES A CERTAINES CLASSES DE PROBLEMES AUX LIMITES

Option :
Equations aux Dérivées Partielles

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Introduction

Integral inequalities play a big role in the study of differential integral equation and partial differential equations. They were introduced by Gronwall in 1919 [13], who gave their applications in the study of some problems concerning ordinary differential equation.

Since that time, the theory of these inequalities knew a fast growth and a great number of monographs were devoted to this subject [4, 14, 15, 38]. The applications of the integral inequalities were developed in a remarkable way in the study of the existence, the uniqueness, the comparison, the stability and continuous dependence of the solution in respect to data. In the last few years, a series of generalizations of those inequalities appeared. Among these generalization, we can quote Pachpatte's work [27].

In the last few years, a number of nonlinear integral inequalities had been established by many scholars, which are motivated by certain applications. For example, we refer the reader to literatures ([1],[31],[33]) and the references therein. However, only a few papers [27, 25, 23] studied the nonlinear delay integral inequalities and integrodifferential inequalities as far as we know.

In fact, in the quantitative of some classes of delay partial differential equations, integral equations and integrodifferential equations, the bounds provided by the earlier inequalities are inadequate and it is necessary to seek some new integral inequalities, delay integral inequalities, integrodifferential inequalities in the case of the functions with one and more than one variables which can be used as tools in this way.

The aim of the present work is to give an exposition of the classical results about integral inequalities with have appeared in the mathematical literature in recent years; and to establish some new integral inequalities, integrodifferential inequalities and also many new retarded integral inequalities. The results given here can be used in the qualitative theory of various classes of boundary value problems of partial differential equations, partial differential equations with a delay, differential equations, integral equations and integrodifferential equations.

The work begging by presenting a number of classical facts in the domain of Gronwall inequalities and some nonlinear inequalities in the general cases, we collected a most of the above inequalities from the book "Integral inequalities and applications" by Bainov and Simeonov [4].

In the chapter 2, we establish some new nonlinear integral inequalities for functions of one variable, with a further generalization of these inequalities in to function with n independent variables. These results (extend the Gronwall type inequalities obtained by Pachpatte [27, 2002] and Oguntuase [25, 2001].We note that our results in this chapter are published in E.J.D.E. (see : [10])

In the chapter 3, we establish some new linear and nonlinear integrodifferential inequalities for functions of n independent variables, which can be used as tools in the theory of partial differential and integrodifferential equations. The present results (in press : [16]) are a generalizations of some inequalities proved in [32].

The purpose of the chapter 4 is to establish some nonlinear retarded integral inequalities in the case of functions of n independent variables which can be used as handy tools in the theory of partial differential and integral equations with time delays. These new inequalities represent a generalization of the results obtained by Ma and Pecaric [23, 2006], Pachpatte [29, 2002] and by Cheung [5, 2006] in the case of the functions with one and two variables.

We illustrate this work, in the end of each chapter (2-4), by applying our new results to certain boundary value problem and some integral equations with a delay .

Chapter 1

Classical Gronwall Inequalities

This chapter is presenting a number of classical facts in the domain of Gronwall inequalities and some nonlinear inequalities in the general cases, we collected a most of the above inequalities from the book "Integral inequalities and applications" by Bainov and Simeonov [4].

1.1 Some Linear Gronwall Inequalities

In the qualitative theory of differential and Volterra integral equation, the Gronwall type inequalities of one variable for the real functions play a very important role. the first use of the Gronwall inequality to establish boundedness and stability is due to Bellman. For the ideas and the methods of Bellman see [3] where further references are given.

In 1919, Gronwall [13] proved a remarkable inequality which has attracted and continues to attract considerable attention in the literature.

Theorem 1 ***(Gronwall-bellman-Bihari)*** *Let* u, Ψ *and* g *be real continuous functions defined in* $[a,b]\,, g(t) \geq 0$ *for* $t \in [a,b]\,.$ *Suppose that on* $[a,b]$ *we have the inequality*

$$u(t) \leq \Psi(t) + \int_a^t g(s)u(s)ds. \tag{1.1}$$

Then

$$u(t) \leq \Psi(t) + \int_a^t g(s)\Psi(s)\exp\left(\int_s^t g(\sigma)d\sigma\right)ds. \tag{1.2}$$

Proof. Define the function

$$y(t) = \int_a^t g(s)u(s)ds, \tag{1.3}$$

for $t \in [a,b]$. Then we have $y(a) = 0$ and

$$\begin{aligned} y'(t) &= g(t)u(t) \\ &\leq g(t)\Psi(t) + g(t)y(t), \quad t \in [a,b]\,. \end{aligned}$$

By multiplication with $\exp\left(-\int_a^t g(s)ds\right) \succ 0$, we obtain

$$\frac{d}{dt}\left(y(t)\exp\left(-\int_a^t g(s)ds\right)\right) \leq \Psi(t)g(t)\exp\left(-\int_a^t g(s)ds\right).$$

By integration on $[a,t]$, one gets

$$y(t)\exp\left(-\int_a^t g(s)ds\right) \leq \int_a^t \Psi(x)g(x)\exp\left(-\int_a^x g(s)ds\right)dx,$$

from where results

$$y(t) \leq \int_a^t \Psi(x)g(x)\exp\left(-\int_x^t g(s)ds\right)dx, \quad t \in [a,b]\,.$$

Since

$$u(t) \leq \Psi(t) + y(t),$$

the theorem is thus proved ■

Next, we shall present some important corollaries resulting from the Gronwall ineqaulity above.

Corollary 2 *If* Ψ *is differentiable, then from (1.1) it follows that*

$$u(t) \leq \Psi(a) \left(\int_a^t g(s)ds \right) + \int_a^t \exp\left(- \int_s^t g(x)dx \right) \Psi'(s)ds,$$

for all $t \in [a, b]$.

Corollary 3 *(**Gronwall ineqaulity**) If* Ψ *is constant, then from*

$$u(t) \leq \Psi + \int_a^t g(s)u(s)ds$$

it follows that

$$u(t) \leq \Psi \exp\left(\int_a^t g(s)ds \right).$$

Filatov [12] proved the following linear generalization of Gronwall's inequality.

Theorem 4 *Let* u *be a continuous nonnegative function such that*

$$u(t) \leq a + \int_{t_0}^t [b + cu(s)]\, ds, \qquad \text{for } t \geq t_{0,}$$

where $a \geq 0, \quad b \geq 0$ *and* $c \succ 0$.

then for $t \geq t_{0,}$ $u(t)$*satisfies*

$$u(t) \leq \left(\frac{b}{c} \right) (\exp(c(t - t_0)) - 1) + a \exp(c(t - t_0)).$$

More general result was given in Willett [37]. Here we give an extended version due to Beesack [2]

Theorem 5 *Let u and k be continuous and a and b Riemann integrable functions on $J = [\alpha, \beta]$ with b and k nonnegative on J.*

(i) If

$$u(t) \leq a(t) + b(t) \int_{\alpha}^{t} k(s)u(s)ds, \qquad t \in J, \tag{1.4}$$

then

$$u(t) \leq a(t) + b(t) \int_{\alpha}^{t} k(s)a(s) \exp\left(\int_{s}^{t} b(x)k(x)dx\right) ds. \tag{1.5}$$

For all $t \in J$. Moreover, equality holds in (1.5) for a subinterval $J_1 = [\alpha, \beta_1]$ of J if equality holds in (1.4) for $t \in J_1$.

(ii) The result remains valid if $\leq$ is replaced by $\geq$ in both (1.4) and (1.5) .

(iii) Both (i) and '(ii) remain valid if $\int_{\alpha}^{t}$ is replaced by $\int_{t}^{\beta}$ and $\int_{s}^{t}$ by $\int_{t}^{s}$ throughout.

Remark 6 *Pachpatte [26] Proved an analogous result on $\mathbb{R}^{+}$ and $(-\infty, 0]$.*

Remark 7 *Willett's paper [37] also contains a linear generalization in which $b(t)k(s)$ is replaced by $\sum_{i=1}^{n} b_i(t)k_i(s)$.*

1.2 Analogues of Gronwall's Inequality

1.2.1 Nonlinear Inequalities

We can consider various nonlinear generalisations of Gornwall's ineqaulity.

The .following theorem is proved in Pachpatte [27]

Theorem 8 *Let u be a nonnegative function that satisfies the integral ineqaulity*

$$u(t) \leq c + \int_{t_0}^{t} (a(s)u(s) + b(s)u^{\alpha}(s))ds, \tag{1.6}$$

for $c \geq 0, \quad \alpha \geq 0$, *where* $a(t)$ *and* $b(t)$ *are continuous nonnegative functions for* $t \geq t_0$.

(i) For $0 \leq \alpha \prec 0$ *we have*

$$u(t) \leq \left\{ c^{1-\alpha} \exp\left[(1-\alpha)\int_{t_0}^{t} a(s)ds\right] + (1-\alpha)\int_{t_0}^{t} b(s)\exp\left[(1-\alpha)\int_{s}^{t} a(x)dx\right]ds \right\}^{\frac{1}{1-\alpha}}.$$

(i) For $\alpha = 1$ *we have*

$$u(t) \leq c\exp\left(\int_{t_0}^{t}[a(s)+b(s)]\,ds\right);$$

(i) For $\alpha \succ 1$ *, with the additional hypothesis*

$$c \prec \left\{\exp\left[(1-\alpha)\int_{t_0}^{t_0+h} a(s)ds\right]\right\}^{\frac{1}{1-\alpha}} \left\{(1-\alpha)\int_{t_0}^{t_0+h} b(s)ds\right\}^{-\frac{1}{1-\alpha}},$$

we also get for $t_0 \leq t \leq t_0 + h$, *for* $h \succ 0$,

$$u(t) \leq c\left\{\exp\left[(1-\alpha)\int_{t_0}^{t} a(s)ds\right] - c^{-1}(\alpha-1)\int_{t_0}^{t} b(s)\exp\left[(1-\alpha)\int_{s}^{t} a(x)dx\right]ds\right\}^{\frac{1}{1-\alpha}}.$$

Remark 9 *Inequality (1.5) is also considered in Willett [37], willett and Wong [36]. and Chu and Metcalf [7]*

The following theorem is modified version of theorem proved in Pachpatte [27] (see also : [25]).

Theorem 10 *If*

$$u(t) \leq f(t) + c\int_{0}^{t} \Phi(s)u^{\alpha}(s)ds,$$

where all functions are continuous and nonnegative on $[0, h]$, $0 \prec \alpha \prec 1$ *and* $c \geq 0$, *then*

$$u(t) \leq f(t) + c\xi_0^\alpha \left(\int_0^t \Phi^{\frac{1}{1-\alpha}}(s) \right)^{1-\alpha},$$

where ξ_0 *is the unique root of* $\xi = a + b\xi^\alpha$.

Pachpatte [27] also proved the following result:

Theorem 11 *If*

$$u(t) \leq c_1 + c_2 \int_0^t \Phi(s)u^\alpha(s)ds + c_3 \int_0^h \Phi(s)u^\alpha(s)ds,$$

$c_1 \geq 0$, $c_2 \geq 0$, $c_3 \succ 0$, *and the functions* $u(t)$ *and* $\Phi(t)$ *are continuous and nonnegative on* $[0, h]$, *the for* $0 \prec \alpha \prec 1$ *we have*

$$u(t) \leq \left(\xi_0^{1-\alpha} + c_2(1-\alpha) \int_0^t \Phi(s)ds \right)^{\frac{1}{1-\alpha}},$$

where ξ_0 *is the unique root of equation*

$$\left[\frac{c_2 + c_3}{c_3}\xi + \frac{c_2 c_1}{c_3} \right]^{1-\alpha} - \xi^{1-\alpha} - c_2(1-\alpha) \int_0^h \Phi(s)ds = 0.$$

If $\alpha \succ 1$ *and* $c_2(1-\alpha) \int_0^h \Phi(s)ds \prec c_1^{1-\alpha}$, *there exists an interval* $[0, \delta] \subset [0, h]$ *where*

$$u(t) \leq \left(c_1^{1-\alpha} + c_2(\alpha - 1) \int_0^t \Phi(s)ds \right)^{\frac{1}{1-\alpha}}.$$

A related result was proved by Stachurska [34].:

Theorem 12 *Let the functions* u, a, b *and* k *be continuous and nonegative of* $J = [\alpha, \beta]$, *and* n *be a positive integer* $(n \geq 2)$ *and* $\frac{a}{b}$ *be nondecreasing function. If*

$$u(t) \leq a(t) + b(t) \int_\alpha^t k(s)u^n(s)ds, \qquad t \in J, \tag{1.7}$$

then

$$u(t) \leq a(t) \left\{ 1 - (n-1) \int_{\alpha}^{t} k(s) b(s) a^{n-1}(s) ds \right\}^{\frac{1}{1-\alpha}}, \tag{1.8}$$

$\alpha \leq t \leq \beta_n$, *where*

$$\beta_n = \sup \left\{ t \in J : (n-1) \int_{\alpha}^{t} k(s) b(s) a^{n-1}(s) ds \right\}.$$

Remark 13 *The ineqaulity (1.7) was considered by Maroni [24], But without the assumption of the monocity of the ratio* $\frac{a}{b}$*. He obtained two estimates, one for* $n \geq 2$ *and another for* $n \geq 3$*. Both are more complicated than (1.8) . For* $n = 2$ *and* $\frac{a}{b}$ *nondecreasing, Stachurska's result can be better than Maroni's on long intervals.*

1.2.2 Inequalities with Kernels of (L)-Type

In this section we present some natural generalisations of Gronwall ineqaulities for real functions of one variable with kernels satisfying a Lipschitz condition (see: (1.9)), which are important in the qualitative theory of differential eqautions.

Theorem 14 ***(Bainov and Simeonov ,[4])*** *; Let* $A, B : [\alpha, \beta[\to \mathbb{R}^+$, $L : [\alpha, \beta[\times \mathbb{R}^+ \to \mathbb{R}^+$ *be continuous functions and*

$$0 \leq L(t,x) - L(t,y) \leq M(t,y)(x-y), \quad t \in [\alpha, \beta[, \ x \geq y \geq 0, \tag{1.9}$$

where M *is nonnegative continuous function on* $[\alpha, \beta[\times \mathbb{R}^+$.

Then for every nonnegative continuous function $u : [\alpha, t] \to \mathbb{R}^+$ *satisfying the ineqaulity*

$$u(t) \leq A(t) + B(t) \int_{\alpha}^{t} L(s, u(s)) ds, \quad t \in [\alpha, \beta[. \tag{1.10}$$

We have the estimation

$$u(t) \leq A(t) + B(t) \int_{\alpha}^{t} L(x, A(x)) \exp\left(\int_{x}^{t} M(s, A(s))B(s)ds \right) dx,$$

for all $t \in [\alpha, \beta[$.

Now, we can give the following two corollaries that are obvious consequences of the above Theorem.

Corollary 15 *Let us suppose that* $A, B : [\alpha, \beta[\to \mathbb{R}^+$, $G : [\alpha, \beta[\times \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous and*

$$0 \leq G(t, x) - G(t, y) \leq N(t)(x - y), \quad t \in [\alpha, \beta[, \; x \geq y \geq 0, \tag{1.11}$$

where N *is nonnegative continuous function on* $[\alpha, \beta[$.

If $u : [\alpha, t] \to \mathbb{R}^+$ *is continuous and satisties the ineqaulity*

$$u(t) \leq A(t) + B(t) \int_{\alpha}^{t} G(s, u(s))ds, \quad t \in [\alpha, \beta[. \tag{1.12}$$

We have the estimation

$$u(t) \leq A(t) + B(t) \int_{\alpha}^{t} G(x, A(x)) \exp\left(\int_{x}^{t} N(s)B(s)ds \right) dx,$$

for all $t \in [\alpha, \beta[$.

Corollary 16 *Let* $A, B, C : [\alpha, \beta[\to \mathbb{R}^+$, $H : \mathbb{R}^+ \to \mathbb{R}^+$ *be continuous and* H *satisfies the following condition of Lipschitz type:*

$$0 \leq H(x) - H(y) \leq M.(x - y), \quad M \succ 0, \quad x \geq y \geq 0, \tag{1.13}$$

If $u : [\alpha, t] \to \mathbb{R}^+$ *is continuous and satisties the ineqaulity*

$$u(t) \leq A(t) + B(t) \int_\alpha^t C(s)H(u(s))ds, \quad t \in [\alpha, \beta[. \tag{1.14}$$

We have the bound

$$u(t) \leq A(t) + B(t) \int_\alpha^t C(x)H(A(x)) \exp\left(M \int_x^t C(s)B(s)ds \right) dx,$$

for all $t \in [\alpha, \beta[$.

For more generalisations of above Theorem, we refer the reader to literatures ([9],[4] and [31]) and the references therein.

Remark 17 *Putting* $H : \mathbb{R}^+ \to \mathbb{R}^+$, $H(x) = x$, *we obtain Lemma 1 of [8] which a natural generalisation of the Gronwall ineqaulity.*

Chapter 2

Some Generalisations of Classical Integral Inequalities

In the present chapter we establish some new nonlinear integral inequalities for functions of one variable, with a further generalization of these inequalities in to function with n independent variables. These results (see: [27]) extend the Gronwall type inequalities obtained by Pachpatte [27] and Oguntuase [25].

In the last section of this chapter, we present some applications of our results to study certain properties of solutions of the nonlinear hyperbolic partial integrodifferential equation.

