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A NEW HIGHER ORDER SHEAR AND NORMAL DEFORMATION THEORY FOR BENDING ANALYSIS OF ADVANCED COMPOSITE PLATES

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ABSTRACT

In this work, a higher-order theory including the stretching effect is developed for the static analysis of advanced composite plates such as functionally graded plates. The number of unknown functions involved in the present theory is only five as against six or more in case of other shear and normal deformation theories. The governing equations are derived by employing the principle of virtual work and the physical neutral surface concept. Navier-type analytical solution is obtained for functionally graded plate subjected to transverse load for simply supported boundary conditions. A comparison with the corresponding results is made to check the accuracy and efficiency of the present theory.

Key Words: *FG plates; New plate theory; Neutral surface position; Stretching effect;*

NOMENCLATURE

Symbols :

V_C	The volume-fraction of ceramic
C	The distance of neutral surface from the mid-
P	The material non-homogeneous properties of FG
E	The elasticity modules
N	The stress resultants
ν	Poisson's ratio
ρ	The mass density
u_s , v_s	Shear components
τ_{xz} , τ_{yz}	Shear stresses

Indices / Exponents :

k	The power law index
n	The material parameter
a/h	Thickness ratio

1. INTRODUCTION

In the past three decades, researches on functionally graded (FG) plates have received great attention, and a variety of plate theories has been introduced based on considering the transverse shear deformation effect. The classical plate theory (CPT), which neglects the transverse shear deformation effect, provides reasonable results for thin plate. To overcome the deficiency of the CPT, many shear deformation plate theories which account for the transverse shear deformation

effects have been developed. The Reissner [1] and Mindlin [2] theories are known as the first -order shear deformation plate theory (FSDT). Since the FSDT violates the equilibrium conditions on the top and bottom surfaces of the plate, a shear correction factor is required to compensate for the error due to a constant shear strain assumption through the thickness. Although the FSDT provides a sufficiently accurate description of response for thin to moderately thick plate, it is not convenient for use due to the difficulty in determination of the correct value of the shear correction factor [3]. To avoid the use of shear correction factors, many refined shear deformation plate theories have been developed such as the third-order shear deformation plate theory (TSDT) of Reddy [4], sinusoidal shear deformation plate theory (SSDT) [5,6], and hyperbolic shear deformation plate theory (HSDT) [7]. However, in most shear deformation theories, FG plates have been analysed neglecting the thickness stretching, being the transverse displacement considered independent by thickness coordinates. The effect of thickness stretching in FG plates has been investigated by Carrera et al. [8], using finite elements. Neves et al. [9] have presented an original hyperbolic sine shear deformation theory for the bending and free vibration analysis of FG plates.
The purpose of this study is to develop a new shear deformation plate theory for FG plates by including the so-called ‘‘stretching effect’’. Just five unknown displacement functions are used in the present theory against six or more unknown displacement functions used in the corresponding ones. This is due to the fact that the stretching – bending coupling in the constitutive equations of an FG plate does not exist when the physical neutral surface is considered as a coordinate system. The theory does not require shear correction factors since the displacement components are expressed by trigonometric series representation through the plate thickness to develop a two-dimensional theory and gives rise to transverse shear stress variation such that the transverse shear stresses vary parabolically across the thickness satisfying shear stress free surface conditions. The effectiveness of the present theory is demonstrated and results are compared with the corresponding FGM solution.

2. MATHEMATICAL FORMULATION

The Having a rectangular plate made of FGMs of thickness h, length a, and width b, referred to the rectangular Cartesian coordinates (x, y, z). The $x-y$ plane is taken to be the un-deformed mid-plane of the plate, and the z axis is perpendicular to the $x-y$ plane. To specify the position of neutral surface of FG plates, two different planes are considered for the measurement of z, namely, z_{ms} and z_{ns} measured from the middle surface and the neutral surface of the plate, respectively, The volume-fraction of ceramic V_C is expressed based on z_{ms} and z_{ns} coordinates as

$$V_C = \left(\frac{z_{ms}}{h}+\frac{1}{2}\right)^k = \left(\frac{z_{ns}+C}{h}+\frac{1}{2}\right)^k \tag{1}$$