2.1 Nonlinear Generalisations in One Variable

Our main results are given in the following theorems:

Theorem 18 *Let $u(t)$, $f(t)$ be nonnegative continuous functions in a real interval $I = [a, b]$. Suppose that $k(t, s)$ and its partial derivatives $k_t(t, s)$ exist and are nonnegative continuous functions for almost every $t, s \in I$. Let $\Phi(u(t))$ be real-valued, positive, continuous, strictly non-decreasing, subadditive, and submultiplicative function for $u(t) \geq 0$ and let $W(u(t))$ be real-valued, positive, continuous, and non-decreasing function defined for $t \in I$. Assume that $a(t)$ is a positive continuous function and nondecreasing for $t \in I$. If*

$$u(t) \leq a(t) + \int_a^t f(s)\, u(s)\, ds + \int_a^t f(s)\, W\left(\int_a^s k(s, \tau)\, \Phi(u(\tau)) d\tau\right) ds, \tag{2.1}$$

for $a \leq \tau \leq s \leq t \leq b$, then for $a \leq t \leq t_1$,

$$\begin{aligned} u(t) \leq\ & p(t)\left\{a(t) + \int_a^t f(s)\, \Psi^{-1}\left(\Psi(\zeta)\right.\right. \\ & \left.\left. + \int_a^s k(s, \tau)\, \Phi(p(\tau)) \Phi(\int_a^\tau f(\sigma)\, d\sigma) d\tau\right) ds\right\}, \end{aligned} \tag{2.2}$$

where

$$p(t) = 1 + \int_a^t f(s) \exp\left(\int_a^s f(\sigma) d\sigma\right) ds, \tag{2.3}$$

$$\zeta = \int_a^b k(b, s)\, \Phi(p(s)\, a(s)) ds, \tag{2.4}$$

$$\Psi(x) = \int_{x_0}^x \frac{ds}{\Phi(W(s))}, \quad x \geq x_0 > 0. \tag{2.5}$$

Where Ψ^{-1} is the inverse of Ψ and $t_1$is chosen so that

$$\Psi(\zeta) + \int_a^s k(s, \tau)\, \Phi(p(\tau)) \Phi(\int_a^\tau f(\sigma)\, d\sigma) d\tau \in Dom(\Psi^{-1}).$$

Proof. *Define a function $z(t)$ by*

$$z(t) = a(t) + \int_a^t f(s)\, W\left(\int_a^s k(s, \tau)\, \Phi(u(\tau)) d\tau\right) ds, \tag{2.6}$$

then (2.6) can be restated as

$$u(t) \le z(t) + \int_a^t f(s)\, u(s)\, ds, \tag{2.7}$$

clearly $z(t)$ is nonnegative and continuous in $t \in I$, using lemma **??** *to (2.7), we get*

$$u(t) \le z(t) + \int_a^t f(s) z(s) \exp\left(\int_a^s f(\sigma) d\sigma\right) ds; \tag{2.8}$$

moreover if $z(t)$ is nondecreasing in $t \in I$, we obtain

$$u(t) \le z(t) p(t), \tag{2.9}$$

where $p(t)$ is defined by (2.3). From (2.6), we have

$$z(t) \le a(t) + \int_a^t f(s)\, W(v(s)) ds, \tag{2.10}$$

where

$$v(t) = \int_a^t k(t,s)\, \Phi(u(s)) ds. \tag{2.11}$$

From (2.9) we observe that

$$\begin{aligned} v(t) &\le \int_a^t k(t,s)\, \Phi\left[p(s)\left(a(s) + \int_a^s f(\tau)\, W(v(\tau)) d\tau\right)\right] ds \\ &\le \int_a^t k(t,s)\, \Phi(p(s)\, a(s)) ds + \int_a^t k(t,s)\, \Phi\left(p(s) \int_a^s f(\tau)\, W(v(\tau)) d\tau\right) ds \\ &\le \int_a^b k(b,s)\, \Phi(p(s)\, a(s))\, ds + \int_a^t k(t,s)\, \Phi\left(p(s) \int_a^s f(\tau)\, d\tau\right) \Phi(W(v(s)))\, ds \\ &\le \zeta + \int_a^t k(t,s)\, \Phi\left(p(s) \int_a^s f(\tau)\, d\tau\right) \Phi(W(v(s)))\, ds. \end{aligned} \tag{2.12}$$

Where ζ is defined by (2.4).

Since Φ is a subadditive and a submultiplicative function, W and $v(t)$ are nondecreasing. Define $r(t)$ as the right side of (2.12), then $r(a) = \zeta$ and $v(t) \le r(t)$, $r(t)$ is *positive nondecreasing in $t \in I$ and*

$$\begin{aligned} r'(t) &= k(t,t)\,\Phi\left(p(t)\int_a^t f(\tau)\,d\tau\right)\Phi\left(W(v(t))\right) \\ &\quad + \int_a^t k_t(t,s)\,\Phi(p(s)\int_a^s f(\tau)\,d\tau)\Phi(W(v(s)))ds, \\ &\le \Phi\left(W(r(t))\right)\left[k(t,t)\,\Phi\left(p(t)\int_a^t f(\tau)\,d\tau\right)\right. \\ &\quad \left. + \int_a^t k_t(t,s)\,\Phi\left(p(s)\int_a^s f(\tau)\,d\tau\right)ds\right], \end{aligned} \tag{2.13}$$

dividing both sides of (2.13) by $\Phi(W(r(t)))$ we obtain

$$\frac{r'(t)}{\Phi(W(r(t)))} \le \left[\int_a^t k(t,s)\,\Phi(p(s)\int_a^s f(\tau)\,d\tau)ds\right]'. \tag{2.14}$$

Note that for

$$\Psi(x) = \int_{x_0}^x \frac{ds}{\Phi(W(s))}, \quad x \ge x_0 > 0,$$

it follows that

$$[\Psi(r(t))]' = \frac{r'(t)}{\Phi(W(r(t)))}. \tag{2.15}$$

From (2.15) and (2.14), we have

$$[\Psi(r(t))]' \le [\int_a^t k(t,s)\,\Phi(p(s)\int_a^s f(\tau)\,d\tau)ds]', \tag{2.16}$$

integrate (2.16) from a to t, leads to

$$\Psi(r(t)) \le \Psi(\zeta) + \int_a^t k(t,s)\,\Phi(p(s)\int_a^s f(\tau)\,d\tau)ds,$$

then

$$r(t) \le \Psi^{-1}\left(\Psi(\zeta) + \int_a^t k(t,s)\,\Phi(p(s))\Phi(\int_a^s f(\tau)\,d\tau)ds\right). \tag{2.17}$$

By (2.17),(2.12) ,2.10) and (2.9) we have the desired result ■

Remark 19 *The preceding Theorem is a generalization of the result obtained by Pachpatte in [27, Theorem 2.1].and the inequality in (2.1) can be considered as a further generalization of the inequality given in ([25],[15]) .*

Theorem 20 *Let* $u(t), f(t), b(t), h(t)$ *be nonnegative continuous functions in a real interval* $I = [a, b]$. *Suppose that* $h(t) \in C^1(I, \mathbb{R}^+)$ *is nondecreasing. Let* $\Phi(u(t)), W(u(t))$ *and* $a(t)$ *be as defined in Theorem 18. If*

$$u(t) \le a(t) + \int_a^t f(s)\, u(s)\, ds + \int_a^t f(s)\, h(s) W\Big(\int_a^s b(\tau)\, \Phi(u(\tau)) d\tau\Big) ds,$$

for $a \le \tau \le s \le t \le b$, *then for* $a \le t \le t_2$,

$$\begin{aligned} u(t) \quad \le \quad & p(t) \left\{ a(t) + \int_a^t f(s)\, h(s) \Psi^{-1} \left(\Psi(\vartheta) \right. \right. \\ & \left. \left. + \int_a^s b(\tau) \Phi \left(p(\tau) \int_a^\tau f(\sigma)\, h(\sigma) d\sigma \right) d\tau \right) ds \right\}. \end{aligned}$$

Where $p(t)$ *is defined by (2.2),* Ψ *is defined by (2.7) and*

$$\vartheta = \int_a^b b(s) \Phi(p(s)\, a(s)) ds,$$

the t_2 *is chosen so that* $\Psi(\vartheta) + \int_a^s b(\tau)\Phi(p(\tau) \int_a^\tau f(\sigma)\, h(\sigma) d\sigma) d\tau \in Dom(\Psi^{-1})$.

The proof of the above theorem follows similar arguments as in the proof of Theorem 18; so we omit it.

Remark 21 *The preceding theorem is a generalization of the result obtained by Oguntuase in [25, Theorem 2.3, 2.9].*

In this section we use the following class of function.

Definition 22 *A function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to the class S if it satisfies the following conditions,*

1. $g(u)$ is positive, nondecreasing and continuous for $u \geq 0$ and

2. $(1/v)g(u) \leq g(u/v),\ u > 0, v \geq 1.$

Theorem 23 *Let $u(t),\ f(t), a(t), k(t,s), \Phi$ and W be as defined in Theorem 18, let $g \in S$. If*

$$u\left(t\right) \leq a(t) + \int_a^t f\left(s\right) g(u\left(s\right))ds + \int_a^t f\left(s\right) W\left(\int_a^s k\left(s,\tau\right)\Phi(u\left(\tau\right))d\tau\right) ds, \quad (2.18)$$

for $a \leq \tau \leq s \leq t \leq b$, then for $a \leq t \leq t_3$,

$$\begin{aligned} u(t) \quad &\leq \quad \overline{p}(t)\left\{a(t) + \int_a^t f\left(s\right)\Psi^{-1}\left(\Psi(\overline{\zeta})\right.\right. \\ &\left.\left. + \int_a^s k\left(s,\tau\right)\Phi(\overline{p}(\tau))\Phi(\int_a^\tau f\left(\sigma\right)d\sigma)d\tau\right) ds\right\}. \end{aligned} \quad (2.19)$$

Where

$$\overline{p}(t) = \Omega^{-1}\left(\Omega(1) + \int_a^t f(s)ds\right), \quad (2.20)$$

$$\overline{\zeta} = \int_a^b k\left(b,s\right)\Phi(\overline{p}\left(s\right)a(s))ds, \quad (2.21)$$

$$\Omega(\delta) = \int_\varepsilon^\delta \frac{ds}{g(s)}, \quad \delta \geq \varepsilon > 0. \quad (2.22)$$

Where Ω^{-1} is the inverse function of Ω, and Ψ, Ψ^{-1} are defined in theorem 18, $t_3$is chosen so that $\Omega(1) + \int_a^t f(s)ds$ is in the domain of Ω^{-1}, and

$$\Psi(\overline{\zeta}) + \int_a^s k\left(s,\tau\right)\Phi(\overline{p}(\tau))\Phi(\int_a^\tau f\left(\sigma\right)d\sigma)d\tau,$$

*is in the domain of Ψ^{-1}.**Proof.** Define the function*

$$z(t) = a(t) + \int_a^t f\left(s\right) W\left(\int_a^s k\left(s,\tau\right)\Phi(u\left(\tau\right))d\tau\right) ds, \quad (2.23)$$

then (2.18) can be restated as

$$u(t) \leq z(t) + \int_a^t f\,(s)\,g(u\,(s))ds. \tag{2.24}$$

When $z(x)$ is a positive, continuous, nondecreasing in $x \in I$ and $g \in S$,then it can be restated as

$$\frac{u(t)}{z(t)} \leq 1 + \int_a^t f\,(s)\,g(\frac{u\,(s)}{z(s)})ds. \tag{2.25}$$

The inequality (2.25) may be treated as one-dimensional Bihari-lasalle inequality (see [4]), which implies

$$u(t) \leq \overline{p}(t)z(t), \tag{2.26}$$

*where $\overline{p}(t)$ is defined by (2.20). By (2.23) and (**??**) we get*

$$u(t) \leq \overline{p}(t)\left[a(t) + \int_a^t f\,(s)\,W(v(s))ds\right],$$

where

$$v(s) = \int_a^s k\,(s,\tau)\,\Phi(u\,(\tau))d\tau.$$

Now, by following the argument as in the proof of Theorem 18, we obtain the desired inequality in (2.19). ■

Remark 24 *Under some suitable conditions in the above Theorem, the inequality in (2.18) can be considered as a further generalization of the inequality given in [27, Theorem 2.1].*

Theorem 25 *Let $u\,(t)\,, f(t), b\,(t)\,,\ h\,(t)$,$\Phi(u(t)), W(u(t))$ and $a(t)$ be as defined in Theorem 20, let $g \in S$. If*

$$u\,(t) \leq a(t) + \int_a^t f\,(s)\,g(u\,(s))ds + \int_a^t f\,(s)\,h(s)W\left(\int_a^s b\,(\tau)\,\Phi(u\,(\tau))d\tau\right)ds, \tag{2.27}$$

for $a \le \tau \le s \le t \le b$, then for $a \le t \le t_4$,

$$u(t) \le \overline{p}(t)\left\{a(t)+\int_a^t f(s)\,h(s)\Psi^{-1}\left(\Psi(\overline{\vartheta})+\int_a^s b(\tau)\Phi(\overline{p}(\tau)\int_a^\tau f(\sigma)\,h(\sigma)d\sigma)d\tau\right)ds\right\}. \tag{2.28}$$

Here $\overline{p}(t)$ is defined by (2.20), Ψ is defined by (2.7) and

$$\overline{\vartheta}=\int_a^b b(s)\Phi(\overline{p}(s)\,a(s))ds,$$

the value t_4 is chosen so that $\Psi(\overline{\vartheta})+\int_a^s b(\tau)\Phi(\overline{p}(\tau)\int_a^\tau f(\sigma)\,h(\sigma)d\sigma)d\tau \in Dom(\Psi^{-1})$.

The proof of the above theorem follows similar argument as in the proof of Theorem 23, we omit it.

2.2 Nonlinear Generalizations in Several Variables

In what follows we denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+=[0,\infty)$. All the functions which appear in the inequalities are assumed to be real valued of n variables which are nonnegative and continuous. All integrals are assumed to exist on their domains of definitions.

Throughout this paper, we assume that $\mathbb{I}=[a;b]$ in any bounded open set in the dimensional Euclidean space $\mathbb{R}^n$ and that our integrals are on $\mathbb{R}^n(n\ge 1)$, where $a=(a_1,a_2,....,a_n), b=(b_1,b_2,....,b_n)\in\mathbb{R}^n_+$. For $x=(x_1,x_2,...x_n), t=(t_1,t_2,...t_n)\in\mathbb{I}$, we shall denote the integral

$$\int_a^x=\int_{a_1}^{x_1}\int_{a_2}^{x_2}...\int_{a_n}^{x_n}...dt_n...dt_1.$$

Furthermore, for $x,t \in \mathbb{R}^n$, we shall write $t \leq x$ whenever $t_i \leq x_i,\ i = 1,2,...,n$.and $0 \leq a \leq x \leq b$,for $x \in \mathbb{I}$ and $D = D_1 D_2 ... D_n$, where $D_i = \frac{\partial}{\partial x_i}$, for $i = 1,2,...,n$.

Let $C(\mathbb{I}, \mathbb{R}_+)$ denote the class of continuous functions from $\mathbb{I}$ to $\mathbb{R}_+$.

Remark 26 *The following theorem deals with n-independent variables versions of the inequalities established in Pachpatte [27, Theorem 2.3].*

Theorem 27 *Let $u(x), f(x), a(x) \in C(\mathbb{I}, \mathbb{R}_+)$ and let $K(x,t)$, $D_i k(x,t)$ be in $C(\mathbb{I} \times \mathbb{I}, \mathbb{R}_+)$ for all $i = 1,2,...,n$ and c be a nonnegative constant.* (**1**) *If*

$$u(x) \leq c + \int_a^x f(s) \left[u(s) + \int_a^s k(s,\tau) u(\tau) d\tau \right] ds, \tag{2.29}$$

for $x \in \mathbb{I}$ and $a \leq \tau \leq s \leq b$, then

$$u(x) \leq c \left[1 + \int_a^x f(t) \exp\left(\int_a^t (f(s) + k(b,s)) ds \right) dt \right]. \tag{2.30}$$

(**2**) *If*

$$u(x) \leq a(x) + \int_a^x f(s) \left[u(s) + \int_a^s k(s,\tau) u(\tau) d\tau \right] ds, \tag{2.31}$$

for $x \in \mathbb{I}$ and $a \leq \tau \leq s \leq b$, then

$$u(x) \leq a(x) + e(x) \left[1 + \int_a^x f(t) \exp\left(\int_a^t (f(s) + k(b,s)) ds \right) dt \right], \tag{2.32}$$

where

$$e(x) = \int_a^x f(s) \left[a(s) + \int_a^s k(s,\tau) a(\tau) d\tau \right] ds. \tag{2.33}$$

Proof. (**1**) *The inequality (2.29) implies the estimate*

$$u(x) \leq c + \int_a^x f(s) \left[u(s) + \int_a^s k(b,\tau) u(\tau) d\tau \right] ds.$$

We define the function

$$z(x) = c + \int_a^x f(s) \left[u(s) + \int_a^s k(b,\tau) u(\tau) d\tau \right] ds,$$

then $z(a_1, x_2, ..., x_n) = c$, $u(x) \leq z(x)$ *and*

$$\begin{aligned} Dz(x) &= f(x) \left[u(x) + \int_a^x k(b,s) u(s) ds \right], \\ &\leq f(x) \left[z(x) + \int_{x^0}^x k(b,s) z(s) ds \right]. \end{aligned}$$

Define the function

$$v(x) = z(x) + \int_a^x k(b,s) z(s) ds,$$

then $z(a_1, x_2, ..., x_n) = v(a_1, x_2, ..., x_n) = c$, $Dz(x) \leq f(x)v(x)$ *and* $z(x) \leq v(x)$, *we have*

$$Dv(x) = Dz(x) + k(b,x)z(x) \leq (f(x) + k(b,x)) v(x). \tag{2.34}$$

Clearly $v(x)$ *is positive for all* $x \in \mathbb{I}$, *hence the inequality (**??**) implies the estimate*

$$\frac{v(x)Dv(x)}{v^2(x)} \leq f(x) + k(b,x);$$

that is

$$\frac{v(x)Dv(x)}{v^2(x)} \leq f(x) + k(b,x) + \frac{(D_n v(x))(D_1 D_2 ... D_{n-1} v(x))}{v^2(x)};$$

hence

$$D_n \left(\frac{D_1 D_2 ... D_{n-1} v(x)}{v(x)} \right) \leq f(x) + k(b,x).$$

Integrating with respect to x_n *from* a_n *to* x_n , *we have*

$$\frac{D_1 D_2 ... D_{n-1} v(x)}{v(x)} \leq \int_{a_n}^{x_n} \left[f(x_1, ..., x_{n-1}, t_n) + k(b, x_1, ..., x_{n-1}, t_n) \right] dt_n;$$

thus

$$\frac{v(x)D_1D_2...D_{n-1}v(x)}{v^2(x)} \leq \int_{a_n}^{x_n} [f(x_1,...,x_{n-1},t_n)+k(b,x_1,...,x_{n-1},t_n)]\,dt_n + \frac{(D_{n-1}v(x))(D_1D_2...D_{n-2}v(x))}{v^2(x)}.$$

That is

$$D_{n-1}\left(\frac{D_1D_2...D_{n-2}v(x)}{v(x)}\right) \leq \int_{a_n}^{x_n} [f(x_1,...,x_{n-1},t_n)+k(b,x_1,...,x_{n-1},t_n)]\,dt_n,$$

integrating with respect to x_{n-1} *from* a_{n-1} *to* x_{n-1} *, we have*

$$\frac{D_1D_2...D_{n-2}v(x)}{v(x)} \leq \int_{a_n-1}^{x_n-1}\int_{a_n}^{x_n} [f(x_1,...,x_{n-2},t_{n-1},t_n) + k(b,x_1,...,x_{n-2},t_{n-1},t_n)]dt_n dt_{n-1}.$$

Continuing this process, we obtain

$$\frac{D_1v(x)}{v(x)} \leq \int_{a_2}^{x_2}...\int_{a_n}^{x_n} [f(x_1,t_2,t_3,...,t_n)+k(b,x_1,t_2,t_3,...,t_n)]\,dt_n...dt_2.$$

Integrating with respect to x_1 *from* a_1 *to* x_1*, we have*

$$\log\frac{v(x)}{v(a_1,x_2,...,x_n)} \leq \int_a^x [f(t)+k(b,t)]\,dt;$$

that is,

$$v(x) \leq c\exp\left(\int_a^x [f(t)+k(b,t)]\,dt\right). \tag{2.35}$$

Substituting (2.35) into $Dz(x) \leq f(x)v(x)$*, we have*

$$Dz(x) \leq cf(x)\exp\left(\int_a^x [f(t)+k(b,t)]\,dt\right), \tag{2.36}$$

integrating (2.36) with respect to the x_n *component from* a_n *to* x_n*, then with respect to the* a_{n-1} *to* x_{n-1}*, and continuing until finally* a_1 *to* x_1*, and noting that*

$z(a_1, x_2, ..., x_n) = c$, *we have*

$$z(x) \leq c\left[1+\int_a^x f(t)\exp\left(\int_a^t [f(s)+k(b,s)]\,ds\right)dt\right].$$

This completes the proof of the first part.

(**2**) *Define a function* $z(x)$ *by*

$$z(x) = \int_a^x f(s)\left[u(s)+\int_a^s k(s,\tau)\,u(\tau)\,d\tau\right]ds. \tag{2.37}$$

Then from (2.31), $u(x) \leq a(x) + z(x)$ *and using this in (2.37), we get*

$$\begin{aligned} z(x) &\leq \int_a^x f(s)\left[a(s)+z(s)+\int_a^s k(s,\tau)\,[a(\tau)+z(\tau)]d\tau\right]ds, \\ &\leq e(x)+\int_a^x f(s)\left[z(s)+\int_a^s k(s,\tau)\,z(\tau)\,d\tau\right]ds. \end{aligned} \tag{2.38}$$

Where $e(x)$ *is defined by (2.33). Clearly* $e(x)$ *is positive, continuous an nondecreasing for all* $x \in \mathbb{I}$. *From (2.38) it is easy to observe that*

$$\frac{z(x)}{e(x)} \leq 1+\int_a^x f(s)\left[\frac{z(s)}{e(s)}+\int_a^s k(s,\tau)\,\frac{z(\tau)}{e(\tau)}d\tau\right]ds.$$

Now, by application the inequality in part (**1**), *we have*

$$z(x) \leq e(x)\left[1+\int_a^x f(t)\exp\left(\int_a^t (f(s)+k(b,s))\,ds\right)dt\right]. \tag{2.39}$$

The desired inequality in (2.32) follows from (2.39) and the fact that $u(x) \leq a(x) + z(x)$. ■

The following theorem deals with n-independent variables versions of the inequalities established in Theorem 23. We need the inequalities in the following lemma (see [15]).

Lemma 28 *([15, Khellaf])Let $u(x)$ and $b(x)$ be nonnegative continuous functions, defined for $x \in \mathbb{I}$.,and let $g \in S$. Assume that $a(x)$ is positive, continuous function , nondecreasing in each of the variables $x \in \mathbb{I}$. Suppose that*

$$u(x) \leq c + \int_a^x b(t) g\left(u(t)\right) dt, \tag{2.40}$$

holds for all $x \in \mathbb{I}$ with $x \geq a$, then

$$u(x) \leq G^{-1}\left[G(c) + \int_a^x b(t) dt\right], \tag{2.41}$$

for all $x \in \mathbb{I}$ such that $G(c) + \int_a^x b(t)dt \in Dom(G^{-1})$, Where $G(u) = \int_{u_0}^u dz/g(z), u > 0 (u_0 > 0)$.

Theorem 29 *Let $u(x), f(x), a(x)$ and $k(x,t)$ be as defined in Theorem 27.Let $\Phi(u(x))$ be real-valued, positive, continuous, strictly non-decreasing, subadditive, and submultiplicative function for $u(x) \geq 0$ and let $W(u(x))$ be real-valued, positive, continuous, and non-decreasing function defined for $x \in \mathbb{I}$. Assume that $a(x)$ is positive continuous function and nondecreasing for $x \in \mathbb{I}$. If*

$$u(x) \leq a(x) + \int_a^x f(t) g(u(t)) dt + \int_a^x f(t) W\left(\int_a^t k(t,s) \Phi(u(s)) ds\right) dt, \tag{2.42}$$

for $a \leq s \leq t \leq x \leq b$, then for $a \leq x \leq x^*$,

$$\begin{gathered} u(x) \leq \beta(x) \left\{a(x) + \textstyle\int_a^x f(t) \right. \\ \left. \times W\left[\Psi^{-1}\left(\Psi(\eta) + \textstyle\int_a^t k(b,s) \Phi\left[\beta(s) \int_a^s f(\tau) d\tau\right] ds\right)\right] dt\right\}, \end{gathered} \tag{2.43}$$

where

$$\beta(x) = G^{-1}\left(G(1) + \int_a^x f(s) ds\right), \tag{2.44}$$

$$\eta = \int_a^b k(b,s) \Phi(\beta(s) a(s)) ds, \tag{2.45}$$

$$G(u) = \int_{u_0}^{u} 1/g(z)dz, \quad u > 0 \ (u_0 > 0), \tag{2.46}$$

$$\Psi(x) = \int_{x_0}^{x} \frac{ds}{\Phi(W(s))}, \quad x \geq x_0 > 0. \tag{2.47}$$

Where G^{-1} *is the inverse function of* G*, and* Ψ *is the inverse function of* Ψ^{-1} *,* x^**is chosen so that* $G(1) + \int_a^x f(s)ds$ *is in the domain of* G^{-1}*, and*

$$\Psi(\eta) + \int_a^t k(b,s)\Phi\left[\beta(s)\int_a^s f(\tau)\,d\tau\right]ds,$$

is in the domain of Ψ^{-1} ***.Proof.*** *Define the function*

$$z(x) = a(x) + \int_a^x f(t)\,W\left(\int_a^t k(t,s)\,\Phi(u(s))ds\right)dt, \tag{2.48}$$

then (2.48) can be restated as

$$u(x) \leq z(x) + \int_a^x f(t)\,g(u(t))dt.$$

We have $z(x)$ *is a positive, continuous, nondecreasing in* $x \in \mathbb{I}$ *and* $g \in S$*,Then the above inequality can be restated as*

$$\frac{u(x)}{z(x)} \leq 1 + \int_a^x f(t)\,g\left(\frac{u(t)}{z(t)}\right)dt. \tag{2.49}$$

by Lemma 28 we have

$$u(x) \leq z(x)\beta(x), \tag{2.50}$$

where $\beta(x)$ *is defined by (2.44). By (2.48) and (2.50) we have*

$$z(x) \leq a(x) + \int_a^x f(t)\,W(v(t))dt, \tag{2.51}$$

where

$$v(x) = \int_a^x k(x,t)\,\Phi(u(t))dt. \tag{2.52}$$

By (2.52) and (2.50) , we observe that

$$\begin{aligned} v(x) &\leq \int_a^x k(b,t)\,\Phi\left[\beta(t)\left(a(t)+\int_a^t f(s)\,W(v(s))ds\right)\right]dt \\ &\leq \int_a^x k(b,s)\,\Phi(\beta(s)\,a(s))ds \\ &\quad + \int_a^t k(b,s)\,\Phi(\beta(s)\int_a^s f(\tau)\,W(v(\tau))d\tau)ds, \\ &\leq \eta + \int_a^x k(b,s)\,\Phi[\beta(s)\int_a^s f(\tau)\,d\tau]\Phi(W(v(s)))ds. \end{aligned} \tag{2.53}$$

Where η *is defined by (2.45). Since* Φ *is subadditive and submultiplicative function,* W *and* $v(x)$ *are nondecreasing for all* $x \in \mathbb{I}$. *Define* $r(x)$ *as the right side of (2.53), then* $r(a_1, x_2, ..., x_n) = \eta$ *and* $v(x) \leq r(x)$, $r(x)$ *is positive and nondecreasing in each of the variables* $x_1, x_2, x_3, ...x_n$ *. Hence*

$$\frac{Dr(x)}{\Phi(W(r(x)))} \leq k(b,x)\,\Phi[\beta(x)\int_a^x f(s)\,ds],$$

since

$$D_n\left(\frac{D_1...D_{n-1}r(x)}{\Phi(W(r(x)))}\right) = \frac{Dr(x)}{\Phi(W(r(x)))} - \frac{D_n\Phi(W(r(x)))D_1...D_{n-1}r(x)}{\Phi^2(W(r(x)))},$$

the above inequality implies

$$D_n\left(\frac{D_1...D_{n-1}r(x)}{\Phi(W(r(x)))}\right) \leq \frac{Dr(x)}{\Phi(W(r(x)))},$$

and

$$D_n\left(\frac{D_1...D_{n-1}r(x)}{\Phi(W(r(x)))}\right) \leq k(b,x)\,\Phi[\theta(x)].$$

Where $\theta(x) = \beta(x)\int_a^x f(s)\,ds$. *Integrating with respect to* x_n *from* a_n *to* x_n*, we have*

$$\frac{D_1...D_{n-1}r(x)}{\Phi(W(r(x)))} \leq \int_{a_n}^{x_n} k(b,x_1,x_2,...,x_{n-1},s_n)\,\Phi[\theta(x_1,x_2,...,x_{n-1},s_n)]ds_n.$$

Repeating this argument, we find that

$$\frac{D_1 r(x)}{\Phi(W(r(x)))} \leq \int_{a_2}^{x_2} ... \int_{a_{n-1}}^{x_{n-1}} \int_{a_n}^{x_n} k\,(b, x_1, s_2, ..., s_n)\, \Phi[\theta(x_1, s_2, ..., s_n)] ds_n ds_{n-1} ... ds_2.$$

Integrating both sides of the above inequality with respect to x_1 from a_1 to x_1, we have

$$\Psi(r(x)) - \Psi(\eta) \leq \int_a^x k(b,s)\Phi\left[\theta(s)\right] ds,$$

and

$$r(x) \leq \Psi^{-1}\left(\Psi(\eta) + \int_a^x k(b,s)\Phi\left[\beta\,(s) \int_a^s f\,(\tau)\, d\tau\right] ds\right).$$

From this we obtain

$$v\,(x) \leq r(x) \leq \Psi^{-1}\left(\Psi(\eta) + \int_a^x k(b,s)\Phi\left[\beta\,(s) \int_a^s f\,(\tau)\, d\tau\right] ds\right). \tag{2.54}$$

By (2.50), (2.51) and (2.54) we obtain the desired inequality in (2.43). ■

2.3 Applications

In this section, our results are applied to the qualitative analysis of two applications. The first is the system of nonlinear differential equations for one variable functions. The second is nonlinear hyperbolic partial integrodifferential equation of n-independent variables.

First we consider the system of nonlinear differential equations

$$\frac{du}{dt} = F_1(t, u(t), \int_{x_0}^t K_1(t, u(s))ds), \tag{2.55}$$

for all $t \in I = [t_0, t_\infty] \subset R_+$, where $u \in C(I, \mathbb{R}^n), F_1 \in C(I \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $K_1 \in C(I \times \mathbb{R}^n, \mathbb{R}^n)$.

In what follows, we shall assume that the Cauchy problem

$$\begin{aligned} \frac{du}{dt} &= F_1(t, u(t), \int_{t_0}^{t} K_1(t, u(s))ds), \quad x \in I, \\ u(t_0) &= u_0 \in \mathbb{R}^n, \end{aligned} \tag{2.56}$$

has a unique solution, for every $t_0 \in I$ and $u_0 \in \mathbb{R}^n$. We shall denote this solution by $u(., t_0, u_0)$. The Following theorem deals the estimate on the solution of the nonlinear Cauchy problem (2.56).

Theorem 30 *Assume that the functions F_1 and K_1 in (2.56) satisfy the conditions*

$$\|K_1(t, u)\| \leq h(t)\Phi(\|u\|), \quad t \in I, \tag{2.57}$$

$$\|F_1(t, u, v)\| \leq \|u\| + \|v\|, \quad u, v \in \mathbb{R}^n, \tag{2.58}$$

where h and Φ are as defined in Theorem 20. Then we have the estimate, for $t_0 \leq t \leq t_2$,

$$\|u(t, t_0, u_0)\| \leq e^{t-t_0}(\|u_0\| + \int_{t_0}^{t} h(s)E_1(s, \|u_0\|)ds), \tag{2.59}$$

where

$$E_1(t, \|u_0\|) = \Psi^{-1}(\Psi(\vartheta) + \int_{t_0}^{t} \Phi(e^{\tau - x_0} \int_{t_0}^{\tau} h(\sigma)d\sigma)d\tau), \tag{2.60}$$

$$\Psi(t) = \int_{a}^{t} \frac{ds}{\Phi(s)}, \quad t \geq a > 0, \tag{2.61}$$

$$\vartheta = \int_{t_0}^{t_\infty} \|u_0\| \, \Phi\left(e^{s-t_0}\right) ds, \tag{2.62}$$

and t_2 is chosen so that $\Psi(\vartheta) + \int_{x_0}^{s} \Phi\left(e^{\tau - t_0} \int_{t_0}^{\tau} h(\sigma)d\sigma\right) d\tau \in Dom(\Psi^{-1})$.

Proof. *Let $t_0 \in I$, $u_0 \in \mathbb{R}^n$ and $u(., t_0, u_0)$ be the solution of the Cauchy problem (2.56). Then we have*

$$u(t, t_0, u_0) = u_0 + \int_{t_0}^{t} F_1(s, u(s, t_0, u_0), \int_{t_0}^{s} K_1(s, u(\tau, t_0, u_0))d\tau)ds. \tag{2.63}$$

Using (2.57) and (2.58) in (2.63), we have

$$\begin{aligned}\|u(t,t_0,u_0)\| &\leq \|u_0\| + \int_{t_0}^{t} f(s)\left[\|u(s,t_0,u_0)\| + \int_{t_0}^{s} \|K_1(s,u(\tau,t_0,u_0))\|\, d\tau\right] ds,\\ &\leq \|u_0\| + \int_{t_0}^{t} f(s)\left(\|u(s,t_0,u_0)\| + h(s)\int_{t_0}^{s} \Phi(\|u(\tau,t_0,u_0)\|)d\tau\right) ds. \quad (2.64)\end{aligned}$$

Now, a suitable application of Theorem 20 with $a(t) = \|u_0\|$, $f(t) = b(t) = 1$ *and* $W(u) = u$ *to (2.64) yields (2.59).* ■

If, in addition, we assume that the function F_1 satisfies the general condition

$$\|F_1(t,u,v)\| \leq f(t)\left(g\left(\|u\|\right) + W\left(\|v\|\right)\right), \quad (2.65)$$

where f , g and W are as defined in Theorem 25, we obtain an estimation for $u(.,t_0,u_0)$, and from any particular conditions of (2.65) and (2.57), we can get some useful results similar to Theorem 30.

Secondly, we shall demonstrate the usefulness of the inequality established in Theorem 29 by obtaining pointwise bounds on the solutions of a certain class of nonlinear equation in n-independent variables. We consider the nonlinear hyperbolic partial integrodifferential equation

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 ... \partial x_n} = F\left(x, u(x), \int_{x^0}^{x} K\left(x,s,u(s)\right) ds\right) + G\left(x,u(x)\right) \quad (2.66)$$

for all $x \in \mathbb{I} = [x^0; x^\infty] \subset \mathbb{R}_+^n$, where $x = (x_1, x_2, ..., x_n)$, $x^0 = (x_1^0, x_2^0, ..., x_n^0)$, $x^\infty = (x_1^\infty, x_2^\infty, ..., x_n^\infty)$ are in $\mathbb{R}_+^n$ and $u \in C(\mathbb{I},\mathbb{R})$, $F \in C(\mathbb{I}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $K \in C(\mathbb{I}\times\mathbb{I}\times\mathbb{R},\mathbb{R})$ and $G \in C(\mathbb{I}\times\mathbb{R},\mathbb{R})$. With suitable boundary conditions, the solution of (2.66) is of the form

$$u(x) = l(x) + \int_{x^0}^{x} F\left(s, u(s), \int_{x^0}^{s} K\left(s,t,u(t)\right) dt\right) ds + \int_{x^0}^{x} G\left(s,u(s)\right) ds. \quad (2.67)$$

The following theorem gives the bound of the solution of (2.66).

Theorem 31 *Assume that the functions $l, F,$ K and G in (2.66) satisfy the conditions*

$$|K(s,t,u(t))| \leq k(s,t)\Phi(|u(t)|), \quad t,s \in \mathbb{I} \text{ and } u \in \mathbb{R}, \tag{2.68}$$

$$|F(t,u,v)| \leq \frac{1}{2}|u| + |v|, \quad u,v \in \mathbb{R} \text{ and } t \in \mathbb{I}, \tag{2.69}$$

$$|G(s,u)| \leq \frac{1}{2}|u|, \quad s \in \mathbb{I} \text{ and } u \in \mathbb{R}, \tag{2.70}$$

$$|l(x)| \leq a(x), \qquad x \in \mathbb{I}, \tag{2.71}$$

where a, f, k and Φ are as defined in Theorem 20, with $f(x) = b(x) + e(x)$ for all $x \in \mathbb{I}$ where $b, e \in C(\mathbb{I}, \mathbb{R}_+)$, then we have the estimate, for $x^0 \leq x \leq x^*$

$$|u(x)| \leq \exp\left(\prod_{i=1}^{n}(x_i - x_i^0)\right)\left(a(x) + \int_a^x E(t)dt\right). \tag{2.72}$$

Here

$$E(t) = \Psi^{-1}\left(\Psi(\eta) + \int_a^t k(x^\infty, s)\Phi\left[\exp\left(\prod_{i=1}^{n}(s_i - x_i^0)\right)\int_a^s f(\tau)\,d\tau\right]ds\right), \tag{2.73}$$

$$\eta = \int_{x^0}^{x^\infty} k(x^\infty, s)\,\Phi\left(a(s)\exp\left(\prod_{i=1}^{n}(s_i - x_i^0)\right)\right)ds, \tag{2.74}$$

$$\Psi(x) = \int_{x^0}^{x} \frac{ds}{\Phi(s)}, \quad x \geq x^0 > 0, \tag{2.75}$$

where $x^*$is chosen so that $\Psi(\eta) + \int_a^t k(x^\infty, s)\Phi\left[\exp\left(\prod_{i=1}^{n}(s_i - x_i^0)\right)\int_a^s f(\tau)\,d\tau\right]ds$,is in the domain of Ψ^{-1}.