Where k is the power law index which takes the value greater or equal to zero and C is the distance of neutral surface from the mid-surface. Thus, using Eq. (1), the material non-homogeneous properties of FG plate P, as a function of thickness coordinate, become

$$P(z) = P_M + P_{CM}\left(\frac{z_{ns}+C}{h}+\frac{1}{2}\right)^k, \quad P_{CM} = P_C - P_M \tag{2}$$

Where P_M and P_C are the corresponding properties of the metal and ceramic. In the present work, we assume that the elasticity modules E and the mass density ρ are described by Eq. (2),
The bending components $\boldsymbol{u_b}$ and $\boldsymbol{v_b}$ are assumed to be similar to the displacements given by the classical plate theory. Therefore, the expression for $\boldsymbol{u_b}$ and $\boldsymbol{v_b}$ can be given as

$$u_b = -z_{ns}\frac{\partial w_b}{\partial x}, \quad v_b = -z_{ns}\frac{\partial w_b}{\partial y} \tag{3}$$

the expression for u_s and v_s can be given as

$$u_s = -f(z_{ns})\frac{\partial w_s}{\partial x}, \quad v_s = -f(z_{ns})\frac{\partial w_s}{\partial y} \tag{4}$$

where

$$f(zns) = z - h\sinh\left(\frac{z}{h}\right) + z\cosh\left(\frac{1}{2}\right) \tag{5}$$

The component due to the stretching effect w_{st} can be given as

$$w_{st}(x, y, z_{ns}) = g(z_{ns})\varphi(x, y) \tag{6}$$

The additional displacement φ accounts for the effect of normal stress is included and $g(z_{ns})$ is given as follows

$$g(z_{ns}) = \cosh\left(\frac{z}{h}\right) - \cosh\left(\frac{1}{2}\right) \tag{7}$$

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs. (7) as

$$u(x, y, z_{ns}) = u_0(x, y) - z_{ns}\frac{\partial w_b}{\partial x} - f(z_{ns})\frac{\partial w_s}{\partial x} \tag{8a}$$

$$v(x, y, z_{ns}) = v_0(x, y) - z_{ns}\frac{\partial w_b}{\partial y} - f(z_{ns})\frac{\partial w_s}{\partial y} \tag{8b}$$

$$w(x, y, z_{ns}) = w_b(x, y) + w_s(x, y) + g(z_{ns})\varphi(x, y) \tag{8c}$$

The kinematic relations can be obtained as follows:

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} + z_{ns}\begin{Bmatrix}k_x^b\\ k_y^b\\ k_{xy}^b\end{Bmatrix} + f(z_{ns})\begin{Bmatrix}k_x^s\\ k_y^s\\ k_{xy}^s\end{Bmatrix}, \quad \begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} = g(z_{ns})\begin{Bmatrix}\gamma_{yz}^0\\ \gamma_{xz}^0\end{Bmatrix}, \quad \varepsilon_z = g'(z_{ns})\varepsilon_z^0 \tag{9}$$

The stress-strain relations for a linear elastic plate are written in the form:

where (σ_x, σ_y, σ_z, τ_{yz}, τ_{xz}, τ_{xy}) and (ε_x, ε_y, ε_z, γ_{yz}, γ_{xz}, γ_{xy}) are the stress and strain components, respectively.

The computation of the elastic constants C_{ij} depends on which assumption of ε_z.

The principle of virtual displacements is used herein to derive the governing equations. The principle can be stated in an analytical form as

$$\begin{aligned}0 &= \int_A \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \left[\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \sigma_z\delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz}\right] dz_{ns}\,dA - \int_A q\delta(w_b + w_s + w_{st})dA \\ &= \int_A \left[N_x\delta\varepsilon_x^0 + N_y\delta\varepsilon_y^0 + N_z\delta\varepsilon_z^0 + N_{xy}\delta\gamma_{xy}^0 + M_x^b\delta k_x^b + M_y^b\delta k_y^b + M_{xy}^b\delta k_{xy}^b \right.\\ &\quad \left. + M_x^s\delta k_x^s + M_y^s\delta k_y^s + M_{xy}^s\delta k_{xy}^s + S_{yz}^s\delta\gamma_{yz} + S_{xz}^s\delta\gamma_{xz} - q\delta(w_b + w_s + w_{st})\right]dA\end{aligned} \tag{10}$$

where q is the transverse load; and N, M, and Q are the stress resultants defined by

$$\begin{Bmatrix}N_x, & N_y, & N_{xy}\\ M_x^b, & M_y^b, & M_{xy}^b\\ M_x^s, & M_y^s, & M_{xy}^s\end{Bmatrix} = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} (\sigma_x, \sigma_y, \tau_{xy})\begin{Bmatrix}1\\ z_{ns}\\ f(z_{ns})\end{Bmatrix} dz_{ns}, \tag{11}$$