Proof. *Using the conditions (2.68), (2.71) in (2.67), we have*

$$\begin{aligned} |u(x)| &\leq a(x) + \int_{x^0}^{x} |G(s,u(s))|\,ds + \int_{x^0}^{x} f(s)\left[|u(s)| + \int_{x^0}^{s} |K(s,t,u(t))|\,dt\right]ds, \\ &\leq a(x) + \int_{x^0}^{x}\left(|u(s)| + \int_{x^0}^{s} k(s,t)\Phi(|u(t)|)dt\right)ds. \end{aligned} \tag{2.76}$$

Now, a suitable application of Theorem 29 with $f(s) = 1, g(u) = u$ *and* $W(u) = u$ *to (2.76) yields (2.72)* ■

Remarks. If we assume that the functions F and G satisfy the general conditions

$$|F(t,u,v)| \leq f(t)\left(g\left(|u|\right) + W\left(|v|\right)\right), \tag{2.77}$$

$$|G\left(t,u\right)| \leq f(t)g\left(|u|\right), \quad \text{for } t \in \mathbb{I} \text{ and } u \in \mathbb{R}, \tag{2.78}$$

we can obtain an estimation of $u(x)$.

From the particular conditions of (2.68), (2.77) and (2.78), we can obtain some useful results similar to Theorem 29. To save space, we omit the details here.

Remark 32 *Under some suitable conditions, the uniqueness and continuous dependence of the solutions of (2.55) and (2.66) can also be discussed using our results (see:[27]).*

Chapter 3

Some New Integrodifferential Ineqaulities

The study of integrodifferential inequalities for functions of one or n independent variables is also very important tool in the study of stability, existence, bounds and other qualitative properties of differential equations solutions', integrodifferential equations and in the theory of hyperbolic partial differential equations.we refer the reader to literatures ([1],[31],[33]) and the references therein. Our aim in this chapter is to establish some integrodifferential inequalities in n independent variables, an application of our results is also given.

3.1 Linear Integrodifferential Ineqaulities in One Variable

One of the most useful inequalities is given in the following lemma (see [1],[32])

Lemma 33 *[1]Let* $\Phi(x,y)$ *and* $c(x,y)$ *be nonnegative continuous functions defined for* $x \geq 0, y \geq 0$, *for which the inequality*

$$\Phi(x,y) \leq a(x) + b(y) + \int_0^x \int_0^y c(s,t)\Phi(s,t)dsdt, \tag{3.1}$$

holds for $x \geq 0, y \geq 0$, *where* $a(x), b(y) > 0$; $a'(x), b'(y) \geq 0$ *are continuous functions defined for* $x \geq 0, y \geq 0$. *Then*

$$\Phi(x,y) \leq \frac{[a(0)+b(y)][a(x)+b(0)]}{[a(0)+b(0)]} \exp\left(\int_0^x \int_0^y c(s,t)dsdt\right), \tag{3.2}$$

for $x \geq 0, y \geq 0$.

Wendroff's inequality has recently evoked a lively interest, as may be seen from the papers of Pachpatte [25]. In [32] Pachpatte considered on some new integrodifferential inequalities of the Wendroff type for functions of two independent variables.

In this section , we will give some linear integrodifferential ineqaulities in one variable.

Let first give the following result :

Theorem 34 *Let* $\Phi(x), c(x), D_i\Phi(x)$ *and* $D\Phi(x)$ *be nonnegative continuous functions for all* $i = 1, 2, ..., n$ *defined for* $x \in S$, $\Phi(x_1^0, x_2, x_3, ..., x_n) = 0$ *and* $\Phi(x_1, ..., x_{i-1}, x_i^0, x_{i+1}, ...x_n) = 0$ *for any* $i = 2, 3, ..., n$. *If*

$$D\Phi(x) \leq \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} c(t)[\Phi(t) + D\Phi(t)]dt, \tag{3.3}$$

holds for $x \in S$, where $a_i(x_i) > 0$; $a_i'(x_i) \geq 0$ are continuous functions defined for $x_i \geq 0$ for all $i = 1, 2, ..., n$. Then

$$D\Phi(x) \leq \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} c(t)[A(t)\exp(\int_{x^0}^{t}[1 + c(\tau)]d\tau)]dt. \tag{3.4}$$

For $x \in S$ with $x \geq t \geq \tau \geq x^0 \geq 0$, where $A(x)$ is defined above .

Proof. *We define the function*

$$u(x) = \sum_{i=1}^{n} a_i(x_i) + \int_{x_0}^{x} c(t)[\Phi(t) + D\Phi(t)]dt, \tag{3.5}$$

and

$$u(x_1^0, x_2, x_3, ..., x_n) = a_1(x_1^0) + \sum_{i=2}^{n} a_i(x_i). \tag{3.6}$$

Then,

$$D\Phi(x) \leq u(x). \tag{3.7}$$

Differentiating

$$Du(x) = c(x)[\Phi(x) + D\Phi(x)]. \tag{3.8}$$

By integrating b to x from x^0 to x , we have

$$\Phi(x) \leq \int_{x_0}^{x} u(t)dt. \tag{3.9}$$

We obtain

$$Du(x) \leq c(x)\left(u(x) + \int_{x^0}^{x} u(t)dt\right). \tag{3.10}$$

If we put

$$v(x) = u(x) + \int_{x^0}^{x} u(t)dt, \tag{3.11}$$

$$v(x_1, ..., x_{i-1}, x_i^0, x_{i+1}, ...x_n) = u(x_1, ..., x_{i-1}, x_i^0, x_{i+1}, ...x_n), \tag{3.12}$$

we have

$$Dv(x) = Du(x) + u(x).$$

Using the facts that $Du(x) \leq c(x)v(x)$ *and* $u(x) \leq v(x)$, *we have*

$$Dv(x) \leq [1 + c(x)]v(x). \tag{3.13}$$

$$D_1 D_2 ... D_{n-1} u(x_1, ..., x_{n-1}, x_n^0) = 0. \tag{3.14}$$

then

$$D_1 D_2 u(x_1, x_2, x_3^0, x_4, ..., x_n) = 0. \tag{3.15}$$

Continuing this process, we obtain

$$\begin{aligned} D_1 D_2 u(x) \quad \leq \quad & \int_{x_3^0}^{x_3} ... \int_{x_n^0}^{x_n} c(x_1, x_2, t_3, ..., t_n) A(x_1, x_2, t_{3,} ..., t_n) \\ & \exp\left(\int_{x^0}^{t} [1 + c(\tau)] d\tau\right) dt_n ... dt_3. \end{aligned} \tag{3.16}$$

we have

$$D_1 u(x_1, x_2^0, x_3, x_4, ..., x_n) = a_1'(x_1). \tag{3.17}$$

Integrating to x_2 *from* x_2^0 *to* x_2 *), we have*

$$\begin{aligned} D_1 u(x) \quad \leq \quad & a_1'(x_1) + \int_{x_2^0}^{x_2} \int_{x_3^0}^{x_3} ... \int_{x_n^0}^{x_n} c(x_1, t_2, ..., t_n) A(x_1, t_2, ..., t_n) \\ & \exp\left(\int_{x^0}^{t} [1 + c(\tau)] d\tau\right) dt_n ... dt_2. \end{aligned} \tag{3.18}$$

Integrating with respect to x_1 *from* x_1^0 *to* x_1, *and we have*

$$u(x) \leq \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} c(t) A(t) \exp\left(\int_{x^0}^{t} [1 + c(\tau)] d\tau\right) dt. \tag{3.19}$$

By the above inequalities above, we obtain the desired bound ■

Remark 35 *We note that in the special case* $n = 2$, $x \in \mathbb{R}^2_+$ *and* $x^0 = (x^0_1, x^0_2) = (0, 0)$ *in Theorem above. then our result reduces to Theorem 1 obtained in [32].*

Theorem 36 *Let* $\Phi(x), c(x), D_i\Phi(x)$ *and* $D\Phi(x)$ *be nonnegative continuous functions for all* $i = 1, 2, ..., n$ *defined for* $x \in S$, $\Phi(x^0_1, x_2, x_3, ..., x_n) = 0$ *and* $\Phi(x_1, ..., x_{i-1}, x^0_i, x_{i+1}, ...x_n) = 0$ *for any* $i = 2, 3, ..., n$. *If*

$$D\Phi(x) \le \sum_{i=1}^{n} a_i(x_i) + M\left[\Phi(x) + \int_{x^0}^{x} c(t)[\Phi(t) + D\Phi(t)]dt\right], \tag{3.20}$$

holds for $x \in S$, *where* $a_i(x_i) > 0$; $a'_i(x_i) \ge 0$ *are continuous functions defined for* $x_i \ge 0$ *for all* $i = 1, 2, ..., n$. *and* $M \ge 0$ *is constant. Then*

$$D\Phi(x) \le A(x)\exp\left(\int_{x^0}^{x} [M + c(t) + Mc(t)]dt\right),$$

for $x \in S$, *with* $x \ge t \ge x^o \ge 0$,

Proof. *We define the function*

$$u(x) = \sum_{i=1}^{n} a_i(x_i) + M\left[\Phi(x) + \int_{x^0}^{x} c(t)[\Phi(t) + D\Phi(t)]dt\right], \tag{3.21}$$

with

$$u(x^0_1, x_2, x_3, ..., x_n) = a_1(x^0_1) + \sum_{i=2}^{n} a_i(x_i). \tag{3.22}$$

we have

$$Du(x) = M\left[D\Phi(x) + c(x)[\Phi(x) + D\Phi(x)]\right]. \tag{3.23}$$

Using the fact that $D\Phi(x) \le u(x)$ *and* $M\Phi(x) \le u(x)$, *we have*

$$Du(x) \le [M + c(x) + Mc(x)]u(x), \tag{3.24}$$

we have

$$u(x) \le A(x)\exp\left(\int_{x^0}^{x} [M + c(t) + Mc(t)]dt\right), \tag{3.25}$$

we obtain the desired bound above) ■

Remark 37 *We note that in the special case* $n = 2$ *,*$x \in \mathbb{R}^2_+$ *and* $x^0 = (x^0_1, x^0_2) = (0, 0)$ *in Theorem above . then our result reduces to Theorem 2 obtained in [32] .*

3.2 linear Generalisations in Several Variables

In this section , All the functions which appear in the inequalities are assumed to be real valued of n-variables which are nonnegative and continuous. All integrals are assumed to exist on their domains of definitions.

Throughout this paper, we shall assume that S in any bounded open set in the dimensional Euclidean space $\mathbb{R}^n$ and that our integrals are on $\mathbb{R}^n (n \geq 1)$.

For $x = (x_1, x_2, ...x_n),\ t = (t_1, t_2, ...t_n),\ x^0 = (x^0_1, x^0_2, ..., x^0_n) \in S$, we shall denote

$$\int_{x^0}^{x} dt = \int_{x^0_1}^{x_1} \int_{x^0_2}^{x_2} ... \int_{x^0_n}^{x_n} ...dt_n...dt_1.$$

Furthermore, for $x, t \in \mathbb{R}^n$, we shall write $t \leq x$ whenever $t_i \leq x_i,\ i = 1, 2, ..., n$ and $x \geq x_0 \geq 0$, for $x, x^0 \in S$.

We note $D = D_1 D_2 ... D_n$, where $D_i = \frac{\partial}{\partial x_i}$, for $i = 1, 2, ..., n$.

We use the usual convention of writing $\sum_{s \in \Psi} u(s) = 0$ if Ψ is the empty set.

Our main results are given in the following theorems.

Theorem 38 *Let* $\Phi(x)$ *and* $c(x)$ *be nonnegative continuous functions defined on* S*, for which the inequality*

$$\Phi(x) \leq \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} c(t)\Phi(t)dt, \tag{3.26}$$

holds for all $x \in S$ *with* $x \geq x^0 \geq 0$, *where* $a_i(x_i) > 0$, $a_i'(x_i)$ *are continuous functions defined for* $x_i \geq 0$ *for all* $i = 1, 2, ..., n$. *Then*

$$\Phi(x) \leq A(x) \exp\Big(\int_{x^0}^{x} c(t)dt\Big), \tag{3.27}$$

for $x \in S$ *with* $x \geq x^0 \geq 0$, *where*

$$A(x) = \frac{\left[a_1(x_1) + a_2(x_2^0) + \sum_{s=3}^{n} a_s(x_s)\right]\left[a_1(x_1^0) + a_2(x_2) + \sum_{s=3}^{n} a_s(x_s)\right]}{\left[a_1(x_1^0) + a_2(x_2^0) + \sum_{s=3}^{n} a_s(x_s)\right]} \tag{3.28}$$

Proof. *We define the function* $u(x)$ *by the right member of (3.26), Then*

$$Du(x) = c(x)\Phi(x) \tag{3.29}$$

and

$$u(x_1^0, x_2, ..., x_n) = a_1(x_1^0) + a_2(x_2) + \sum_{s=3}^{n} a_s(x_s), \tag{3.30}$$

$$u(x_1, x_2^0, x_3, ..., x_n) = a_1(x_1) + a_2(x_2^0) + \sum_{s=3}^{n} a_s(x_s). \tag{3.31}$$

Using $\Phi(x) \leq u(x)$ *in (3.29), we have*

$$Du(x) \leq c(x)u(x). \tag{3.32}$$

From (3.32), we observe that

$$\frac{u(x)Du(x)}{u^2(x)} \leq c(x),$$

that is

$$\frac{u(x)Du(x)}{u^2(x)} \leq c(x) + \frac{(D_n u(x))(D_1 ... D_{n-1} u(x))}{u^2(x)},$$

hence

$$D_n\left(\frac{D_1 ... D_{n-1} u(x)}{u(x)}\right) \leq c(x). \tag{3.33}$$

Integrating (3.33) with respect to x_n from x_n^0 to x_n, we have

$$\frac{(D_1...D_{n-1}u(x))}{u(x)} \leq \int_{x_n^0}^{x_n} c(x_1,...,x_{n-1},t_n)dt_n,$$

thus

$$\frac{u(x)D_1...D_{n-1}u(x)}{u^2(x)} \leq \int_{x_n^0}^{x_n} c(x_1,...,x_{n-1},t_n)dt_n + \frac{(D_{n-1}u(x))(D_1...D_{n-2}u(x))}{u^2(x)},$$

that is

$$D_{n-1}\left(\frac{D_1...D_{n-2}u(x)}{u(x)}\right) \leq \int_{x_n^0}^{x_n} c(x_1,...,x_{n-1},t_n)dt_n. \tag{3.34}$$

Integrating (3.34) with respect to x_{n-1} from x_{n-1}^0 to x_{n-1}, we have

$$\frac{D_1...D_{n-2}u(x)}{u(x)} \leq \int_{x_{n-1}^0}^{x_{n-1}}\int_{x_n^0}^{x_n} c(x_1,...x_{n-2},t_{n-1},t_n)dt_n dt_{n-1}.$$

Continuing this process, we obtain

$$\frac{D_1D_2u(x)}{u(x)} \leq \int_{x_3^0}^{x_3}...\int_{x_n^0}^{x_n} c(x_1,x_2,t_{3,}...,t_n)dt_n dt_{n-1}....dt_3,$$

from this we obtain

$$D_2\left(\frac{D_1u(x)}{u(x)}\right) \leq \int_{x_3^0}^{x_3}...\int_{x_n^0}^{x_n} c(x_1,x_2,t_{3,}...,t_n)dt_n dt_{n-1}....dt_3. \tag{3.35}$$

Integrating (3.35) with respect to x_2 from x_2^0 to x_2 and by (3.31) we have

$$\begin{aligned}\frac{D_1u(x)}{u(x)} \quad \leq \quad & \frac{a_1'(x_1)}{a_2(x_0^0)+a_1(x_1)+\sum_{s=3}^n a_s(x_s)} \\ & + \int_{x_2^0}^{x_2}...\int_{x_n^0}^{x_n} c(x_1,t_2,t_{3,}...,t_n)dt_n dt_{n-1}....dt_2.\end{aligned} \tag{3.36}$$

Integrating (3.36) with respect to x_1 from x_1^0 to x_1 and by (3.30), we have

$$\log\frac{u(x)}{u(x_1^0,x_2,....,x_n)} \leq \int_{x_1^0}^{x_1}\frac{a_1'(t_1)}{a_2(x_2^0)+a_1(t_1)+\sum_{s=3}^n a_s(x_s)}dt_1 + \int_{x^0}^{x} c(t)dt,$$

that is

$$u(x) \leq A(x)\exp(\int_{x^0}^{x} c(t)dt). \tag{3.37}$$

By (3.37) and $\Phi(x) \leq u(x)$*, we obtain the desired bound in (3.27).* ■

Remark 39 *We note that in the special case* $n = 2$, $x \in \mathbb{R}_+^2$ *and* $x^0 = (x_1^0, x_2^0) = (0,0)$ *in Theorem 38. our estimate reduces to Lemma 33 (see: [32]).*

Theorem 40 *Let* $\Phi(x), c(x), D_i\Phi(x)$ *and* $D\Phi(x)$ *be nonnegative continuous functions for all* $i = 1, 2, ..., n$ *defined for* $x \in S$, $\Phi(x_1^0, x_2, x_3, ..., x_n) = 0$ *and* $\Phi(x_1, ..., x_{i-1}, x_i^0, x_{i+1}, ...x_n) = 0$ *for any* $i = 2, 3, ..., n$ *. If*

$$D\Phi(x) \leq \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} c(t)[\Phi(t) + D\Phi(t)]dt, \tag{3.38}$$

holds for $x \in S$, *where* $a_i(x_i) > 0$; $a_i'(x_i) \geq 0$ *are continuous functions defined for* $x_i \geq 0$ *for all* $i = 1, 2, ..., n$. *Then*

$$D\Phi(x) \leq \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} c(t)[A(t)\exp(\int_{x^0}^{t}[1 + c(\tau)]d\tau)]dt. \tag{3.39}$$

For $x \in S$ *with* $x \geq t \geq \tau \geq x^0 \geq 0$, *where* $A(x)$ *is defined in (3.28).*

Proof. *We define the function*

$$u(x) = \sum_{i=1}^{n} a_i(x_i) + \int_{x_0}^{x} c(t)[\Phi(t) + D\Phi(t)]dt, \tag{3.40}$$

and

$$u(x_1^0, x_2, x_3, ..., x_n) = a_1(x_1^0) + \sum_{i=2}^{n} a_i(x_i). \tag{3.41}$$

Then, (3.38) can be restated as

$$D\Phi(x) \leq u(x). \tag{3.42}$$

Differentiating (3.40)

$$Du(x) = c(x)[\Phi(x) + D\Phi(x)]. \tag{3.43}$$

Integrating both sides of (3.43) to x *from* x^0 *to* x *, we have*

$$\Phi(x) \le \int_{x_0}^{x} u(t)dt. \tag{3.44}$$

Now, using (3.44) and (3.42) in (3.43) we obtain

$$Du(x) \le c(x)\left(u(x) + \int_{x^0}^{x} u(t)dt\right). \tag{3.45}$$

If we put

$$v(x) = u(x) + \int_{x^0}^{x} u(t)dt, \tag{3.46}$$

$$v(x_1, ..., x_{i-1}, x_i^0, x_{i+1}, ...x_n) = u(x_1, ..., x_{i-1}, x_i^0, x_{i+1}, ...x_n), \tag{3.47}$$

then by (3.46), we have

$$Dv(x) = Du(x) + u(x).$$

Using the facts that $Du(x) \le c(x)v(x)$ *and* $u(x) \le v(x)$*, we have*

$$Dv(x) \le [1 + c(x)]v(x). \tag{3.48}$$

Which by following an argument to that in the proof of Theorem 38 yields the estimate for $v(x)$ *such that*

$$v(x) \le A(x)\exp\left(\int_{x^0}^{x} [1 + c(t)]dt\right). \tag{3.49}$$

By (3.49) and (3.45), we have

$$Du(x) \le c(x)A(x)\exp\left(\int_{x^0}^{x} [1 + c(t)]dt\right), \tag{3.50}$$

and

$$D_1 D_2 ... D_{n-1} u(x_1, ..., x_{n-1}, x_n^0) = 0. \tag{3.51}$$

Integrating both sides of (3.50) to x_n *from* x_n^0 *to* x_n *and by (3.51), we have*

$$D_1 D_2 ... D_{n-1} u(x) \leq \int_{x_n^0}^{x_n} c(x_1, ...x_{n-1}, t_n) A(x_1, ...x_{n-1}, t_n) \exp\left(\int_{x^0}^{t} [1 + c(\tau)] d\tau \right) dt_n.$$

By (3.40), we have

$$D_1 D_2 u(x_1, x_2, x_3^0, x_4, ..., x_n) = 0. \tag{3.52}$$

Continuing this process, and by (3.52), we obtain

$$\begin{aligned} D_1 D_2 u(x) \;\; \leq \;\; & \int_{x_3^0}^{x_3} ... \int_{x_n^0}^{x_n} c(x_1, x_2, t_{3,}..., t_n) A(x_1, x_2, t_{3,}..., t_n) \\ & \exp\left(\int_{x^0}^{t} [1 + c(\tau)] d\tau \right) dt_n ... dt_3. \end{aligned} \tag{3.53}$$

By (3.40), we have

$$D_1 u(x_1, x_2^0, x_3, x_4, ..., x_n) = a_1'(x_1). \tag{3.54}$$

Integrating both sides of (3.53) to x_2 *from* x_2^0 *to* x_2 *and by (3.54), we have*

$$\begin{aligned} D_1 u(x) \;\; \leq \;\; & a_1'(x_1) + \int_{x_2^0}^{x_2} \int_{x_3^0}^{x_3} ... \int_{x_n^0}^{x_n} c(x_1, t_2, ..., t_n) A(x_1, t_2, ..., t_n) \\ & \exp\left(\int_{x^0}^{t} [1 + c(\tau)] d\tau \right) dt_n ... dt_2. \end{aligned} \tag{3.55}$$

Integrating (3.55) with respect to x_1 *from* x_1^0 *to* x_1*, and by (3.41), we have*

$$u(x) \leq \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} c(t) A(t) \exp\left(\int_{x^0}^{t} [1 + c(\tau)] d\tau \right) dt. \tag{3.56}$$

By (3.56) and (3.42), we obtain the desired bound in (3.39) ■

Remark 41 *We note that in the special case* $n = 2$, $x \in \mathbb{R}_+^2$ *and* $x^0 = (x_1^0, x_2^0) = (0, 0)$ *in Theorem 40. then our result reduces to Theorem 1 obtained in [32].*

Theorem 42 *Let* $\Phi(x), c(x), D_i\Phi(x)$ *and* $D\Phi(x)$ *be nonnegative continuous functions for all* $i = 1, 2, ..., n$ *defined for* $x \in S$, $\Phi(x_1^0, x_2, x_3, ..., x_n) = 0$ *and* $\Phi(x_1, ..., x_{i-1}, x_i^0, x_{i+1}, ...x_n) = 0$ *for any* $i = 2, 3, ..., n$. *If*

$$D\Phi(x) \leq \sum_{i=1}^{n} a_i(x_i) + M\left[\Phi(x) + \int_{x^0}^{x} c(t)[\Phi(t) + D\Phi(t)]dt\right], \tag{3.57}$$

holds for $x \in S$, *where* $a_i(x_i) > 0$; $a_i'(x_i) \geq 0$ *are continuous functions defined for* $x_i \geq 0$ *for all* $i = 1, 2, ..., n$. *and* $M \geq 0$ *is constant. Then*

$$D\Phi(x) \leq A(x) \exp\left(\int_{x^0}^{x} [M + c(t) + Mc(t)]dt\right), \tag{3.58}$$

for $x \in S$, *with* $x \geq t \geq x^o \geq 0$, *where* $A(x)$ *is defined in (3.28).* ***Proof.*** *We define the function*

$$u(x) = \sum_{i=1}^{n} a_i(x_i) + M\left[\Phi(x) + \int_{x^0}^{x} c(t)[\Phi(t) + D\Phi(t)]dt\right], \tag{3.59}$$

with

$$u(x_1^0, x_2, x_3, ..., x_n) = a_1(x_1^0) + \sum_{i=2}^{n} a_i(x_i). \tag{3.60}$$

Differentiating (3.59), we have

$$Du(x) = M\left[D\Phi(x) + c(x)[\Phi(x) + D\Phi(x)]\right]. \tag{3.61}$$

Using the fact that $D\Phi(x) \leq u(x)$ *and* $M\Phi(x) \leq u(x)$, *we have*

$$Du(x) \leq [M + c(x) + Mc(x)]u(x), \tag{3.62}$$

by (3.62), we have

$$u(x) \leq A(x) \exp\left(\int_{x^0}^{x} [M + c(t) + Mc(t)]dt\right), \tag{3.63}$$

where $A(x)$ is defined in (3.28).

By (3.63) and using the fact that $D\Phi(x) \le u(x)$ from (3.57), we obtain the desired bound in (3.58) ■

Remark 43 *We note that in the special case $n = 2$,$x \in \mathbb{R}_+^2$ and $x^0 = (x_1^0, x_2^0) = (0,0)$ in Theorem 42. then our result reduces to Theorem 2 obtained in [32] .*

Theorem 44 *Let $\Phi(x), p(x)$, and $q(x)$ be nonnegative continuous functions defined for $x \in S$. If*

$$\Phi(x) \le \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} p(t)\Phi(t)dt + \int_{x^0}^{x} p(t)\left(\int_{x^0}^{t} q(s)\Phi(s)ds\right)dt,$$

holds for $x \ge x^0 \ge 0$, where $a_i(x_i) > 0$; $a_i'(x_i) \ge 0$ are continuous functions defined for $x_i \ge 0$ for all $i = 1, 2, ..., n$. Then

$$\Phi(x) \le \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} p(t)Q(t)dt,$$

for all $x \ge x^0 \ge 0$, where

$$Q(x) = A(x)\exp\left(\int_{x^0}^{x} (p(t) + q(t))dt\right),$$

with $A(x)$ defined in (3.28).

Proof. *The proof of this Theorem follows by an argument similar to that in Theorem 38, We omit the details.* ■

Remark 45 *We note that in the special case $n = 2$,$x \in \mathbb{R}_+^2$ and $x^0 = (x_1^0, x_2^0) = (0,0)$ in Theorem 44, our result reduces to Theorem 2 obtained in [32].*

3.3 Nonlinear Integrodifferential in n-independents Variables

In this section we will give some new nonlinear integrodifferential inequalities for the functions of n-independent variables.

We can also give the following lemma.

Lemma 46 *[10]Let $u(x), a(x)$ and $b(x)$ be nonnegative continuous functions, defined for $x \in S$.*

Assume that $a(x)$ is positive, continuous function and nondecreasing in each of the variables $x \in S$. If

$$u(x) \leq a(x) + \int_{x^0}^{x} b(t)u(t)dt, \tag{3.64}$$

holds for all $x \in S$, with $x \geq x^0 \geq 0$. Then

$$u(x) \leq a(x) \exp(\int_{x^0}^{x} b(t)dt). \tag{3.65}$$

Theorem 47 *Let $\Phi(x), a(x), b(x), c(x), f(x), D_i\Phi(x)$, and $D\Phi(x)$ be nonnegative continuous functions for all $i = 1, 2, ..., n$ defined for $x \in S$, $\Phi(x_1^0, x_2, x_3, ..., x_n) = 0$ and $\Phi(x_1, ..., x_{i-1}, x_i^0, x_{i+1}, ...x_n) = 0$ for any $i = 2, 3, ..., n$. Let $K(\Phi(x))$ be real-valued, positive, continuous, strictly non-decreasing, subadditive and submultiplicative function for $\Phi(x) \geq 0$ and let $H(\Phi(x))$ be real-valued, continuous positive and non-decreasing function defined for $x \in S$. Assume that $a(x), f(x)$ are positive and* *nondecreasing in each of the variables* $x \in S$. *If*

$$D\Phi(x) \le a(x) + f(x)H\left(\int_{x^0}^{x} c(t)K(\Phi(t))dt\right) + \int_{x^0}^{x} b(t)D\Phi(t)dt, \tag{3.66}$$

holds ,for $x \in S$ *with* $x \ge x^0 \ge 0$. *Then*

$$\begin{aligned} D\Phi(x) \;\le\; & \left\{a(x) + f(x)H\left(G^{-1}\left[G(\xi) + \int_{x_0}^{x} c(t)K(p(t)f(t))dt\right]\right)\right\} \\ & \times \exp\left(\int_{x^0}^{x} b(t)dt\right), \end{aligned} \tag{3.67}$$

for $x \in S$,*where*

$$p(x) = \int_{x^0}^{x} \exp(\int_{x^0}^{t} b(s)ds)dt, \tag{3.68}$$

$$\xi = \int_{x_0}^{\infty} c(t)K(a(t)p(t))dt, \tag{3.69}$$

$$G(z) = \int_{z^0}^{z} \frac{ds}{K(H(s))}, \quad z \ge z^0 > 0. \tag{3.70}$$

Where G^{-1} *is the inverse function of* G, *and*

$$G(\xi) + \int_{x_0}^{x} c(t)K(p(t)f(t))dt,$$

is in the domain of G^{-1} *for* $x \in S$.

Proof. *We define the function*

$$z(x) = a(x) + f(x)H\left(\int_{x^0}^{x} c(t)K(\Phi(t))dt\right), \tag{3.71}$$

then (3.28) can be restated

$$D\Phi(x) \le z(x) + \int_{x^0}^{x} b(t)D\Phi(t)dt. \tag{3.72}$$

Clearly $z(x)$ *is positive, continuous function and nondecreasing in each of the variables* $x \in S$, *using (3.64) of lemma 46 to (3.72), we have*

$$D\Phi(x) \le z(x)\exp\left(\int_{x^0}^{x} b(t)dt\right). \tag{3.73}$$

Integrating to x from x^0 to x, we have

$$\Phi(x) \leq z(x)p(x), \tag{3.74}$$

where

$$p(x) = \int_{x^0}^{x} \exp\left(\int_{x^0}^{t} b(s)ds\right) dt. \tag{3.75}$$

By (3.71), we have

$$z(x) = a(x) + f(x)H\left(v(x)\right), \tag{3.76}$$

where

$$v(x) = \int_{x^0}^{x} c(t)K(\Phi(t))dt. \tag{3.77}$$

By (3.74) and (3.77), we have

$$\Phi(x) \leq \left\{a(x) + f(x)H\left(v(x)\right)\right\} p(x). \tag{3.78}$$

From (3.78), (3.77) and since K is subadditive and submultiplicative function, we notice that

$$\begin{aligned} v(x) &\leq \int_{x^0}^{x} c(t)K\left[\left\{a(t) + f(t)H\left(v(t)\right)\right\} p(t)\right] dt, \\ &\leq \int_{x^0}^{x} c(t)K(a(t)p(t))dt \\ &\quad + \int_{x^0}^{x} c(t)K(f(t)p(t))K(H(v(t)))dt, \\ &\leq \int_{x^0}^{\infty} c(t)K(a(t)p(t))dt \\ &\quad + \int_{x^0}^{x} c(t)K(f(t)p(t))K(H(v(t)))dt. \end{aligned} \tag{3.79}$$

We define $\Psi(x)$ as the right side of (3.78), then

$$\Psi(x_1^0, x_2, x_3, ..., x_n) = \int_{x^0}^{\infty} c(t)K(a(t)p(t))dt, \tag{3.80}$$

and

$$v(x) \leq \Psi(x). \tag{3.81}$$

$\Psi(x)$ *is positive nondecreasing in each of the variables* $x_2, ..., x_n \in R_+^{n-1}$, *then*

$$\begin{aligned} D_1\Psi(x) &= \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}...\int_{x_n^0}^{x_n} c(x_1, t_2, ..., t_n) \\ &\quad \times K\left(p(x_1, t_2, ..., t_n) f(x_1, t_2, ..., t_n)\right) K\left(H(v(x_1, t_2, ..., t_n))\right) dt_n...dt_2, \\ &\leq \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}...\int_{x_n^0}^{x_n} d(x_1, t_2, ..., t_n) \\ &\quad \times K\left(p(x_1, t_2, ..., t_n) f(x_1, t_2, ..., t_n)\right) K\left(H(\Psi(x_1, t_2, ..., t_n))\right) dt_n...dt_2. \qquad (3.82) \\ &\leq K\left(H(\Psi(x))\right) \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}...\int_{x_n^0}^{x_n} c(x_1, t_2, ..., t_n) \\ &\quad \times K\left(p(x_1, t_2, ..., t_n) f(x_1, t_2, ..., t_n)\right) dt_n...dt_2. \end{aligned}$$

Dividing both sides of (3.82) by $K\left(H(\Psi(x))\right)$, *we get*

$$\begin{aligned} \frac{D_1\Psi(x)}{K\left(H(\Psi(x))\right)} &\leq \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}...\int_{x_n^0}^{x_n} c(x_1, t_2, ..., t_n) \\ &\quad K\left(p(x_1, t_2, ..., t_n) f(x_1, t_2, ..., t_n)\right) dt_n...dt_2. \end{aligned} \tag{3.83}$$

We note that, for

$$G(z) = \int_{z^0}^{z} \frac{ds}{K(H(s))}, \quad z \geq z^0 > 0. \tag{3.84}$$

Thus it follows that

$$D_1 G(\Psi(x)) = \frac{D_1\Psi(x)}{K(H(\Psi(x)))}. \tag{3.85}$$

From (3.83), (3.84) and (3.85), we have

$$\begin{aligned} D_1 G(\Psi(x)) &\leq \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}...\int_{x_n^0}^{x_n} c(x_1, t_2, ..., t_n) \\ &\quad \times K\left(p(x_1, t_2, ..., t_n) f(x_1, t_2, ..., t_n)\right) dt_n...dt_2. \end{aligned} \tag{3.86}$$

Now setting $x_1 = s$ *in (3.86) and then integrating with respect from* x_1^0 *to* x_1*, we obtain*

$$G(\Psi(x)) \leq G(\Psi(x_1^0, x_2, ..., x_n)) + \int_{x^0}^{x} c(t)K(p(t)f(t))dt, \tag{3.87}$$

by (3.87), we have

$$\Psi(x) \leq G^{-1}\left[G\left(\int_{x_0}^{\infty} c(t)K(a(t)p(t))dt\right) + \int_{x_0}^{x} c(t)K(p(t)f(t))dt\right]. \tag{3.88}$$

The required inequality in (3.67) follows from the fact (3.73), (3.76), (3.81) and (3.88)

■

Many interesting corollaries can be obtained from Theorem 47.

Corollary 48 *Let* $\Phi(x), a(x), b(x), c(x), D_i\Phi(x)$, $D\Phi(x)$ *and* $K(\Phi(x))$ *be as defined in Theorem 47. If*

$$D\Phi(x) \leq a(x) + \int_{x^0}^{x} c(t)g(\Phi(t))dt) + \int_{x^0}^{x} b(t)D\Phi(t)dt,$$

holds, for $x \in \mathbb{R}_+^n$ *with* $x \geq x^0 \geq 0$*. Then*

$$\begin{aligned} D\Phi(x) \quad \leq \quad & \left\{a(x) + T^{-1}\left[T(\xi) + \int_{x_0}^{x} c(t)K(p(t))dt\right]\right\} \\ & \times \exp\left(\int_{x^0}^{x} b(t)dt\right), \end{aligned}$$

for $x \in \mathbb{R}_+^n$ *with* $x \geq x^0 \geq 0$*, where*

$$\begin{aligned} p(x) &= \int_{x^0}^{x} \exp(\int_{x^0}^{t} b(s)ds)dt, \\ \xi &= \int_{x_0}^{\infty} c(t)K(a(t)p(t))dt, \\ T(z) &= \int_{z^0}^{z} \frac{ds}{K(s)}, \quad z \geq z^0 > 0, \end{aligned}$$

where T^{-1} is the inverse function of T and

$$T(\xi) + \int_{x_0}^{x} c(t)K(p(t))dt,$$

is in the domain of T^{-1} for $x \in \mathbb{R}_+^n$.