By substituting the results of Eq. (5) into Eq. (11), the stress resultants are obtained as

$$\begin{Bmatrix} N \\ M^b \\ M^s \end{Bmatrix} = \begin{bmatrix} A & 0 & B^a \\ 0 & D & D^a \\ B^a & D^a & F^a \end{bmatrix} \begin{Bmatrix} \varepsilon \\ k^b \\ k^s \end{Bmatrix} + \begin{bmatrix} L \\ L^a \\ R \end{bmatrix} \varepsilon_z^0, \quad S = A^s \gamma, \tag{12a}$$

$$N_z = R^a \varphi + L\left(\varepsilon_x^0 + \varepsilon_y^0\right) + L^a\left(k_x^b + k_y^b\right) + R\left(k_x^s + k_y^s\right), \tag{12b}$$

where

$$N = \left\{N_x, N_y, N_{xy}\right\}, \quad M^b = \left\{M_x^b, M_y^b, M_{xy}^b\right\}, \quad M^s = \left\{M_x^s, M_y^s, M_{xy}^s\right\}, \tag{13}$$

where the stiffness coefficients A_{ij} and D_{ij} ,… etc., are defined as

$$\begin{Bmatrix} A_{11} & D_{11} & B_{11}^s & D_{11}^s & H_{11}^s \\ A_{12} & D_{12} & B_{12}^s & D_{12}^s & H_{12}^s \\ A_{12} & D_{66} & B_{66}^s & D_{66}^s & H_{66}^s \end{Bmatrix} = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \lambda(z_{ns})\left(1, z_{ns}^2, f(z_{ns}), z_{ns}\ f(z_{ns}), f^2(z_{ns})\right) \begin{Bmatrix} \frac{1-\nu}{\nu} \\ 1 \\ \frac{1-2\nu}{2\nu} \end{Bmatrix} dz_{ns}, \tag{14}$$

Integrating the expressions in Eq. (10) by parts and collecting the coefficients of δu_0, δv_0, δw_b, δw_s and $\delta \varphi$, one obtains the following governing equations

$$\begin{aligned}
&\delta u_0: \quad \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\
&\delta v_0: \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \\
&\delta w_b: \quad \frac{\partial^2 M_x^b}{\partial x^2} + 2\frac{\partial^2 M_{xy}^b}{\partial x \partial y} + \frac{\partial^2 M_y^b}{\partial y^2} + q = 0 \\
&\delta w_s: \quad \frac{\partial^2 M_x^s}{\partial x^2} + 2\frac{\partial^2 M_{xy}^s}{\partial x \partial y} + \frac{\partial^2 M_y^s}{\partial y^2} + \frac{\partial S_{xz}^s}{\partial x} + \frac{\partial S_{yz}^s}{\partial y} + q = 0 \\
&\delta \varphi: \quad \frac{\partial S_{xz}^s}{\partial x} + \frac{\partial S_{yz}^s}{\partial y} - N_z = 0
\end{aligned} \tag{15}$$

By substituting Eq. (12) into Eq. (15), the equations of motion can be expressed in terms of displacements (δu_0, δv_0, δw_b, δw_s, $\delta \varphi$) as

$$A_{11} d_{11} u_0 + A_{66} d_{22} u_0 + \left(A_{12} + A_{66}\right) d_{12} v_0 - \left(B_{12}^s + 2B_{66}^s\right) d_{122} w_s - B_{11}^s d_{111} w_s + L d_1 \varphi = 0 \tag{16a}$$

$$A_{22} d_{22} v_0 + A_{66} d_{11} v_0 + \left(A_{12} + A_{66}\right) d_{12} u_0 - \left(B_{12}^s + 2B_{66}^s\right) d_{112} w_s - B_{22}^s d_{222} w_s + L d_2 \varphi = 0 \tag{16b}$$