Corollary 49 *Let $\Phi(x), b(x), c(x), D_i\Phi(x)$and $D\Phi(x)$ be as defined in Theorem 47. If*

$$D\Phi(x) \leq M + \int_{x^0}^{x} c(t)\Phi(t)dt + \int_{x^0}^{x} b(t)D\Phi(t)dt,$$

holds, for $x \in \mathbb{R}_+^n$ with $x \geq x^0 \geq 0$, where $M > 0$ is a constant, then

$$\Phi(x) \leq M\left\{1 + \exp\left[\log(\int_{x_0}^{\infty} c(t)p(t)dt) + \int_{x_0}^{x} c(t)p(t)dt\right]\right\}p(x),$$

for $x \in \mathbb{R}_+^n$ with $x \geq x^0 \geq 0$,where

$$p(x) = \int_{x^0}^{x} \exp(\int_{x^0}^{t} b(s)ds)dt$$

Similarly, we can obtain many other kinds of estimates.

Remark 50 *Our results can be generalized to integrodifferential inequalities with a time delay for functions of one or n independent variables, this is under study.*

3.4 An Application

In this section we present an immediate simple example of application (Theorem 47) to study the boundless of the solution of partial integrodifferential equation.

Consider the nonlinear partial integrodifferential equation

$$\begin{cases} Du(x) = f(x) + \int_0^x h(x,t,u(t),Du(t))dt \\ u(...,x_i,0,x_{i+2},...) = 0 \text{ for all } i = 1,2,...,n \end{cases}, \tag{3.89}$$

for $x \in \mathbb{R}_+^n$ .Were $h : \mathbb{R}_+^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(x) : \mathbb{R}_+^n \to \mathbb{R}$ are continuous functions. Assume that these functions are defined and continuous on their respective domains of definition such that

$$|f(x)| \leq M, \tag{3.90}$$

and

$$|h(x,t,u(t),Du(t))| \leq c(t)\,|u(t)| + b(t)\,|Du(t)|\,, \tag{3.91}$$

for $x \in \mathbb{R}_+^n$, where $M > 0$ is a constant and $c(x), b(x)$ are nonnegative, continuous functions defined for $x \in \mathbb{R}_+^n$. If $\Phi(x)$ is any solution of the boundary value problem (3.89), then

$$D\Phi(x) = f(x) + \int_0^x h(x,t,\Phi(t),D\Phi(t))dt, \tag{3.92}$$

for $x \in \mathbb{R}_+^n$, by (3.90) and (3.91) we have

$$|D\Phi(x)| = M + \int_0^x c(t)\,|\Phi(x)| + b(t)\,|D\Phi(x)|\,dt. \tag{3.93}$$

Now by a suitable application of Corollary 49 of Theorem 47, we obtain the bound on the solution $\Phi(x)$ of (3.89).

$$|\Phi(x)| \leq Mp(x)\left\{1 + \exp\left[\log(\int_0^\infty c(t)p(t)dt) + \int_0^x c(t)p(t)dt\right]\right\}, \tag{3.94}$$

for $x \in \mathbb{R}_+^n$, where

$$p(x) = \int_0^x \exp(\int_0^t b(s)ds)dt.$$

Chapter 4

Some New Integral Ineqaulities With Delay

The purpose of this chapter is to establish some nonlinear retarded integral inequalities in the case of functions of n independent variables which can be used as handy tools in the theory of partial differential and integral equations with time delays. These new inequalities represent a generalization of the results obtained by Ma and Pecaric [23], Pachpatte [29] and by Cheung [5] in the case of the functions with one and two variables.

4.1 Introduction

Now, let us first list the main results of [23, 29, 5], for the functions with two variables for $u(x, y) \in (\Delta \in \mathbb{R}_+^2, \mathbb{R}_+)$, as the following:

A Inequality of Ma and Pecaric [23, Theorem 2.1] :

$$\begin{aligned} u^p(x,y) &= k + \sum_{i=1}^{m} \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s,t) u^q(s,t) dtds \\ &+ \sum_{j=1}^{n} \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s,t) u^q(s,t) w(u(s,t)) dtds. \end{aligned} \tag{4.1}$$

B Pachpatte's inequality [29, Theorem 4] :

$$\begin{aligned} u^p(x,y) &= k + \int_{x_0}^{x} \int_{y_0}^{y} a(s,t) g_1(u(s,t)) dtds \\ &+ \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(s,t) g_2(u(s,t)) dtds. \end{aligned} \tag{4.2}$$

C Cheung's inequality [5, Theorem 2.4] :

$$\begin{aligned} u^p(x,y) &= k + \frac{p}{p-q} \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) u^q(s,t) dtds \\ &+ \int_{\gamma(x_0)}^{\gamma(x)} \int_{\gamma(y_0)}^{\delta(y)} b(s,t) u^q(s,t) \varphi(u(s,t)) dtds. \end{aligned} \tag{4.3}$$

However, sometimes we need to study such inequalities with a function $c(x)$ in place of the constant term k and for functions of several variables .

Our main result is given in the flowing inequality in the case of functions with n independent variables :

$$\begin{aligned} \varphi(u(x)) &\leq c(x) + \sum_{j=1}^{n_1} d_j(x) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t) \Phi(u(t)) w_1(u(t)) dt \\ &+ \sum_{k=1}^{n_2} l_k(x) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t) \Phi(u(t)) w_2(u(t)) dt, \end{aligned} \tag{4.4}$$

in a general form, where $c(x)$ is a function and all the functions which appear in this inequality are assumed to be real valued of $n-$variables.

Furthermore, we show that the results (4.1)- (4.3) can be deduced from our inequality (4.4) in some special cases. As applications we give the estimate solution of retarded partial differential equation.

We note that the inequality (4.4) is also a generalization of the main results in [21, 35].

In this chapter, we suppose $\mathbb{R}^n_+ = [0, \infty)$ a subset of $\mathbb{R}^n$. All the functions which appear in the inequalities are assumed to be real valued of n-variables which are nonnegative and continuous. All integrals are assumed to exist on their domains of definitions.

For $x = (x_1, x_2, ...x_n),\ t = (t_1, t_2, ...t_n),\ x^0 = (x_1^0, x_2^0, ..., x_n^0) \in \mathbb{R}^n_+$, we shall denote :

$$\begin{aligned}
\int_{\widetilde{\alpha}_i(x^0)}^{\widetilde{\alpha}_i(x)} dt &= \int_{\alpha_{j1}(x_1^0)}^{\alpha_{j1}(x_1)} \int_{\alpha_{j2}(x_2^0)}^{\alpha_{j2}(x_2)} \cdots \int_{\alpha_{jn}(x_n^0)}^{\alpha_{jn}(x_n)} ...dt_n...dt_1, \quad j = 1, 2, ..., n_1, \\
\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} dt &= \int_{\beta_{k1}(x_1^0)}^{\beta_{k1}(x_1)} \int_{\beta_{k2}(x_2^0)}^{\beta_{k2}(x_2)} \cdots \int_{\beta_{kn}(x_n^0)}^{\beta_{kn}(x_n)} ...dt_n...dt_1, \quad k = 1, 2, ..., n_2,
\end{aligned}$$

with $n_1, n_2 \in \{1, 2, ...,\} : .$

Furthermore, for $x, t \in \mathbb{R}^n_+$, we shall write $t \leq x$ whenever $t_i \leq x_i,\ i = 1, 2, ..., n$ and $x \geq x_0 \geq 0$, for $x, x^0 \in \mathbb{R}^n_+$.

We note $D = D_1 D_2 ... D_n$, where $D_i = \frac{\partial}{\partial x_i}$, for $i = 1, 2, ..., n$,

We use the usual convention of writing $\sum_{s \in \Psi} u(s) = 0$ if Ψ is the empty set.

$\widetilde{\alpha}_j(t) = (\alpha_{j1}(t_1), \alpha_{j2}(t_2), ..., \alpha_{jn}(t_n)) \in \mathbb{R}^n_+$ for $j = 1, 2, ..., n_1$

$\widetilde{\beta}_k(t) = (\alpha_{k1}(t_1), \alpha_{k2}(t_2), ..., \alpha_{kn}(t_n)) \in \mathbb{R}^n_+$ for $k = 1, 2, ..., n_1$

we note $\widetilde{\alpha}_j(t) \leq t$ for $j = 1, 2, ..., n_1$ whenever $\alpha_{ji}(t_i) \leq t_i$ for any $i = 1, 2, ..., n$ and $j = 1, 2, ..., n_1$, and $\widetilde{\beta}_k(t) \le t$ for $k = 1, 2, ..., n_2$ whenever $\beta_{ki}(t_i) \le t_i$ for any $i = 1, 2, ..., n$ and $k = 1, 2, ..., n_2$

4.2 Generalization of the Integral Ineqaulities With Delay

Now, let us list the our main results as the following :

Theorem 51 *Let $c \in C(\mathbb{R}_+^n, \mathbb{R}_+), w_1, w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $w_1(u), w_2(u) > 0$ on $(0, \infty)$ and let $a_j(x,t)$ and $b_k(x,t) \in C(\mathbb{R}_+^n \times \mathbb{R}_+^n, \mathbb{R}_+)$ be nondecreasing functions in x for every t fixed for any $j = 1, 2, ..., n_1,\ k = 1, 2, ..., n_2$. Let $\alpha_{ji}, \beta_{ki} \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\alpha_{ji}(t_i) \le t_i$ and $\beta_{ki}(t_i) \le t_i$ on $\mathbb{R}_+$ for $i = 1, 2, ..., n;\ j = 1, 2, ..., n_1,\ k = 1, 2, ..., n_2$ and $p > q \ge 0$.*

(a_1) *If $u \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ and*

$$\begin{aligned} u^p(x) \ &\le \ c(x) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t) u^q(t) dt \\ &\quad + \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t) u^q(t) w_1(u(t)) dt, \end{aligned} \tag{4.5}$$

for any $x \in \mathbb{R}_+^n$ with $x^0 \le t \le x$, then there exists $x^* \in \mathbb{R}_+^n$, such that for all $x^0 \le t \le x^*$, we have

$$u(x) \le \left(\Psi_1^{-1} \left[\Psi_1(p(x)) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t) dt \right] \right)^{\frac{1}{p-q}}. \tag{4.6}$$

Where

$$p(x) = c^{(p-q)/p}(x) + \frac{p-q}{p} \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t) dt, \tag{4.7}$$

and

$$\Psi_1(\delta) = \int_{\delta_0}^{\delta} \frac{ds}{w_1(s^{\frac{1}{p-q}})}, \qquad \delta > \delta_0 > 0. \tag{4.8}$$

Here, Ψ^{-1} *is the inverse function of* Ψ*, and the real numbers* x^* *are chosen so that* $\Psi_1(p(x)) + \frac{p-q}{p}\sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)dt \in dom(\Psi_1^{-1})$.

(a$_2$***)*** *If* $u \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ *and*

$$\begin{aligned} u^p(x) \quad &\leq \quad c(x) + \sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t)u^q(t)w_1(u(t))dt \\ &+ \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)u^q(t)w_2(u(t))dt. \end{aligned} \tag{4.9}$$

(i) For the case $w_2(u) \leq w_1(u)$*, for any* $x \in \mathbb{R}_+^n$ *with* $x^0 \leq t \leq x$*, then there exists* $\xi_1 \in \mathbb{R}_+^n$*, such that for all* $x^0 \leq t \leq \xi_1$*, we have*

$$u(x) \leq \left(\Psi_1^{-1}\left(\Psi_1(c^{(p-q)/p}(x)) + e(x)\right)\right)^{\frac{1}{p-q}}.$$

(ii) For the case $w_1(u) \leq w_2(u)$*, for any* $x \in \mathbb{R}_+^n$ *with* $x^0 \leq t \leq x$*, then there exists* $\xi_2 \in \mathbb{R}_+^n$*, such that for all* $x^0 \leq t \leq \xi_2$*, we have*

$$u(x) \leq \left(\Psi_2^{-1}\left(\Psi_2(c^{(p-q)/p}(x)) + e(x)\right)\right)^{\frac{1}{p-q}}.$$

Where

$$e(x) = \frac{p-q}{p}\left[\sum_{j=1}^{n_1}\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} a_j(x,t)dt + \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)dt\right], \tag{4.10}$$

$$\Psi_i(\delta) = \int_{\delta_0}^{\delta} \frac{ds}{w_i(s^{\frac{1}{p-q}})}, \qquad \delta > \delta_0 > 0, \quad for \quad i = 1, 2. \tag{4.11}$$

Here, Ψ_i^{-1} *is the inverse function of* Ψ_i *and the real numbers* ξ_i *are chosen so that* $\Psi_2(c^{(p-q)/p}(x)) + e(x) \in dom(\Psi_i^{-1})$ *for* $i = 1, 2$ *respectively.*

The proof of the theorem will be given in the next section.

Corollary 52 *Let the functions* u, c, w_1, a_j, b_k $(\ j = 1, 2, ..., n_1;\ k = , 2, ..., n_1)$ *and the constants* p, q *be defined as in Theorem 51 and*

$$\begin{aligned} u^p(x,y) &\leq c(x,y) + \sum_{j=1}^{n_1} \int_{\alpha_j(x_0)}^{\alpha_j(x)} \int_{\alpha_j(y_0)}^{\alpha_j(y)} a_j(x,y,s,t) u^q(s,t) ds dt \\ &+ \sum_{k=1}^{n_2} \int_{\beta_k(x_0)}^{\beta_k(x)} \int_{\beta_k(y_0)}^{\beta_k(y)} b_k(x,y,s,t) u^q(t) w_1(u(t)) dt, \end{aligned} \tag{4.12}$$

for any $(x, y) \in \mathbb{R}_+^2$ *with* $x_0 \leq s \leq x$ *and* $y_0 \leq t \leq y$, *then there exists* $(x^*, y^*) \in \mathbb{R}_+^n$, *such that for all* $x_0 \leq s \leq x^*$ *and* $y_0 \leq s \leq y^*$, *then*

$$u(x,y) \leq \left(\Psi^{-1} \left[\Psi(p_1(x,y)) + \frac{p-q}{p} B_1(x,y) \right] \right)^{\frac{1}{p-q}}. \tag{4.13}$$

Where

$$\begin{aligned} p_1(x,y) &= c^{(p-q)/p}(x,y) + \frac{p-q}{p} A_1(x,y), &(4.14)\\ A_1(x,y) &= \sum_{j=1}^{n_1} \int_{\alpha_j(x_0)}^{\alpha_j(x)} \int_{\alpha_j(y_0)}^{\alpha_j(y)} a_j(x,y,s,t) ds dt, &(4.15)\\ B_1(x,y) &= \sum_{k=1}^{n_2} \int_{\beta_k(x_0)}^{\beta_k(x)} \int_{\beta_k(y_0)}^{\beta_k(y)} b_k(x,y,s,t) ds dt, &(4.16) \end{aligned}$$

and

$$\Psi(\delta) = \int_{\delta_0}^{\delta} \frac{ds}{w_1(s^{\frac{1}{p-q}})}, \quad \delta > \delta_0 > 0. \tag{4.17}$$

Here, Ψ^{-1} *is the inverse function of* Ψ, *and the real numbers* (x^*, y^*) *are chosen so that* $\Psi(p_1(x,y)) + \frac{p-q}{p} B_1(x,y) \in dom(\Psi^{-1})$.

Remark 53 *Setting* $a_j(x,y,s,t) = a_j(s,t)$, $b_k(x,y,s,t) = b_k(s,t)$ *and* $c(x,y) = k \geq 0$ *in Corollary 52, we obtain Ma and Pecaric's Theorem 2.1 [23].*

Remark 54 *Define* $a_j(x,y,s,t) = \frac{p}{p-q}a_j(s,t)$, $b_k(x,y,s,t) = \frac{p}{p-q}b_k(s,t)$ $c(x,y) = k > 0$ *(Constant) and* $j = k = 1$ *in Corollary 52, we obtain Cheung's Theorem 2.4 [5].*

Remark 55 *Obviously, (4.1)–(4.3) are special cases of Theorem 51. So our result includes the main results in [23, 29, 5].*

Using Theorem 51, we can get some more generalized results as follows.

Theorem 56 *Let the functions* u, c, w_i, a_j, b_k $(i = 1,2,\ \ j = 1,2,...,n_1,\ k = ,2,...,n_1)$ *be defined as in Theorem 51. Moreover, let* $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be a strictly increasing function such that* $\lim_{x\to\infty}\varphi(x) = \infty$, *and let* $\Phi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing function with* $\Phi(x) > 0$ *for all* $x \in \mathbb{R}^n_+$

(b$_1$***)*** *If* $u \in C(\mathbb{R}^n_+, \mathbb{R}_+)$ *and*

$$\begin{aligned}\varphi(u(x)) \quad &\leq \quad c(x) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t)\Phi(u(t))dt \\ &\quad + \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)\Phi(u(t))w_1(u(t))dt \qquad (4.18)\end{aligned}$$

for any $x \in \mathbb{R}^n_+$ *with* $x^0 \leq t \leq x$, *then there exists* $x^* \in \mathbb{R}^n_+$, *such that for all* $x^0 \leq t \leq x^*$, *we have*

$$u(x) \leq \varphi^{-1}\left(G^{-1}\left[\Psi_1^{-1}\left(\Psi_1(\pi(x)) + B(x)\right)\right]\right) \qquad (4.19)$$

where

$$\pi(x) = G(c(x)) + A(x), \tag{4.20}$$

$$A(x) = \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t)dt, \tag{4.21}$$

$$B(x) = \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)dt, \tag{4.22}$$

and

$$G(x) = \int_{x_0}^{x} \frac{ds}{\Phi(\varphi^{-1}(s))}, \qquad x > x_0 > 0, \tag{4.23}$$

$$\Psi_i(\delta) = \int_{\delta_0}^{\delta} \frac{ds}{w_i(\varphi^{-1}(G^{-1}(s)))}, \qquad \delta > \delta_0 > 0, \quad i = 1,2. \tag{4.24}$$

The real number x^* is chosen so that $\Psi_1(\pi(x)) + B(x) \in dom(\Psi_1^{-1})$.