$$\begin{aligned} &- D_{11} d_{1111} w_b - 2\left(D_{12} + 2D_{66}\right) d_{1122} w_b - D_{22} d_{2222} w_b - D_{11}^s d_{1111} w_s - 2\left(D_{12}^s + 2D_{66}^s\right) d_{1122} w_s \\ &- D_{22}^s d_{2222} w_s + L^a\left(d_{11}\varphi + d_{22}\varphi\right) + q = 0 \end{aligned} \tag{16c}$$

$$\begin{aligned} &B_{11}^s d_{111} u_0 + \left(B_{12}^s + 2B_{66}^s\right) d_{122} u_0 + \left(B_{12}^s + 2B_{66}^s\right) d_{112} v_0 + B_{22}^s d_{222} v_0 - D_{11}^s d_{1111} w_b - 2\left(D_{12}^s + 2D_{66}^s\right) d_{1122} w_b \\ &- D_{22}^s d_{2222} w_b - H_{11}^s d_{1111} w_s - 2\left(H_{12}^s + 2H_{66}^s\right) d_{1122} w_s - H_{22}^s d_{2222} w_s + A_{44}^s d_{11} w_s + A_{55}^s d_{22} w_s \\ &+ R\left(d_{11}\varphi + d_{22}\varphi\right) + A_{44}^s d_{11}\varphi + A_{55}^s d_{22}\varphi + q = 0 \end{aligned} \tag{16d}$$

$$L\left(d_1 u_0 + d_2 v_0\right) - L^a\left(d_{11} w_b + d_{22} w_b\right) + \left(R - A_{44}^s\right) d_{11} w_s + \left(R - A_{55}^s\right) d_{22} w_s + R^a \varphi - A_{44}^s d_{11}\varphi - A_{55}^s d_{22}\varphi = J_1^s\left(\ddot{w}_b + \ddot{w}_s\right) + K_2^s \ddot{\varphi} \tag{16e}$$

Rectangular plates are generally classified according to the type of support used. Here, we are concerned with the exact solutions of Eqs. (16) for a simply supported FG plate. The following boundary conditions are imposed at the side edges:

$$v_0 = w_b = w_s = \frac{\partial w_s}{\partial y} = \varphi = N_x = M_x^b = M_x^s = 0 \quad \text{at} \quad x = 0, a \tag{17a}$$

$$u_0 = w_b = w_s = \frac{\partial w_s}{\partial x} = \varphi = N_y = M_y^b = M_y^s = 0 \quad \text{at} \quad y = 0, b \tag{17b}$$

Following the Navier solution procedure, we assume the following solution form for u_0, v_0, w_b, w_s and φ that satisfies the boundary conditions given in Eq. (17),

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_b \\ w_s \\ \varphi \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn}\cos(\lambda x)\sin(\mu y) \\ V_{mn}\sin(\lambda x)\cos(\mu y) \\ W_{bmn}\sin(\lambda x)\sin(\mu y) \\ W_{smn}\sin(\lambda x)\sin(\mu y) \\ \Phi_{mn}\sin(\lambda x)\sin(\mu y) \end{Bmatrix} \tag{18}$$

where U_{mn}, V_{mn}, W_{bmn}, W_{smn} and Φ_{mn} are arbitrary parameters to be determined, and $\lambda = m\pi / a$ and $\mu = n\pi / b$.

The transverse load q is also expanded in the double-Fourier sine series as

$$q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\sin(\lambda x)\sin(\mu y) \tag{19}$$

For the case of a sinusoidally distributed load, we have $m = n = 1$ and $q_{11} = q_0$ where q_0 represents the intensity of the load at the plate centre.

3. NUMERICAL RESULTS

The results are presented for the simply supported plate under bi-sinusoidal transverse loads of intensity q. The static analysis was conducted using aluminium (bottom, Al) and alumina (top, Al_2O_3).

Figure 1. presents the variation of non-dimensional parameter C/h versus the material parameter n of Al/Al_2O_3 functionally graded plate. It can be observed when the material parameter of FGM becomes zero (fully ceramic) or infinity (fully metallic); the neutral surface coincides on the middle surface, as expected.