(b$_2$) *If $u \in C(\mathbb{R}^n_+, \mathbb{R}_+)$ and*

$$\varphi(u(x)) \leq c(x) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t)\Phi(u(t))w_1(u(t))dt + \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)\Phi(u(t))w_2(u(t))dt.$$

(i) For the case $w_2(u) \leq w_1(u)$, for any $x \in \mathbb{R}^n_+$ with $x^0 \leq t \leq x$, then there exists $\xi_1 \in \mathbb{R}^n_+$, such that for all $x^0 \leq t \leq \xi_1$, we have

$$u(x) \leq \varphi^{-1}\left(G^{-1}\left[\Psi_1^{-1}\left(\Psi_1(G(c(x))) + A(x) + B(x)\right)\right]\right).$$

(ii) For the case $w_1(u) \leq w_2(u)$, for any $x \in \mathbb{R}^n_+$ with $x^0 \leq t \leq x$, then there exists $\xi_2 \in \mathbb{R}^n_+$, such that for all $x^0 \leq t \leq \xi_2$, we have

$$u(x) \leq \varphi^{-1}\left(G^{-1}\left[\Psi_2^{-1}\left(\Psi_2(G(c(x))) + A(x) + B(x)\right)\right]\right).$$

Where A, B, G *and* $\Psi_i (i = 1, 2)$ *are defined in (4.21)-(4.24),* Ψ_i^{-1} *is the inverse function of* Ψ_i *and the real numbers* ξ_i *are chosen so that* $\Psi_i(G(c(x))) + A(x) + B(x) \in dom(\Psi_i^{-1})$ *for* $i = 1, 2$ *respectively.*

Many interesting corollaries can also obtained for the above theorms (in the case of one variable or in the case of n independent variables). For example :

Corollary 57 ***(Inequality in one variable)***

Let $p > q \geq 0,\quad c > 0$ *be some constants and* w_1, w_2 *be defined as in Theorem 51. Moreover, let* $a_j(x,t)$ *and* $b_k(x,t) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing functions in* x *for every* t *fixed and* $\alpha_j, \beta_k \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing functions with* $\alpha_j(t) \leq t$ *and* $\beta_k(t) \leq t_i$ *on* $\mathbb{R}_+$ *for* $j = 1, 2, ..., n_1,\ k = 1, 2, ..., n_2$ *for any* $j = 1, 2, ..., n_1,\ k = 1, 2, ..., n_2$

(c_1) *Let* $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and*

$$\begin{aligned} u(x)^p \;\leq\; & c^{\frac{p}{p-q}} + \frac{p}{p-q} \sum_{j=1}^{n_1} \int_0^{\alpha_j(x)} a_j(x,t) u(t)^q dt \\ & + \frac{p}{p-q} \sum_{k=1}^{n_2} \int_0^{\beta_k(x)} b_k(x,t) u(t)^q w_1(u(t)) dt \end{aligned}$$

for any $x \in \mathbb{R}_+$ *with* $0 \leq t \leq x$ *, then there exists* $(x^*) \in \mathbb{R}_+$*, such that for all* $0 \leq t \leq x^*$ *, we have*

$$u(x) \leq \left(\left[\Psi_1^{-1}\left(\Psi_1(\pi(x)) + B(x)\right)\right]\right)^{\frac{1}{p-q}}. \tag{4.25}$$

Where

$$\pi(x) = c + A(x),$$

and

$$A(x) = \sum_{j=1}^{n_1} \int_0^{\alpha_j(x)} a_j(x,t)dt, \quad (4.26)$$

$$B(x) = \sum_{k=1}^{n_2} \int_0^{\beta_k(x)} b_k(x,t)dt, \quad (4.27)$$

$$\Psi_i(\delta) = \int_{\delta_0}^{\delta} \frac{ds}{w_i\left(s^{\frac{1}{p-q}}\right)} \qquad \delta > \delta_0 > 0,\ i = 1,2 \quad (4.28)$$

Where the real number x^* *is chosen so that* $\Psi_1(\pi(x)) + B(x) \in dom(\Psi_1^{-1})$.

(c$_2$) *If* $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and*

$$u(x)^p \leq c^{\frac{p}{p-q}} + \frac{p}{p-q}\sum_{j=1}^{n_1}\int_0^{\alpha_j(x)} a_j(x,t)u(t)^q w_1(u(t))dt + \frac{p}{p-q}\sum_{k=1}^{n_2}\int_0^{\beta_k(x)} b_k(x,t)u(t)^q w_2(u(t))dt. \quad (4.29)$$

(i) For the case $w_2(u) \leq w_1(u)$, *for any* $x, t \in \mathbb{R}_+$ *with* $0 \leq t \leq x$, *then*

$$u(x) \leq u(x) \leq \left(\left[\Psi_1^{-1}\left(\Psi_1(c) + A(x) + B(x)\right)\right]\right)^{\frac{1}{p-q}} .$$

(ii) For the case $w_1(u) \leq w_2(u)$, *for any* $x, t \in \mathbb{R}_+$ *with* $0 \leq t \leq x$, *then we have :*

$$u(x) \leq u(x) \leq \left(\left[\Psi_2^{-1}\left(\Psi_2(c) + A(x) + B(x)\right)\right]\right)^{\frac{1}{p-q}} .$$

Where $\Psi_i, A and\ B\ (i = 1,2)$*are defined in (4.26)-(4.28).*

Remark 58 *(i) Corollary 57 (**c**$_1$) reduces to the Sun's inequality [35, Theorem 2.1] in the case of one variable* $(n = 1)$ *when* $a_j(x,t) = a_j(t),\ b_k(x,t) = b_k(t),\ \beta_k(x) = \alpha_j(x)$ *and* $j = k = 1$.

*(ii) Corollary 57 (**c**$_2$) reduces to the Sun's inequality [35, Theorem 2.2] in the case of*

one variable $(n = 1)$ *when* $a_j(x,t) = a_j(t),\ b_k(x,t) = b_k(t),\ \beta_k(x) = x$ *and* $j = k = 1$ *and* $w_1 = w_2$.

Remark 59 *Under a suitable conditions in* (b_1)*, the inquality (4.18) gives a new estimate for the inquality (4.5) in*$(\boldsymbol{a}_1)$*.*

Corollary 60 ***(Inequality in two variables)***

Let $a, b \in C(\Delta \subset \mathbb{R}_+^2, \mathbb{R}_+)\quad \alpha \in C^1(J_1, J_1),\ \beta \in C^1(J_2, J_2)$ *be nondecreasing functions with* $\Delta = J_1 \times J_2$ *and* $J_1 = [x_0, a] \in \mathbb{R}_+,\ J_1 = [y_0, b] \in \mathbb{R}_+$*where* $\alpha(x) \le x$ *on* $J_1,\ \beta(y) \le y$ *on* J_2*. Let* $k \ge 0$*.*

If $u(x,y) \in C(\Delta \subset \mathbb{R}_+^2, \mathbb{R}_+)$ *and*

$$\begin{aligned} u(x,y) &\le k + \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)u(s,t)dsdt \\ &\quad + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)u(s,t)dsdt, \end{aligned} \tag{4.30}$$

for $(x,y)\ \in \Delta,$ *then*

$$u(x,y) \le k \exp\left[A(x,y) + B(x,y)\right],$$

for $(x,y)\ \in \Delta,$ *where*

$$A(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)dsdt, \tag{4.31}$$

$$B(x,y) = \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)dsdt; \tag{4.32}$$

for $(x,y)\ \in \Delta.$

Theorem 61 *Let the functions* $u, c, , \varphi, \Phi, w_i, a_j, b_k$ $(i = 1, 2, \quad j = 1, 2, ..., n_1, \ k = , 2, ..., n_1)$ *be defined as in Theorem 56*

$$\begin{aligned}\varphi(u(x)) \ &\leq \ c(x) + \sum_{j=1}^{n_1} d_j(x) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t)\Phi(u(t))w_1(u(t))dt \\ &+ \sum_{k=1}^{n_2} l_k(x) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)\Phi(u(t))w_2(u(t))dt,\end{aligned}$$

then

$$u(x) \leq \varphi^{-1}\left(G^{-1}\left[\Psi^{-1}\left(\Psi(G(c(x))) + \widetilde{A}(x) + \widetilde{B}(x)\right)\right]\right)$$

where

$$\begin{aligned}\widetilde{A}(x) &= \sum_{j=1}^{n_1} d_j(x) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t)dt, \\ \widetilde{B}(x) &= \sum_{k=1}^{n_2} l_k(x) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)dt.\end{aligned}$$

Corollary 62 *If*

$$u^p(x) \leq c(x) + \int_0^{\widetilde{\alpha}(x)} a(t)u^q(t) + b(t)u^p(t)dt$$

for any $x \in \mathbb{R}_+^n$ *with* $x^0 \leq t \leq x$, *then there exists* $x^* \in \mathbb{R}_+^n$, *such that for all* $x^0 \leq t \leq x^*$, *we have*

$$u(x) \leq \frac{p}{p-q} c^{\frac{p-q}{p}}(x) \exp\left[\frac{p}{p-q}\int_0^{\widetilde{\alpha}(x)} a(t) + b(t)dt\right]$$

Remark 63 *(i) Theorem 61 reduces to Theorem 2.2 of Lipovan [21] in the case of one variable, when* $\varphi(x) = x$, $b_k(x,t) = 0, w_1(t) = 1, \ j = 1$ *and* $n = 1$.

(ii)Theorem 61 is also a generalization of the main result in Lipovan [21, Theorem 2.1] in the case of one variable variable, when $\varphi(x) = x$, $b_k(x,t) = 0, w_1(t) = 1, \Phi(t) = 1,$ *for any* $x, t \in \mathbb{R}_+ (n = 1)$ *and for* $j = 1$.

Remark 64 *(i)Under a suitable conditions, Theorem 61 reduces to Theorem 2.3 and Theorem 2-4 in the case of tow variables of the main results in Zhang and Meng [40] .*

(ii) Under a suitable conditions in Theorem 61, we can also obtain an other estimations of the Ma and Pecaric's inequality (4.1) and the main results in [23].

Remark 65 *Theorem 61 further reduces to the man results in [5, Therem 2.1, 2.2, 2.4] and the results in [28].*

4.3 Proof of Theorems

Since the proofs resemble each other, we give the details for $(\mathbf{a}_1)$ and Theorem 61 only; the proofs of the remaining inequalities can be completed by following the proofs of the above-mentioned inequalities.

Proof. Theorem 51 $(\mathbf{a}_1)$ Fixing any arbitrary numbers $y = (y_1, ..., y_n) \in \mathbb{R}_+^n$ with

$x^0 < y \leq x^*$, we define on $[x^0; y]$ a function $z(x)$ by

$$\begin{aligned} z(x) &= c(y) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(y,t) u^q(t) dt \\ &\quad + \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(y,t) u^q(t) w_1(u(t)) dt, \end{aligned} \tag{4.33}$$

$z(x)$ is a positive and nondecreasing function and $z(x^0) = c(y)$, then

$$u(x) \leq z(x)^{1/p}, \quad x \in [x^0; y]. \tag{4.34}$$

We know that

$$\begin{aligned} D_1D_2...D_nz(x) &= \sum_{j=1}^{n_1} a_j(y,\widetilde{\alpha}_j(x))u^q(\widetilde{\alpha}_j(x))\alpha_{j1}^{'}\alpha_{j2}^{'}...\alpha_{jn}^{'} \\ &\quad + \sum_{k=1}^{n_2} b_j(y,\widetilde{\beta}_j(x))u^q(\widetilde{\beta}_j(x))w_1(u(\widetilde{\beta}_j(x)))\beta_{k1}^{'}\beta_{k2}^{'}...\beta_{kn}^{'} \\ &\leq z^{q/p}(x)\left[\sum_{j=1}^{n_1} a_j(y,\widetilde{\alpha}_j(x))\alpha_{j1}^{'}(x_1)\alpha_{j2}^{'}(x_2)...\alpha_{jn}^{'}(x_n)\right. \\ &\quad \left. + \sum_{k=1}^{n_2} b_j(y,\widetilde{\beta}_j(x))w_1(z^{1/p}(\widetilde{\beta}_j(x)))\beta_{k1}^{'}\beta_{k2}^{'}...\beta_{kn}^{'}\right]. \end{aligned} \tag{4.35}$$

Using (4.35), we have

$$\begin{aligned} \frac{D_1D_2...D_nz(x)}{z^{q/p}(x)} &\leq \left[\sum_{j=1}^{n_1} a_j(y,\widetilde{\alpha}_j(x))\alpha_{j1}^{'}(x_1)\alpha_{j2}^{'}(x_2)...\alpha_{jn}^{'}(x_n)\right. \\ &\quad \left. + \sum_{k=1}^{n_2} b_j(y,\widetilde{\beta}_j(x))w_1(z^{1/p}(\widetilde{\beta}_j(x)))\beta_{k1}^{'}\beta_{k2}^{'}...\beta_{kn}^{'}\right]. \end{aligned} \tag{4.36}$$

Using $D_1D_2...D_{n-1}z(x) \geq 0$, $\frac{q}{p}z^{(q-p)/p}(x) \geq 0$, $D_n(x) \geq 0$ and by (4.36), then

$$\begin{aligned} D_n\left(\frac{D_1D_2...D_{n-1}z(x)}{z^{q/p}(x)}\right) &\leq \frac{D_1D_2...D_nz(x)}{z^{q/p}(x)} \\ &\leq \sum_{j=1}^{n_1} a_j(y,\widetilde{\alpha}_j(x))\alpha_{j1}^{'}(x_1)\alpha_{j2}^{'}(x_2)...\alpha_{jn}^{'}(x_n) \\ &\quad + \sum_{k=1}^{n_2} b_k(y,\widetilde{\beta}_k(x))w_1(z^{1/p}(\widetilde{\beta}_k(x)))\beta_{k1}^{'}\beta_{k2}^{'}...\beta_{kn}^{'}. \end{aligned} \tag{4.37}$$

Fixing $x_1, x_2, ..., x_{n-1}$,setting $x_n = t_n$ and integrating (4.37) from x_n^0 to x_n, we obtain

$$\begin{aligned} &\frac{D_1D_2...D_{n-1}z(x)}{z^{q/p}(x)} \\ \leq\ & \sum_{j=1}^{n_1}\int_{\alpha_{jn}(x_n^0)}^{\alpha_{jn}(x_n)} a_j(y,\alpha_{j1}(x_1),\alpha_{j2}(x_2),...,\alpha_{jn-1}(x_{n-1}),\alpha_{jn}(t_n))\alpha_{j1}^{'}\alpha_{j2}^{'}...\alpha_{jn-1}^{'}dt_n \\ &+ \sum_{k=1}^{n_2}\int_{\beta_{kn}(x_n^0)}^{\beta_{kn}(x_n)} b_k(y,\beta_{k1}(x_1),\beta_{k2}(x_2),...,\beta_{kn-1}(x_{n-1}),t_n)w_1(z^{1/p}(\beta_{k1},\beta_{k2},...,\beta_{kn-1},t_n)) \\ &\beta_{k1}^{'}(x_1)\beta_{k2}^{'}(x_2)...\beta_{kn-1}^{'}(x_{n-1})dt_n. \end{aligned}$$

Using the same method, we reduce that

$$\begin{aligned}
&\frac{D_1 z(x)}{z^{q/p}(x)} \\
\leq\ & \sum_{j=1}^{n_1}\left[\int_{\alpha_{jn}(x_n^0)}^{\alpha_{jn}(x_n)} \ldots \int_{\alpha_{jn}(x_n^0)}^{\alpha_{jn}(x_n)} a_j(y, \alpha_{j1}(x_1), t_2, \ldots, t_n)\alpha_{j1}^{'}(x_1)dt_n \ldots dt_2\right] \\
& + \sum_{k=1}^{n_2}\left[\int_{\beta_{jn}(x_n^0)}^{\beta_{jn}(x_n)} \ldots \int_{\beta_{jn}(x_n^0)}^{\beta_{jn}(x_n)} b_k(y, \beta_{k1}(x_1), t_2, \ldots, t_n)\right. \\
& \left. .\ w_1(z^{1/p}(\beta_{k1}(x_1), t_2, \ldots, t_n))\beta_{k1}^{'}(x_1)dt_n \ldots dt_2\right]. \qquad (4.38)
\end{aligned}$$

Integrating (4.38) form x_1^0 to x_1, we obtain

$$\begin{aligned}
\frac{p}{p-q} z^{(p-q)/p}(x) \quad \leq \quad & \frac{p}{p-q} c^{(p-q)/p}(y) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(y)} a_j(y,t)dt \\
& + \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(y,t) w_1(z^{1/p}(t))dt,
\end{aligned}$$

for all $x \in [x^0; y]$, which implies that

$$\begin{aligned}
z^{(p-q)/p}(x) \quad \leq \quad & c^{(p-q)/p}(y) + \frac{p-q}{p} \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(y)} a_j(y,t)dt \\
& + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(y,t) w_1(z^{1/p}(t))dt. \qquad (4.39)
\end{aligned}$$

We setting $r_1(x) = z^{(p-q)/p}(x)$, the (4.39) can be rewritten as

$$r_1(x) \leq p(y) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(y,t) w_1(r_1^{1/(p-q)}(t))dt.$$

Defining $v(x)$ on $[x^0; y]$, by

$$v(x) = p(y) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(y,t) w_1(r_1^{1/(p-q)}(t))dt, \qquad (4.40)$$

by (4.40), we have $v(x^0) = p(y)$ and

$$z^{(p-q)/p}(x) \leq v(x), \tag{4.41}$$

and

$$\begin{aligned} D_1 D_2 ... D_n v(x) &= \frac{p-q}{p} \sum_{k=1}^{n_2} b_k(y, \widetilde{\beta}_k(x)) w_1(r_1^{1/(p-q)}(\widetilde{\beta}_k(x))) \beta'_{k1} \beta'_{k2} ... \beta'_{kn} \\ &\leq \frac{p-q}{p} \sum_{k=1}^{n_2} b_k(y, \widetilde{\beta}_k(x)) w_1(v^{1/(p-q)}(\widetilde{\beta}_k(x))) \beta'_{k1} \beta'_{k2} ... \beta'_{kn}. \end{aligned}$$

By using the same method above, we obtain

$$\begin{aligned} &\frac{D_1 v(x)}{w_1(v(x)^{1/p-q})} \\ &\leq \frac{p-q}{p} \sum_{k=1}^{n_2} \left[\int_{\beta_{jn}(x_n^0)}^{\beta_{jn}(x_n)} ... \int_{\beta_{jn}(x_n^0)}^{\beta_{jn}(x_n)} b_k(y, \beta_{k1}(x_1), t_2, ..., t_n) \beta'_{k1}(x_1) dt_n ... dt_2 \right]. \end{aligned} \tag{4.42}$$