Table 1. contains dimensionless transverse displacement and normal stresses of FG plate for various values of thickness ratio a/h, and material parameter n. The present theory with $\varepsilon_z \neq 0$ is compared with analytical solutions by Carrera et al. [10], the quasi-3D sinusoidal shear deformation theory of Neves et al. [9], the classical plate theory (CPT) [9], and the first-order shear deformation theory (FSDT) [9]. It can be seen that the dimensionless displacement and stresses predicted by the new trigonometric higher-order theory with the stretching effect are almost identical with those generated by the quasi-3D sinusoidal theories of Neves et al. [9]. It should be noted that the present theory involves five unknowns as against six or more in other quasi-3D shear deformation theory.

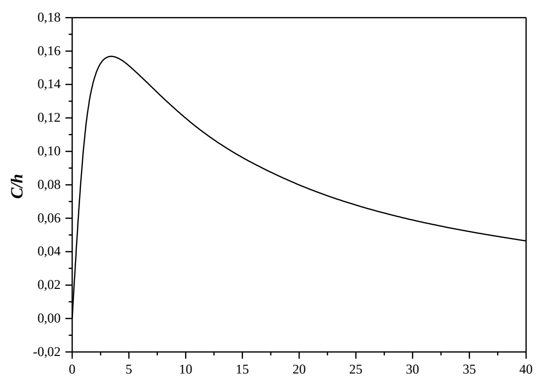

FIGURE 1. Variation of the neutral surface position versus the material parameter n.

n	Theory	ε_z	$\bar{\sigma}_x(a/2, b/2, h/3)$			$\bar{w}(a/2, b/2, 0)$		
			$a/h = 4$	$a/h = 10$	$a/h = 100$	$a/h = 4$	$a/h = 10$	$a/h = 100$
1	CPT [16]	0	0.8060	2.0150	20.150	0.5623	0.5623	0.5623

	FSDT k = 5/6 [16]	0	0.8060	2.0150	20.150	0.7291	0.5889	0.5625
	Neves et al [16]	$\neq 0$	0.5925	1.4945	14.969	0.6997	0.5845	0.5624
	Carrera et al [21]	$\neq 0$	0.6221	1.5064	14.969	0.7171	0.5875	0.5625
	Present	$\neq 0$	0.6021	1.5001	14.659	0.6919	0.5795	0.5562
4	CPT [16]	0	0.6420	1.6049	16.049	0.8281	0.8281	0.8281
	FSDT k = 5/6 [16]	0	0.6420	1.6049	16.049	1.1125	0.8736	0.828
	Neves et al [16]	$\neq 0$	0.4404	1.1783	11.932	1.1178	0.8750	0.8286
	Carrera et al [21]	$\neq 0$	0.4877	1.1971	11.923	1.1585	0.8821	0.8286
	Present	$\neq 0$	0.4366	1.1335	11.409	1.0991	0.8562	0.8021
10	CPT [16]	0	0.4796	1.1990	11.990	0.9354	0.9354	0.9354
	FSDT k = 5/6 [16]	0	0.4796	1.1990	11.990	1.3178	0.9966	0.9360
	Neves et al [16]	$\neq 0$	0.3227	1.1783	11.932	1.3490	0.8750	0.8286
	Carrera et al [21]	$\neq 0$	0.1478	0.8965	8.9077	1.3745	1.0072	0.9361
	Present	$\neq 0$	0.3206	0.8511	8.6055	1.3369	0.9820	0.9142

TABLE 1. Effect of normal strain ε_z on the dimensionless in-plane longitudinal stress $\bar{\sigma}_x$ and displacement $\bar{w}$ for FG square plate.

4. CONCLUSIONS

A higher order shear and normal deformation theory based on neutral surface position for the bending analysis of advanced composite plates is presented. The theory accounts for the stretching and shear deformation effects without requiring a shear correction factor. By dividing the transverse displacement into bending, shear and stretching components, the number of unknowns and governing equations of the present theory is reduced to five as against six or more unknown in the corresponding ones. Based on the present plate theory and the neutral surface concept, the governing equations are derived from the principle of virtual work. The accuracy of the present model is ascertained by comparing it with existing solutions and excellent agreement was observed. It is relevant to notice the strong effect of considering the non-zero transverse normal strain.

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