Integrating (4.42) form x_1^0 to x_1, we obtain

$$\Psi_1(v(x)) \leq \Psi_1(p(y)) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(y, t) dt \tag{4.43}$$

and from (4.43) and for any arbitrary y, we get

$$v(y) \leq \Psi_1^{-1} \left[\Psi_1(p(y)) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(y)} b_k(y, t) dt \right] \tag{4.44}$$

From (4.44) and (4.41), we reduce to :

$$z(y) \leq \left(\Psi_1^{-1} \left[\Psi_1(p(y)) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(y)} b_k(y, t) dt \right] \right)^{\frac{p}{p-q}}. \tag{4.45}$$

By (4.45) and (4.34), then

$$u(y) \leq \left(\Psi_1^{-1} \left[\Psi_1(p(y)) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(y)} b_k(y, t) dt \right] \right)^{\frac{1}{p-q}}$$

Since $y \leq x^*$is arbitrary we are done with the proof. ■

Proof. (Theorem 61) Fxing any arbitrary numbers $\tau = (\tau_1, ..., \tau_n) \in \mathbb{R}_+^n$ with $x^0 < \tau \leq \xi$, we define on $[x^0; \tau]$ a function $z(x)$ by

$$\begin{aligned} z(x) &= c(\tau) + \sum_{j=1}^{n_1} d_j(\tau) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(\tau, t)\Phi(u(t))w_1(u(t))dt \\ &\quad + \sum_{k=1}^{n_2} l_k(\tau) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(\tau, t)\Phi(u(t))w_2(u(t))dt, \end{aligned}$$

$z(x)$ is a positive and nondecreasing function and $z(x^0) = c(\tau)$, then

$$u(x) \leq \varphi^{-1}(z(x)); \quad x \in [x^0; \tau].$$

We know that

$$\begin{aligned} D_1 D_2 ... D_n z(x) &= \sum_{j=1}^{n_1} d_j(\tau) a_j(\tau, \widetilde{\alpha}_j(x))\Phi(u(\widetilde{\alpha}_j(x)))w_1(u(\widetilde{\alpha}_j(x)))\alpha'_{j1}\alpha'_{j2}...\alpha'_{jn} \\ &\quad + \sum_{k=1}^{n_2} l_k(\tau) b_k(\tau, \widetilde{\beta}_k(x))\Phi(u(\widetilde{\beta}_k(x)))w_2(u(\widetilde{\beta}_k(x)))\beta'_{k1}\beta'_{k2}...\beta'_{kn}, \\ &\leq \Phi(\varphi^{-1}(z(x)) \left[\sum_{j=1}^{n_1} d_j(\tau) a_j(\tau, \widetilde{\alpha}_j(x)) w_1(\varphi^{-1}(z(\widetilde{\alpha}_j(x)))\alpha'_{j1}\alpha'_{j2}...\alpha'_{jn} \right. \\ &\quad \left. + \sum_{k=1}^{n_2} l_k(\tau) b_k(\tau, \widetilde{\beta}_k(x)) w_2(\varphi^{-1}(z(\widetilde{\beta}_k(x)))\beta'_{k1}\beta'_{k2}...\beta'_{kn} \right]. \end{aligned}$$

Using the same method in proof of the Theorem 51, and for all $x \in [x^0; \tau]$, which implies that :

$$\begin{aligned} z(x) &\leq G^{-1} \left[G(c(\tau)) + \sum_{j=1}^{n_1} d_j(\tau) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(\tau, t) w_1(u(t))dt \right. \\ &\quad \left. + \sum_{k=1}^{n_2} l_k(\tau) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(\tau, t) w_2(u(t))dt \right]. \end{aligned}$$

Defining $v(x)$ on $[x^0; \tau]$ by

$$\begin{aligned} v(x) &= G(c(\tau)) + \sum_{j=1}^{n_1} d_j(\tau) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(\tau, t) w_1(u(t))dt \\ &\quad + \sum_{k=1}^{n_2} l_k(\tau) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(\tau, t) w_2(u(t))dt. \end{aligned}$$

We have $v(x^0) = G(c(\tau))$ and

$$z(x) \leq G^{-1}(v(x)),$$

and

$$u(x) \leq \varphi^{-1}(G^{-1}(v(x))), \tag{4.46}$$

and we can obtain :

$$\begin{aligned}\frac{D_1 D_2 ... D_n v(x)}{w_1(\varphi^{-1}(G^{-1}(v(x))))} \leq & \left[\sum_{j=1}^{n_1} d_j(\tau) a_j(\tau, \widetilde{\alpha}_j(x)) \alpha'_{j1}(x_1) \alpha'_{j2}(x_2) ... \alpha'_{jn}(x_n)\right. \\ & \left. + \sum_{k=1}^{n_2} l_k(\tau) b_k(\tau, \widetilde{\beta}_k(x)) \beta'_{k1}(x_1) \beta'_{k2}(x_2) ... \beta'_{kn}(x_n)\right].\end{aligned}$$

By using the same method above, we obtain :

$$\begin{aligned}\Psi_1(v(x)) \leq & \Psi_1(G(c(\tau))) + \sum_{j=1}^{n_1} d_j(\tau) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(\tau, t) dt \\ & + \sum_{k=1}^{n_2} l_k(\tau) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(\tau, t) dt.\end{aligned}$$

From which we get

$$\begin{aligned}v(\tau) \leq & \Psi_1^{-1}\left[\Psi_1(G(c(\tau))) + \sum_{j=1}^{n_1} d_j(\tau) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(\tau)} a_j(\tau, t) dt\right. \\ & \left. + \sum_{k=1}^{n_2} l_k(\tau) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(\tau)} b_k(\tau, t) dt\right].\end{aligned} \tag{4.47}$$

for any arbitrary numbers $\tau \in \mathbb{R}^n_+$, with $x^0 < \tau \leq \xi$.

from (4.46) and (4.47), we reduce to :

$$\begin{aligned}u(\tau) \leq & \varphi^{-1}\left\{G^{-1}\left(\Psi_1^{-1}\left[\Psi_1(G(c(\tau))) + \sum_{j=1}^{n_1} d_j(\tau) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(\tau)} a_j(\tau, t) dt\right.\right.\right. \\ & \left.\left.\left. + \sum_{k=1}^{n_2} l_k(\tau) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(\tau)} b_k(\tau, t) dt\right]\right)\right\}.\end{aligned}$$

Since τ is arbitrary and $\tau \leq \xi$, we obtain the result in the Theorem 61. ■

4.4 Applications

4.4.1 Partial delay differential equation in $\mathbb{R}^2$

In this section we present applications of the inequality (4.30) Corollary 60 to study the boundedness and uniqueness of the solutions of the initial boundary value problem for partial delay differential equations in two variables of the form

$$D_2D_1u(x,y) = f\left(x,y,u(x,y),u(x-h_1(x),y-h_2(y))\right), \tag{4.48}$$

$$u(x,y_0) = a_1(x),\quad u(x_0,y)=a_2(y),\quad a_1(x_0)=a_2(y_0), \tag{4.49}$$

where $f \in C(\Delta\times\mathbb{R}^2 \ ,\mathbb{R}\ )$, $a_1 \in C^1(J_1\ ,\mathbb{R}\ )$, $a_2 \in C^1(J_2\ ,\mathbb{R}\ )$, $h_1 \in C^1(J_1\ ,\mathbb{R}_+\ )$, $h_2 \in C^1(J_2\ ,\mathbb{R}_+\ )$ such that $x-h_1(x)\geq 0$, $y-h_2(y)\geq 0$, $h_1'(x)\prec 1$, $h_2'(x)\prec 1$ and $h_1(x_0)=h_2(y_0)=0$.

Where $J_1=[x_0,a]\in\mathbb{R}_+$, $J_2=[x_0,b]\in\mathbb{R}_+$ and $\Delta=J_1\times\ J_2$.

Our first result gives the bound on the solution of the problem (4.48),(4.49).

Theorem 66 *Suppose that*

$$\left|f\left(x,y,u,v\right)\right| \leq a(x,y)\left|u\right| + b(x,y)\left|v\right|, \tag{4.50}$$

and

$$\left|a_1'x) + a_2(y)\right| \leq k, \tag{4.51}$$

where $a,b\in C(\Delta\ ,\mathbb{R}_+\ )$ and $k\geq 0$ is a constant, and let

$$M_1 = \max_{x\in J_1}\frac{1}{1-h_1'(x)},\qquad M_2=\max_{y\in J_2}\frac{1}{1-h_1'(y)}. \tag{4.52}$$

If $u(x,y)$ is any solution of (4.48)-(4.49), then

$$|u(x,y)| \leq k \exp\left(A(x,y) + \overline{B}(x,y)\right), \tag{4.53}$$

where

$$A(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)dsdt, \tag{4.54}$$

$$\overline{B}(x,y) = M_1 M_2 \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} \overline{b}(\sigma,\tau)d\tau d\sigma \tag{4.55}$$

in which $\alpha(x) = x - h_1(x)$ for $x \in J_1$ and $\beta(y) = y - h_2(y)$ for $y \in J_2$ and

$$\overline{b}(\sigma,\tau) = b(\sigma + h_1(s), \tau + h_2(t)), \quad \text{for } \sigma, s \in J_1, \quad \tau, t \in J_2.$$

Proof. The solution $u(x,y)$ of the problem (4.48)-(4.49) satisfies the equivalent integral equation

$$u(x,y) = a_1(x) + a_2(y) + \int_{x_0}^{x}\int_{y_0}^{y} f\left(s,t,u(s,t),u(s-h_1(s),t-h_2(t))\right)dsdt. \tag{4.56}$$

Using (4.50), (4.51) and (4.52) in (4.56) and making the change of variables, we have

$$\begin{aligned} |u(x,y)| \leq\ & k + \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)\,|u(x,y)|\,dsdt \\ & + M_1 M_2 \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} \overline{b}(\sigma,\tau)\,|u(x,y)|\,d\tau d\sigma. \end{aligned} \tag{4.57}$$

Now a suitable application of the inequality (4.30) given in Corollary 60 to (4.57) yields (4.53). The right-hand side of (4.53) gives us the bound on the solution $u(x,y)$ of (4.48)-(4.49) in terms of the known functions. Thus, if the right-hand side of (4.53) is bounded, then we assert that the solution of (4.48)-(4.49) is bounded for $x, y \in \Delta$.

■

The next result deals with uniqueness of the solutions of the problem (4.48)-(4.49).

Theorem 67 *Suppose that the function f in (4.48) satisfies the condition*

$$|f(x,y,u,v) - f(x,y,\overline{u},\overline{v})| \leq a(x,y)\,|u-\overline{u}| + b(x,y)\,|v-\overline{v}|\,, \tag{4.58}$$

where $a, b \in C(\Delta\ , \mathbb{R}_+\ )$, and let $M_1, M_2, \alpha, \beta, \overline{b}$ be as in Theorem 61.

Then the problem (4.48)-(4.49 has at most one solution on Δ.

Proof. Let $u(x,y)$ and $\overline{u}(x,y)$ be two solutions of (4.48)-(4.49) on Δ, the we have

$$\begin{aligned} u(x,y) - \overline{u}(x,y) &= \int_{x_0}^{x}\int_{y_0}^{y} [f(s,t,u(s,t),u(s-h_1(s),t-h_2(t))) \\ &\quad - f(s,t,\overline{u}(s,t),\overline{u}(s-h_1(s),t-h_2(t)))]\,dsdt. \end{aligned} \tag{4.59}$$

Using (4.58) in (4.59) and making the change of variables, we have

$$\begin{aligned} |u(x,y) - \overline{u}(x,y)| &\leq \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)\,|u(s,t) - \overline{u}(s,t)|\,dsdt \\ &\quad + M_1M_2\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} \overline{b}(\sigma,\tau)\,|u(\sigma,\tau) - \overline{u}(\sigma,\tau)|\,d\tau d\sigma \end{aligned} \tag{4.60}$$

Now a suitable application of the inequality (4.30) given in Corollary 60 to (4.60).

Therefore $u(x,y) = \overline{u}(x,y)$; there is at most one solution of the problem (4.48)-(4.49) on Δ. ■

4.4.2 Partial delay differential equation in $\mathbb{R}^n$

In this section we present an immediate application of our results (Theorem 51 and Corollary 62) to study the boundless of the solution of delay partial differential equation.

First we consider the nonlinear partial delay differential equation in $\mathbb{R}^n$:

$$\begin{cases} Du^p(x) = h(x, u(x), u(x - \widetilde{\alpha}(x)), \\ u^p(0, x_2, x_3, ..., x_n) = c_1(x_1), \\ u^p(0, x_2, x_3, ...x_{n-1}, x_n) = c_n(x_n) \\ u^p(..., x_{i-1}, 0, x_{i+1}, ...) = c_i(x_i) \text{ for all } i = 2, 3, ..., n-1, \\ c_i(0) = 0 \text{ for all } i = 1, 2, ..., n. \end{cases} \tag{4.61}$$

For $x = (x_1; x_2, ..., x_n) \in \mathbb{R}_+^n$ and $\widetilde{\alpha}(x) = (\alpha_1(x_1), \alpha_2(x_2), ..., \alpha_n(x_n)) \in \mathbb{R}_+^n$ for α_i , $c_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ for $i = 1, 2, ..., n$where $h : \mathbb{R}_+^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is continuo function. Assume that those functions are defined and continuous on their respective domains of definition such that

$$\widetilde{\alpha}(x) \leq x, \quad \text{for all } x = (x_1; x_2, ..., x_n) \in \mathbb{R}_+^n, \tag{4.62}$$

and

$$|h(x, u, v)| \leq a(x) |v(x)|^q + b(x) |v(x)|^p, \tag{4.63}$$

for $x \in \mathbb{R}_+^n$, where $p > q \geq 0$. is a constants and $a(x), b(x)$ are nonnegative, continuous functions defined for $x \in \mathbb{R}_+^n$. If $u(x)$ is any solution of the boundary value problem (4.61), then

$$u^p(x) = \sum_{i=1}^{n} c_i(x_i) + \int_0^x h(t, u(t), u(t - \widetilde{\alpha}(t))dt, \tag{4.64}$$

For all $x, t \in \mathbb{R}_+^n$ with $0 \leq t \leq x$. using (4.61),(4.63) and by making the change of variables in (4.64), we have

$$|u^p(x)| \leq c(x) + \int_0^{\widetilde{\alpha}(x)} \widetilde{a}(t) |u(t)|^q + \widetilde{b}(t) |u(t)|^p dt, \tag{4.65}$$

with $c(x)=\sum_{i=1}^{n}|c_i(x_i)|$, $\widetilde{a},\widetilde{b}\in C^1(\mathbb{R}_+^n,\mathbb{R}_+)$.

(e$_1$) Now a suitable application of **(a$_1$)** in Theorem 51 to (4.65), when $\widetilde{\alpha}_j=\widetilde{\beta}_k$, $a_j(x,t)=\widetilde{a}(t),\ b_k(x,t)=\widetilde{b}(t)$ with $j=k=1$ and $w_1(u)=u^{p-q}$, then we obtain the boundless of the solution $u(x)$:

$$u(x)\leq\left(c^{(p-q)/p}(x)+\frac{p-q}{p}\int_0^{\widetilde{\alpha}(x)}\widetilde{a}(t)dt\right)^{\frac{1}{p-q}}\exp\left(\frac{1}{p}\int_0^{\widetilde{\alpha}(x)}\widetilde{b}(t)dt\right). \tag{4.66}$$

(e$_2$) Or by an application direct of Corollary 62 to (4.65), then

$$u(x)\leq\frac{p}{p-q}c^{\frac{p-q}{p}}\exp\left[\frac{p}{p-q}\int_0^{\widetilde{\alpha}(x)}\left[\widetilde{a}(t)+\widetilde{b}(t)\right]dt\right]. \tag{4.67}$$

Remark 68 *In the special case ($p=2$ and $q=1$) in the boundary value problem (4.61), we can obtain :*

(i) by using (4.66),we obtain

$$u(x)\leq\left(\sqrt{c(x)}+\frac{1}{2}\int_0^{\widetilde{\alpha}(x)}\widetilde{a}(t)dt\right)\exp\left(\frac{1}{2}\int_0^{\widetilde{\alpha}(x)}\widetilde{b}(t)dt\right).$$

(ii) Or by using (4.67), then

$$u(x)\leq 2\sqrt{c(x)}\exp\left[2\int_0^{\widetilde{\alpha}(x)}\left[\widetilde{a}(t)+\widetilde{b}(t)\right]dt\right].$$

We note that the results given here can be very easily generalized to obtain explicit bounds on integral inequalities involving several retarded arguments.

Remark 69 *Using similar method of those in the proof of Theorems above, we can also obtain a* ***new reversed inequalities*** *of our results.*

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Résumé

Le but de ce travail est de donner une exposition des résultats classiques de certaines inégalités intégrales apparus dans la littérature mathématique dans ces dernières années; et de établir quelques nouvelles inégalités intégrales, integrodifférentielles inégalités et aussi quelques nouvelles inégalités intégrales avec un terme de retard.

Les résultats donnés ici, peuvent être utilisés dans la théorie qualitative de certaines classes des problèmes de valeur aux limites pour les EDP, EDP avec un retard, équations différentielles, équations intégrales et les équations integrodifférentielles.

Abstract

The aim of the present work is to give an exposition of the classical results about integral inequalities with have appeared in the mathematical literature in recent years; and to establish some new integral inequalities, integrodifferential inequalities and also many new retarded integral inequalities. The results given here can be used in the qualitative theory of various classes of boundary value problems of partial differential equations, partial differential equations with a delay, differential equations, integral equations and integrodifferential equations.

ملـــخـــص

الهدف من هذا العمل هو تقديم عرض حول بعض النتائج الكلاسيكية لبعض المتراجحات التكاملية و التي دخلت مجال الرياضيات في الأعوام الأخيرة و إيجاد بعض المتراجحات التكاملية الجديدة ، المتراجحات التفاضلية التكاملية و أيضا بعض المتراجحات التكاملية ذات حد متأخر .

النتائـ ـ ـ ـج المعروضة هنا ، يمكن أن تطبق على نظرية الكـم لكثير من المسائل الحدية لبعض المعادلات التفاضلية الجزئية ، المعادلات التفاضلية الجزئية ذات حد متأخر ، المعادلات التفاضلية ، المعادلات التكاملية و المعادلات التفاضلية التكاملية .